

UNCERTAINTY ANALYSIS IN VOLTERRA SERIES APPLIED IN A NONLINEAR SYSTEM

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Abstract: The goal of this paper is to verify the influence of uncertainties in the identification of Volterra kernels applied in a single degree-of-freedom nonlinear model with cubic stiffness. Stochastic modelling and Monte-Carlo simulations were performed for the identification of the Volterra kernels considering variations in the parameters of the motion equations, with the aim of verify how the kernels change with the presence of uncertainties. The results are evaluated by establishing confidence intervals in the kernels. These results allow to propose a statistical decision if the the kernels are representative of the nonlinear behavior of the systems even with uncertainties.

Keywords: Volterra series, Uncertainty Quantification, Duffing oscillator, Nonlinear Systems.

1. INTRODUCTION

It is known that many engineering structures have geometric and operating conditions, significant relationships in materials and their excitations applications and loads leading to highly nonlinear effects. Thus, it should necessarily be considered in the analysis of the dynamic behavior of such structures, according to many authors (Virgin, 2000; Worden and Tomlinson, 2001; Kerschen *et al.*, 2006). Volterra series expanded in Kautz filters is a powerful technique to identify nonlinear systems due to several reasons (Shiki *et al.*, 2013b; da Silva, 2011a). Several papers as Shiki *et al.* (2013a), da Silva *et al.* (2010), Shiki *et al.* (2014), Hansen *et al.* (2014a) e Hansen *et al.* (2014b) have shown the practical application of this approach.

However, experimental tests are full of uncertainties due to imperfections in measuring instruments or the difficulty data acquisition (Oden *et al.*, 2010). There are two types of uncertainties in mathematical-mechanical modelling of real systems, the data uncertainties and the model uncertainties (Soize, 2005). One of the goals of quantifying uncertainties is to predict the possible variations of the responses, obtained by the model, making it possible to identify errors that may be obtained when compared with physical results and then establish confidence limits (Iaccarino, 2009).

Thus, this paper deals with the uncertainty analysis in the models used in the Volterra kernels identification. Analysis also, as the uncertainties may interfere in the functions and Kautz parameters used in the estimation of Volterra kernels. Hence, it must defined confidence limits for identifying towards a future damage detection in mechanical systems with nonlinear behavior, assuming uncertainties in its modeling.

2. VOLTERRA SERIES

The response $x(k)$ of a nonlinear system can be approximated by discrete-time Volterra series using multiple convolutions (Schetzen, 1980):

$$x(k) = \sum_{\eta=1}^{+\infty} \mathcal{H}_{\eta}(k) = x_1(k) + x_2(k) + x_3(k) + \dots \quad (1)$$

where $x_1(k)$, $x_2(k)$, $x_3(k)$, $\dots$ are the linear, quadratic, cubic and so on contributions of the output $x(k)$ in $k = 1, \dots, K$ (K is the number of time samples) and $\mathcal{H}_{\eta}(k)$ is the Volterra functional given by multidimensional convolutions:

$$\mathcal{H}_{\eta}(k) = \sum_{n_1=0}^{N_1} \dots \sum_{n_{\eta}=0}^{N_{\eta}} \mathcal{H}_{\eta}(n_1, \dots, n_{\eta}) \prod_{i=1}^{\eta} u(k - n_i)$$

where $u(k)$ is the input signal and $\mathcal{H}_{\eta}(n_1, \dots, n_{\eta})$ are the η *th*-order Volterra kernels considering the truncated values $N_1, \dots, N_{\eta}$ for each kernel.

The drawback of Volterra series is the difficulty of convergence of the series when using a large number of terms $N_1, \dots, N_{\eta}$. Fortunately, the expansion of Volterra kernels in some orthonormal basis minimize the problem, especially with the use of Kautz functions (Kautz, 1954; Heuberger *et al.*, 2005; da Silva, 2011b). Thus, the Volterra kernels can be written as:

$$\mathcal{H}_{\eta}(n_1, \dots, n_{\eta}) \approx \sum_{i_1=1}^{J_1} \dots \sum_{i_{\eta}=1}^{J_{\eta}} \mathcal{B}_{\eta}(i_1, \dots, i_{\eta}) \prod_{j=1}^{\eta} \psi_{i_j}(n_j) \quad (2)$$

where $J_1, \dots, J_{\eta}$ are the number of samples in each orthonormal projections of the Volterra kernels $\mathcal{B}_{\eta}(i_1, \dots, i_{\eta})$ and $\psi_{i_j}(n_i)$ are the Kautz functions.

It is important to see that the order of projection $\mathcal{B}_{\eta}(i_1, \dots, i_{\eta})$ is lower and easier to obtain than the order of Volterra kernel given by $\mathcal{H}_{\eta}(n_1, \dots, n_{\eta})$.

Thus, it is possible to rewrite the eq. (1) based on the orthonormal Kautz basis:

$$x(k) \approx \sum_{\eta=1}^{+\infty} \mathcal{B}_{\eta}(k) \quad (3)$$

where $\mathcal{B}_{\eta}(k)$ is the η -*th* orthonormal Volterra functional operator:

$$\mathcal{B}_{\eta}(k) \approx \sum_{i_1=1}^{J_1} \dots \sum_{i_{\eta}=1}^{J_{\eta}} \mathcal{B}_{\eta}(i_1, \dots, i_{\eta}) \prod_{j=1}^{\eta} l_{i_j}(k)$$

that is a multiple convolution between the orthonormal kernel given by $\mathcal{B}_{\eta}(i_1, \dots, i_{\eta})$ and $l_{i_j}(k)$, that is a simple filtering of input signal $u(k)$ by the Kautz function $\psi_{i_j}(n_i)$:

$$l_{i_j}(k) = \sum_{n_i=0}^{V-1} \psi_{i_j}(n_i) u(k - n_i) \quad (4)$$

where $V = \max\{J_1, \dots, J_{\eta}\}$.

The values of orthonormal Volterra kernel $\mathcal{B}_{\eta}(i_1, \dots, i_{\eta})$ can be grouped in a vector Φ and can be found by solving:

$$\Phi = (\Gamma^T \Gamma)^{-1} \Gamma^T \mathbf{x} \quad (5)$$

where the matrix Γ contains $l_{i_j}(k)$ and $\mathbf{x} = [x(1) \dots x(K)]$. It is worth to note that η can be usually truncated in 3 kernels to represent the most part of the structural nonlinearities with smooth behavior. In this work will be used only the first and third kernels for symmetry in the response of the studied system.

2.1 Kautz functions

The Kautz functions perform well in representing the orthonormal kernels to identify the Volterra kernels in oscillatory dynamic models, so they are used in this work (Kautz, 1954).

The Kautz functions $\psi(z)$ are defined by conjugated complex parameters represented by $\beta_{2g-1} = \sigma + j\omega$ and $\beta_{2g} = \sigma - j\omega$, where $|\beta_{2g-1}|, |\beta_{2g}| < 1$ for a stable system, where g represents the number of the kernel, varying in this case 1 and 3. Thus, the Kautz functions of generalized form are given by:

$$\psi_{2j-1}(z) = \frac{\sqrt{1-b^2}\sqrt{1-c^2}}{z^2 + b(c-1)z - c} [H_{b,c}(z)]^{j-1} \quad (6)$$

$$\psi_{2j}(z) = \psi_{2j-1}(z) \frac{z-b}{\sqrt{1-b^2}} \quad (7)$$

where $H_{b,c}(z) = \frac{-cz^2 + b(c-1)z + 1}{z^2 + b(c-1)z - c}$ and the values of b and c , for the parameters β_{2g-1}, β_{2g} considered are obtained through relationships:

$$b = \frac{\beta_{2g-1} + \beta_{2g}}{1 + \beta_{2g-1}\beta_{2g}} \quad (8)$$

$$c = -\beta_{2g-1}\beta_{2g} \quad (9)$$

where the definition parameters of the Kautz functions in continuous domain are a function of frequency (ω_{ng}) and the damping factor (ξ_g):

$$\beta_g = -\xi_g \omega_{ng} \pm j \omega_{ng} \sqrt{1 - \xi_g^2} \quad (10)$$

A procedure for optimizing the choice of setting parameters of the Kautz functions (β_g) must be used (da Rosa *et al.*, 2007; da Rosa, 2009).

3. APPLICATION IN A NONLINEAR BEAM

3.1 Nominal Model

Simulations were performed in a Duffing oscillator, representing the model of the cantilever beam (Figure 1). The motion equation of the beam can be described by:

$$\frac{3m'L}{8} \ddot{x} + \left[\frac{2EI\pi^4}{L^3} + k \right] x + \frac{EA\pi^4}{8L^3} x^\alpha = F(t) \quad (11)$$

where E is the Young's modulus, A is the area of the transversal section, I is the inertia's moment, L is the length, k is the stiffness of the resilient element, m' is the distribution of mass $m(x)$, $F(t)$ is the excitation and $\ddot{x}$, $\dot{x}$ e x are the acceleration, speed and displacement of the beam center, respectively. This model represents the vibration in nonlinear regime in first mode region.

The equation for a Duffing oscillator also considering a viscous damping, is described by (Brennan and Kovacic, 2011):

$$m\ddot{x} + c\dot{x} + k_1x + k_3x^\alpha = F(t) \quad (12)$$

where m is the the equivalent mass, c is the damping, k_1 is the linear stiffness and k_3 is the cubic stiffness.

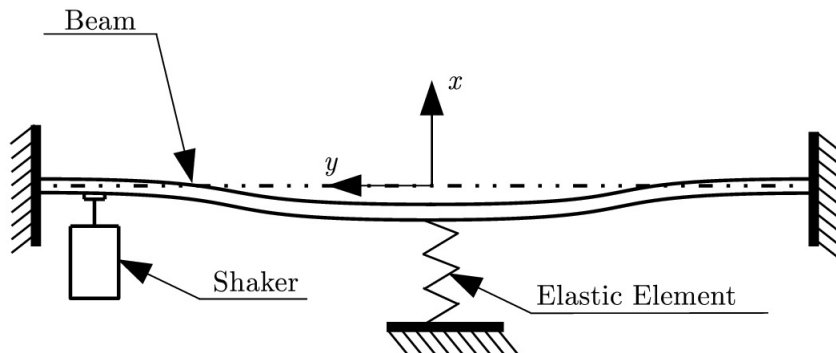


Figure 1. Cantilever Beam (1st mode).

By dividing the eq. (12) by m and doing $\beta = \frac{k_3}{m}$:

$$\ddot{x} + 2\zeta\omega_n\dot{x} + \omega_n^2x + \beta x^\alpha = F(t) \quad (13)$$

where ω_n is the undamped natural frequency of the corresponding linear system, ζ is the damping ratio and β is the nonlinear factor.

By comparing the three expressions of motion, it can find the system parameters through the following expressions:

$$m = \frac{3m'L}{8} \quad \omega_n = \sqrt{\frac{Ehb^3\pi^4}{6mL^3} + \frac{k}{m}} \quad \beta = \frac{bhE\pi^4}{8mL^3} \quad (14)$$

where b is the width and h is the thickness. The beam properties used in the simulations of this study are shown in table 1.

Table 1. Geometricla and physcal properties of the cantilever beam.

Parameter	Value
Young's modulus (E)	62 GPa
Beam length (L)	0.92 m
Beam width (b)	25.4 mm
Beam thickness (h)	3.3 mm
Constant (k)	50 N/m

The parameters shown in the table 1 and the equations shown in 14 are used to determine the parameters of the motion equation for the Duffing oscillator. The results are described in the table 2.

Table 2. Parameters to simulate the Duffing oscillator.

Parameter	Value
α	3
ζ	0.01
ω_n [Hz]	20
β [N/kg m ³]	1.04×10^9

This model is considered the nominal. It is worth observe that some parameters can be unknown or have some uncertainty associated. Next sections describe some numerical simulation considering a sthochastic modelling of some parameters to observe the changes in the Volterra kernels extracted in the system.

3.2 Stochastic model

Simulations were performed considering uncertainty in the Young's modulus, which was modeled as a random variable $\mathbb{E}$. Considering that the Young's modulus cannot be negative, we assumed the interval $(0, \infty)$ as the support of this random variable. Also, it was considered that the expect value of $\mathbb{E}$ is a known real number $\mu_{\mathbb{E}}$. For technical reasons, see Soize (2005) for details, we also suppose that the expected value of $\ln \mathbb{E}$ is finite. Using these conditions as known information, such as done in Cunha Jr and Sampaio (2015), the principle of maximum entropy says that the probability density function (PDF) of this random variable is given by

$$p_{\mathbb{E}}(e) = \mathbb{1}_{(0,\infty)} \frac{1}{\mu_{\mathbb{E}}} \left(\frac{1}{\delta_{\mathbb{E}}^2} \right)^{\left(\frac{1}{\delta_{\mathbb{E}}^2} \right)} \frac{1}{\Gamma(1/\delta_{\mathbb{E}}^2)} \left(\frac{e}{\mu_{\mathbb{E}}} \right)^{\left(\frac{1}{\delta_{\mathbb{E}}^2} - 1 \right)} \exp \left(- \frac{e}{\delta_{\mathbb{E}}^2 \mu_{\mathbb{E}}} \right) \quad (15)$$

where $\delta_{\mathbb{E}}$ is the dispersion of the parameter, Γ indicates de gamma function, and $\mathbb{1}_{(0,\infty)}$ denotes the indicator function of the interval $(0, \infty)$. This PDF corresponds to a gamma distribution.

To compute the propagation of uncertainties of the random parameter $\mathbb{E}$ through the model, the Monte Carlo (MC) method (Robert and Casella, 2010) is employed, using 2048 realizations of the deterministic dynamic system, with $\mu_{\mathbb{E}} = 62$ GPa and $\delta_{\mathbb{E}} = 0.1$. All numerical simulations were performed using a sampling frequency of 500 Hz and 4096 samples. The identification of the Volterra kernels was performed in two steps by using a chirp signal with low amplitude (0.05 N), and high amplitude (0.5 N), both with frequency range from 15 to 25 Hz. The identification of the kernels was performed considering the variation of the system response, caused by the presence of uncertainty in parameter $\mathbb{E}$.

3.3 Study of MC convergence

Cunha Jr and Sampaio (2014) used a method for determining the number of samples needed for the convergence of the Monte-Carlo (MC) method when we consider the dynamic response of a mechanical system. The method is based on Euclidean norm of the response signal used, be described as follows:

$$conv(n_s) = \sqrt{\frac{1}{n_s} \sum_{n=1}^{n_s} \int_{t=t_0}^{t_f} \|x(t, n)\|^2 dt} \quad (16)$$

being, n_s the number of samples used in MC simulations, $\|\cdot\|$ represents the standard Euclidean norm and $x(t, n)$ the response signal of the system.

For more details about this criteria can be found in Soize (2005). The convergence metric for MC simulations, as a function of the number of realizations, can be seen in Figure 2. In these tests the output signal was obtained by the Volterra kernels considering the high level chirp excitation. Note that convergence was achieved.

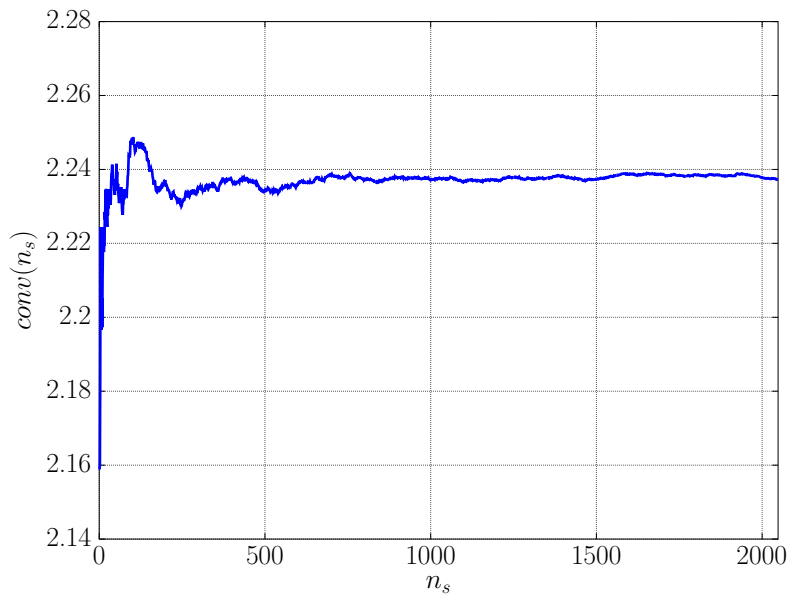


Figure 2. Convergence test for the number of samples used in MC simulations.

3.4 Propagation of uncertainties through the system

Estimations for the normalized PDFs of ω_n and β are presented in Figure 3. In this context, normalized means a distribution with zero mean and unit standard deviation. Also, in Figure 4 is shown the normalized PDFs of the Kautz functions parameters ω_1 , ω_3 , ξ_1 , and ξ_3 , which were obtained by solving an optimization process. We can observe certain similarity in the shape of ω_1 , ω_3 , and ω_n distributions. However, the distribution of ξ_1 presents bimodal behavior, with maxima occurring at the (compact) support border. This support is compact because limits were imposed for it, in the optimization process, to ensure that the model describes the system behavior in linear terms. On the other hand, the PDF of ξ_3 has multi-modal behavior and is completely different from those observed in other distributions. This occurs, possibly, because this parameter is influenced by ω_n and β during its determination process.

A band of reliability, defined as the mean value plus or minus one standard deviation, for the first kernel and the main diagonal of the third kernel can be seen in Figure 5. One can note a certain level of dispersion in the first kernel, due to the dependency of ω_n on $\mathbb{E}$. The third kernel also shows significant dispersion around the mean, because the Young's modulus is related with β , and this parameter is related only with the nonlinear fraction response.

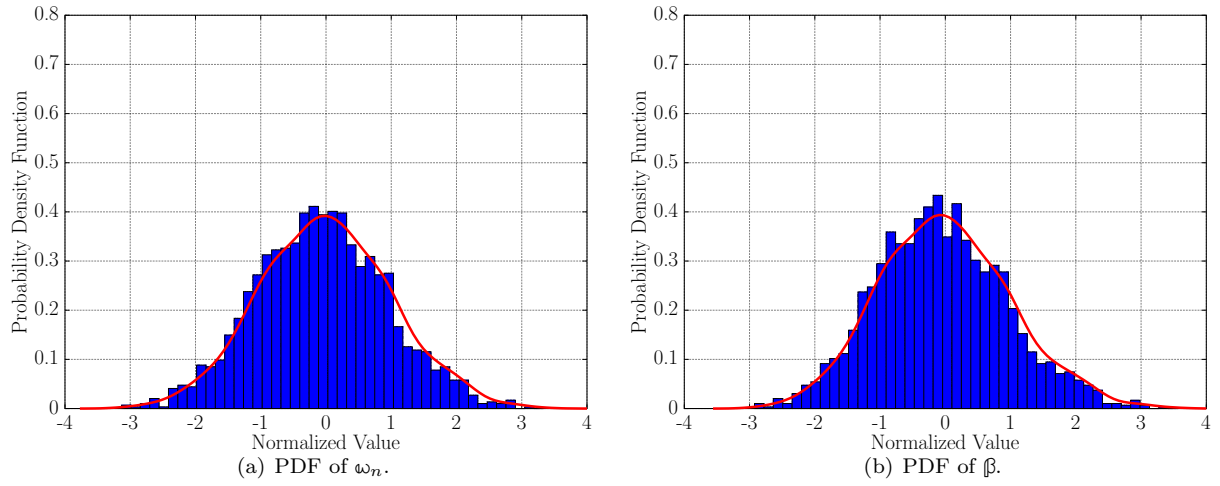


Figure 3. Estimations to the PDFs of ω_n and β .

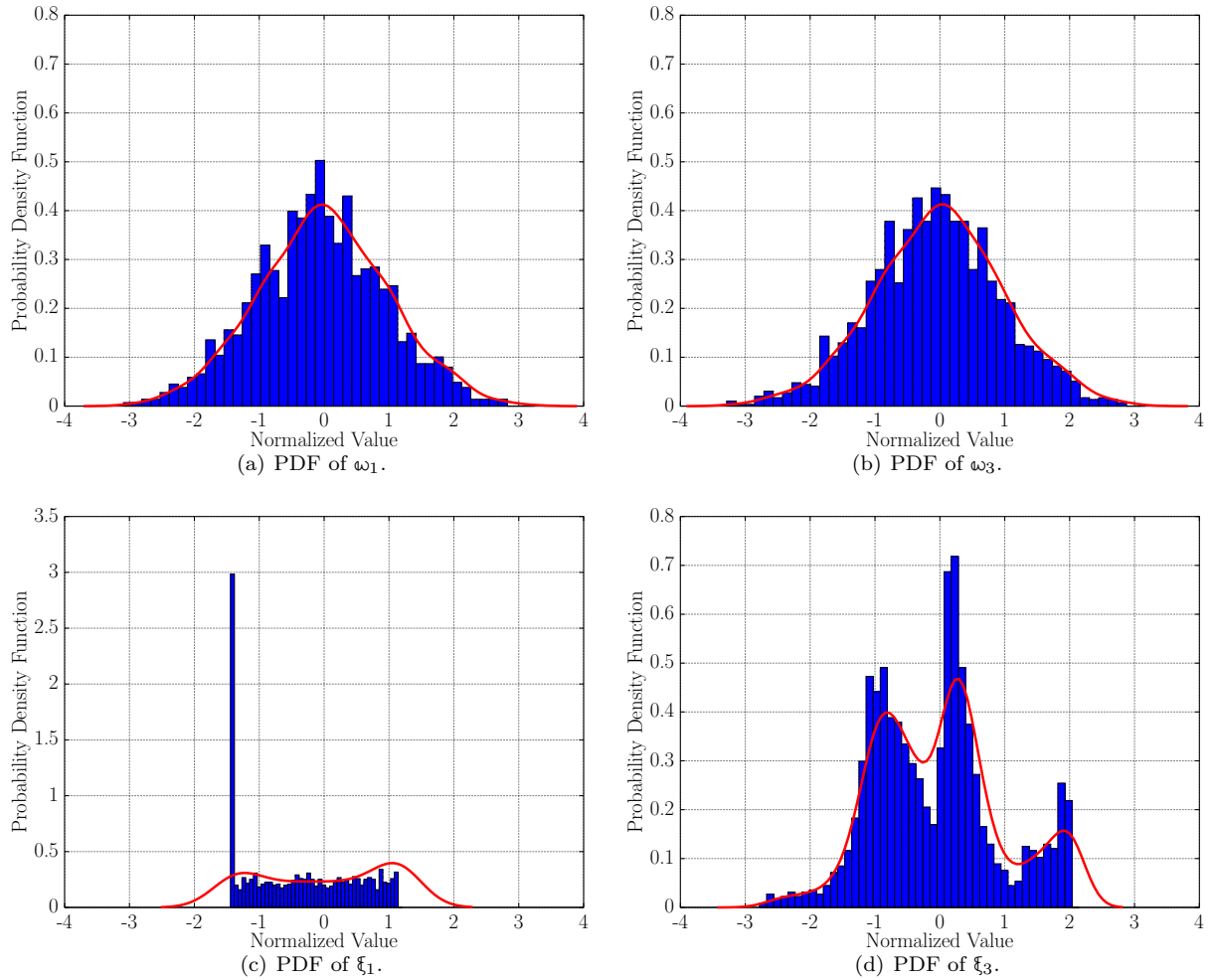


Figure 4. Estimations to the PDFs of ω_1 , ω_3 , ξ_1 and ξ_3 .

The confidence band shown in Figure 6 are associated with the response and prediction of error obtained using the Volterra kernels, for a chirp excitation, with frequency ranging from 15 to 25 Hz and low amplitude (0.05 N). It may be noted a great variation in the output, which was expected by the large change of the first kernel. The variation of the prediction error is acceptable, which justifies the limits used in the optimization process.

In Figure 7 is presented the linear and cubic contribution of the total response, obtained through the Volterra kernels for a low input, and a confidence band (similar to the previous one) around then. Clearly, the system behavior is linear in this case. The large variation in the linear contribution is due to the large dispersion observed in the first kernel.

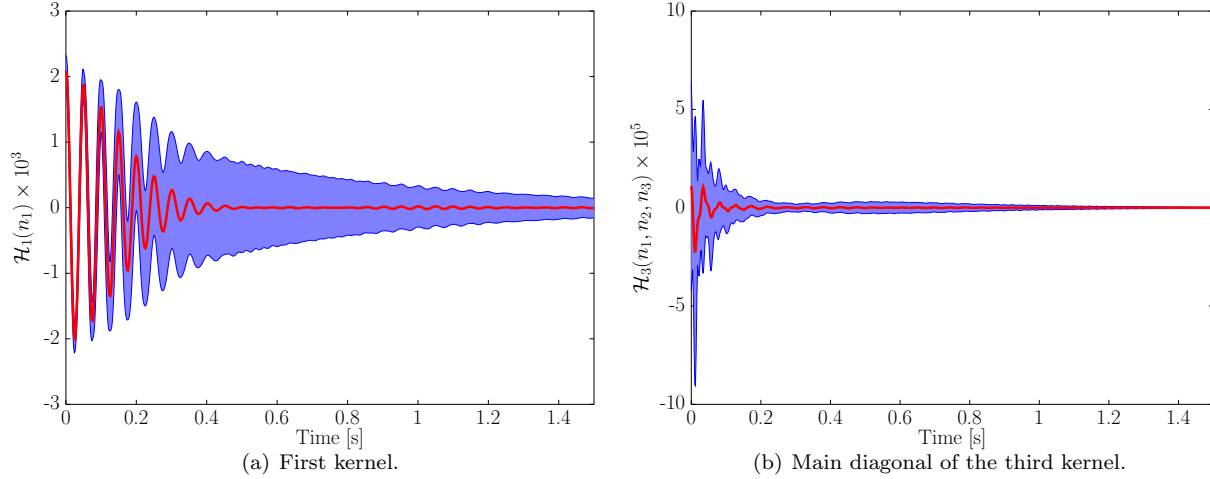


Figure 5. Confidence band for Volterra kernels considering the mean $\pm$ one standard deviation. The mean is represented by -.

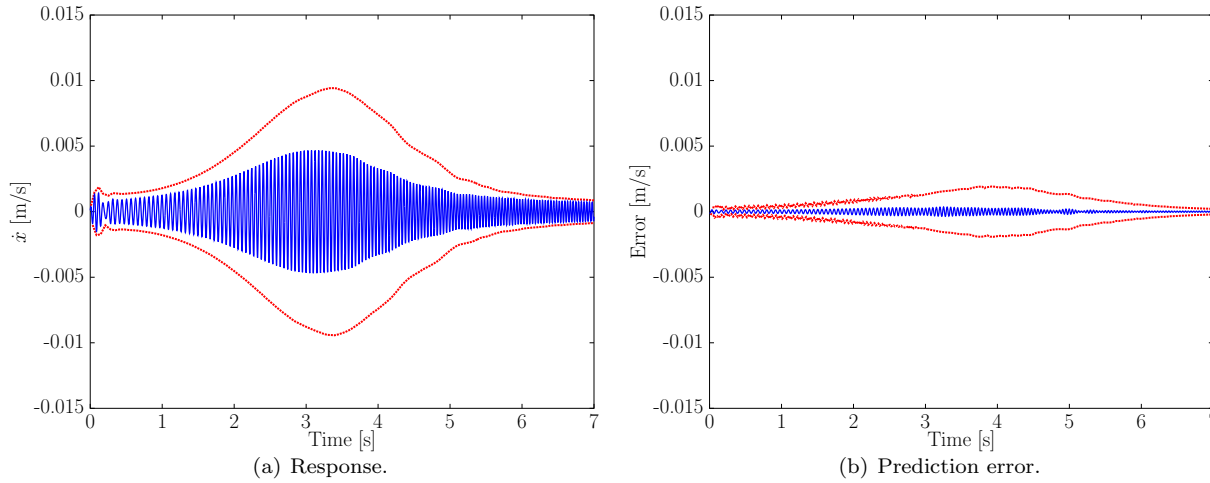


Figure 6. Confidence band for the response obtained by Volterra kernels considering low input. The mean is represented by - and the mean $\pm$ one standard deviation is represented by -.

The confidence bands for the response and prediction error, obtained using the Volterra kernels for a chirp excitation with frequency sweeping up from 15 to 25 Hz, considering the high force amplitude (0.5 N) are shown in Figure 8. In Figure 9 is shown the linear and cubic contribution of the response for the same input. There is no quadratic contribution by the symmetry of response. The prediction error is small, which shows a good description of the model. There is large variation in the response which reflects the wide variation both linear as cubic contribution. Because of the dispersion values of Young's modulus are large, there is a big propagation of uncertainty, leading to large variations in the response obtained, which should be reflected in difficulty in damage detection.

4. FINAL REMARKS

The results obtained have shown that the Volterra kernels are sensible and with a large variation in function of the parametric uncertainty in the value of the Young's modulus. Experimental errors obtained in the data acquisition have shown that the estimation of the Volterra kernel could difficult the process of identification seeking damage detection. Further tests need to be performed to establish confidence limits to identify the Volterra kernels.

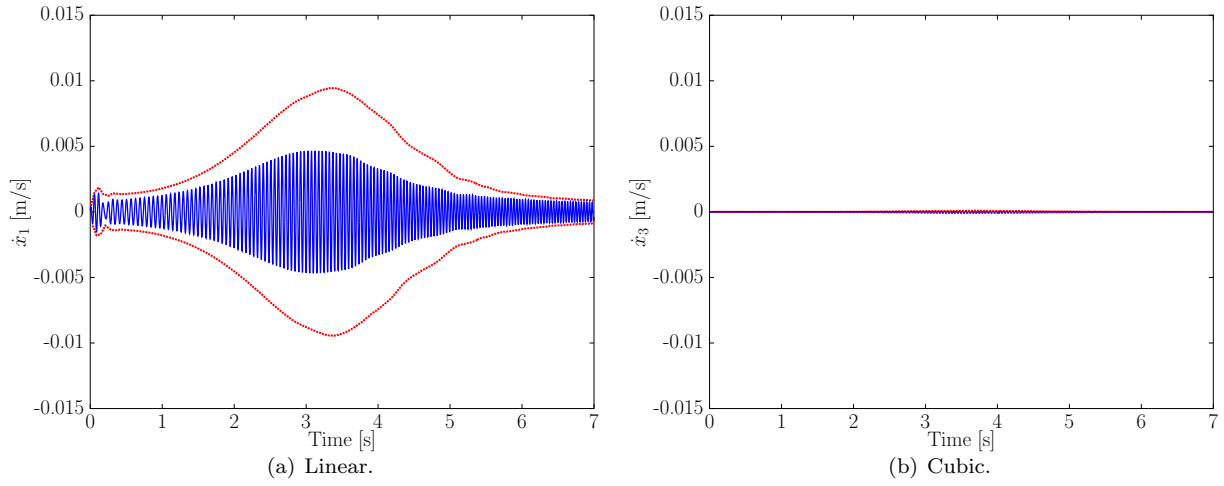


Figure 7. Confidence band for the contribution of Volterra kernels considering low input. The mean is represented by - and the mean $\pm$ one standard deviation is represented by -.

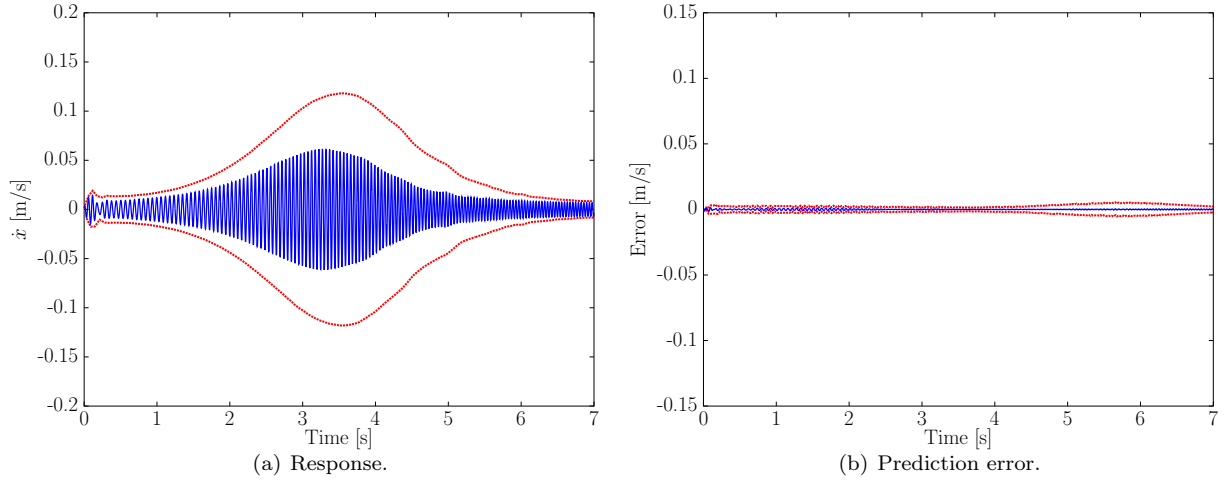


Figure 8. Confidence band for the response obtained by Volterra kernels considering high input. The mean is represented by - and the mean $\pm$ one standard deviation is represented by -.

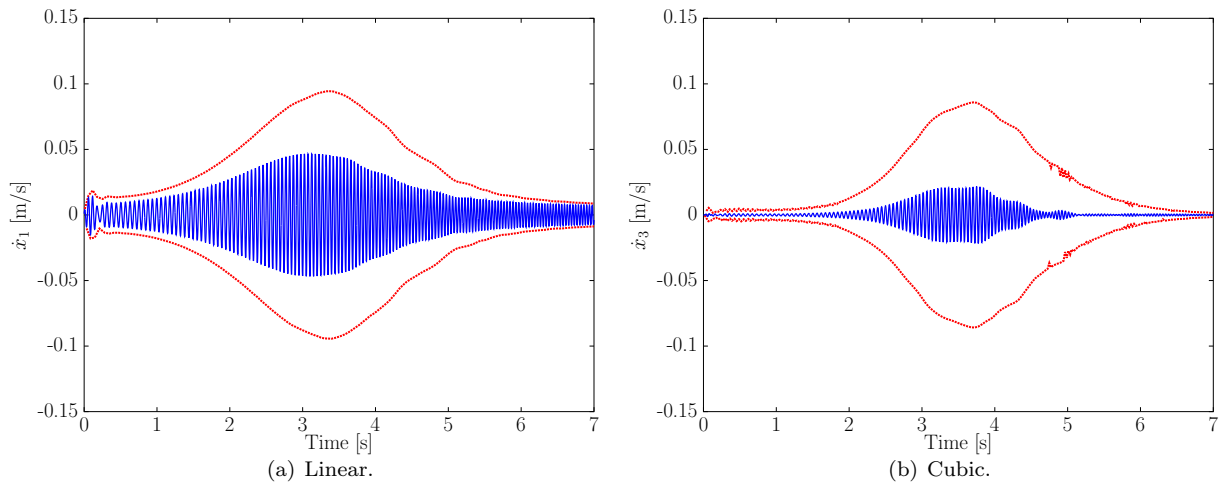


Figure 9. Confidence band for the contribution of Volterra kernels considering high input. The mean is represented by - and the mean $\pm$ one standard deviation is represented by -.

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