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HOW TO DESIGN NONLINEAR OSCILLATORS WITH AN AMPLITUDE-INDEPENDENT PERIOD

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Abstract. *It is well-known that the dynamics of nonlinear oscillators in general involves a relationship between the amplitude of periodic motion and its period/frequency. There may be some situations where such dependence is undesirable. This raises the question of designing a system which is nonlinear, but whose period is constant and amplitude-independent, corresponding to the so-called isochronous motion. The solution to this problem is provided, both in terms of general mathematical and mechanical models by starting from a simple pendulum. The examples when the simple pendulum and its cubic approximation perform isochronous motion are also given. In addition, the issue of design of nonlinear systems that have a similar equation of motion but are damped is also addressed and illustrated.*

Keywords: *pendulum, isochronous motion, nonlinear oscillators, potential energy, kinetic energy*

1. INTRODUCTION

A pendulum is one of the classical paradigms in modeling and studying oscillatory motion and associated phenomena in physics and engineering. Being inspired by swinging motion of a chandelier in the Pisa cathedral, Italian scientist Galileo Galilei was the first to study the properties of a pendulum, finding that its period is independent of the mass of the bob, but depends on the constant of gravity and on the pendulum length. Beginning around 1602 (Stillman, 2003), Galileo observed that the pendulum does not change its period even if its amplitude changes, i.e. that it has the so-called isochronism property/isochronicity. Galileo discovered that this property holds for small angles, and extended this, erroneously, even to larger swings. Despite this, he could determine the period of every pendulum with an acceptable approximation for that time. Following Galileo's epochal analysis of pendulum motion, Christian Huygens utilised the pendulum in clockwork in 1673, and so provided the world's first accurate measure of time (Matthews, *et al*., 2005). Huygens realized that to make oscillation isochronous, the pendulum should describe a cycloid arc.

Although there have been almost three and a half centuries since then, this property of isochronicity is still intriguing to researchers and engineers. In the field of nonlinear dynamics, isochronicity has been investigated from the viewpoint of beneficial practical applications in vibration absorbers (Show and Geist, 2010; Legeza, 2011; Mayet and Ulbricht, 2014). The author and her collaborators have recently contributed to the detected shortage of general models of nonlinear mechanical systems and the corresponding equations of motion whose response is isochronous (Kovacic and Rand, 2013a; Kovacic and Rand, 2013b; Kovacic and Rand, 2014; Kovacic and Mickens, 2015). The study presented herein relies on these results, but puts them in context of a pendulum. Thus, Section 1 contains known results related to the expressions for the period of a large-amplitude simple pendulum, its cubic and linear approximations, emphasizing their (in)dependences on the amplitude and illustrating them graphically. Section 2 is concerned with Huygens' pendulum, and provides its links with the case of a simple pendulum. In Section 4, these links are extended to a general class of nonlinear isochronous oscillators in terms of their kinetic and potential energy as well as the equations of motion. In Section 5, further generalisation is presented for the design of damped nonlinear oscillators that have a constant period, but a decreasing amplitude.

2. PENDULUM AND ITS PERIOD: EXACT, APPROXIMATED AND LINEAR

A simple pendulum (SP) consists of mass m attached to a hinged weightless rope of length L as shown in Fig. 1a. Taking the angle φ between the rope and the vertical line as a generalized coordinate, the kinetic E_k and potential energy E_p are given by

$$E_{k\mathrm{SP}} = \frac{1}{2}m(L\dot{\varphi})^2, \quad E_{p\mathrm{SP}} = -mgL\cos\varphi. \tag{1a,b}$$

Lagrange's equation $d\left(\partial E_k/\partial\dot{\varphi}\right)/dt - \partial E_k/\partial\varphi + \partial E_p/\partial\varphi = 0$, where t is time and the dot stand for differentiation with respect to it, yields the following equation of motion

$$\ddot{\varphi}+\omega_0^2\sin\varphi=0,\quad \omega_0^2=\frac{g}{L}. \tag{2a,b}$$

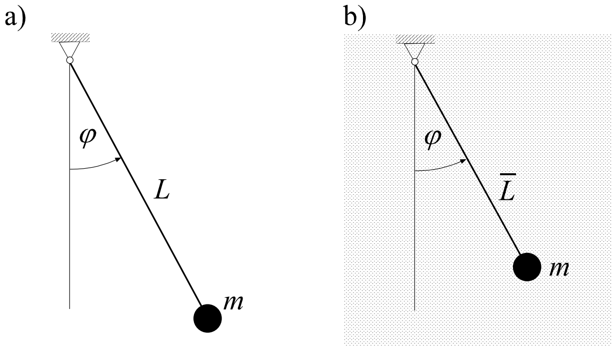

Figure 1. a) Simple pendulum; b) Undamped pendulum

Assuming that the initial conditions are $\varphi(0)=A>0$, $\dot{\varphi}(0)=0$, the first integral (energy conservation law) reads as

$$\frac{\dot{\varphi}^2}{2}-\omega_0^2\cos\varphi=-\omega_0^2\cos A, \tag{3}$$

and can be used to write down the following expression for the period of vibrations T

$$T=\frac{4}{\omega_0}\int_0^A\frac{d\varphi}{\sqrt{2(\cos\varphi-\cos A)}}. \tag{4}$$

By introducing the substitution $\sin(\varphi/2)=\sin(A/2)\sin\psi$, Eq. (4) transforms into

$$T_{\mathrm{SP}}=\frac{4}{\omega_0}\int_0^{\pi/2}\frac{d\psi}{\sqrt{1-\sin^2\frac{A}{2}\sin^2\psi}}. \tag{5}$$

The integral corresponds to the definition of the elliptic integral of the first kind $K(k)$ (Gradshteyn and Ryzhik, 2000), with the elliptic modulus k being $k=\sin(A/2)$, which gives

$$T_{\mathrm{SP}}=\frac{4K\left(\sin\frac{A}{2}\right)}{\omega_0}. \tag{6}$$

It is clearly seen that the period depends on the length of the pendulum via ω_0, but also on its initial amplitude, which is the consequence of the nonlinearity in Eq. (2a).

Let us now analyze the approximate case of a cubic pendulum (CP), when the potential energy is truncated to the fourth order

$$E_{p\mathrm{CP}}\approx -mgL\left(1-\frac{\varphi^2}{2}+\frac{\varphi^4}{24}\right). \tag{7}$$

The kinetic energy stays the same as given by Eq. (1a) and the equation of motion gets the form of the Duffing softening oscillator (Kovacic and Brennan, 2011)

$$\ddot{\varphi}+\omega_0^2\left(\varphi-\frac{\varphi^3}{6}\right)=0. \tag{8}$$

Although being nonlinear, this differential equation has the exact solution that can be expressed in terms of the Jacob sn elliptic function $y=A\mathrm{sn}\left[\omega_{\mathrm{sn}}t\middle|k^2\right]$ (Kovacic and Brennan, 2011). To find the values of the parameters ω_{sn} and k, this solution needs to be differentiated twice and substituted into Eq. (8). By collecting terms next to the sn function and its cube, and equating them with zero, one can derive $\omega_{\mathrm{sn}}=\omega_0\sqrt{1-A^2/12}$ and $k=\sqrt{A^2/(12-A^2)}$. The period T of the sn function used is $4K(k)/\omega_{\mathrm{sn}}$, so that the period of vibrations of the CP becomes

$$T_{\mathrm{CP}}=\frac{4K\left(\sqrt{\frac{A^2}{12-A^2}}\right)}{\omega_0\sqrt{1-\frac{A^2}{12}}}. \tag{9}$$

It is again seen that the period is amplitude-dependent.

Let us now consider the case of a linear pendulum (LP), whose potential energy is obtained by truncating the potential energy given by (1b) to the quadratic order

$$E_{p\mathrm{LP}} = -mgL(1-\frac{\varphi^2}{2}). \tag{10}$$

Using the fact that $E_{k\mathrm{SP}} = E_{k\mathrm{LP}}$, the well-known equation of motion of a harmonic oscillator is derived

$$\ddot{\varphi} + \omega_0^2 \varphi = 0. \tag{11}$$

Its harmonic solution has the period

$$T_{\mathrm{LP}} = \frac{2\pi}{\omega_0}, \tag{12}$$

which is, unlike Eqs. (6) and (9), constant and amplitude-independent. The change of the ratios $T_{\mathrm{SP}}/T_{\mathrm{LP}}$ and $T_{\mathrm{CP}}/T_{\mathrm{LP}}$ with the amplitude A is shown in Fig. 2. This dependence can be inconvenient in certain applications and the solutions for amplitude-independent periods may be needed. This brings about the question of designing a nonlinear isochronous oscillator (NIO) for which the ratio $T_{\mathrm{NIO}}/T_{\mathrm{LP}}$ will be equal to unity both in an undamped and damped case, i.e. the corresponding graphical presentation will be a horizontal line as shown in Fig. 2. The very first known answer to this question was provided by Huygens and is usually referred to as Huygens' pendulum. This pendulum is described subsequently.

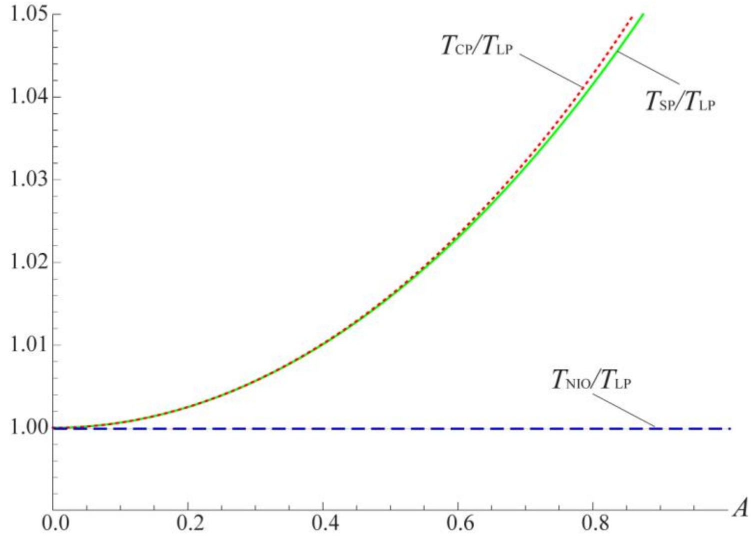

Figure 2. Variation of the ratios $T_{\mathrm{SP}}/T_{\mathrm{LP}}$ (green solid line), $T_{\mathrm{CP}}/T_{\mathrm{LP}}$ (red dotted line) and $T_{\mathrm{NIO}}/T_{\mathrm{LP}}$ (blue dashed line) with the amplitude A

3. HUYGENS' PENDULUM

Huygens' pendulum (HP) (THEP, 2015) wraps around a cycloid

$$x = \frac{L}{4}(\theta - \sin\theta), \quad y = -\frac{L}{4}(\cos\theta - 1). \tag{13a,b}$$

The corresponding parametric equations of motion of the bob are (THEP, 2015)

$$X = \frac{L}{4}(\theta + \sin\theta), \quad Y = -\frac{L}{4}(3 + \cos\theta). \tag{14a,b}$$

Both of these cycloids are shown in Fig. 3 for L=4.

By using Eqs. (14a,b), the corresponding kinetic energy $E_{k\mathrm{HP}} = m(\dot{X}^2 + \dot{Y}^2)/2$ is found to be

$$E_{k\mathrm{HP}} = \frac{1}{2} m(\frac{L}{2}\cos\frac{\theta}{2})^2 \dot{\theta}^2. \tag{15}$$

Its potential energy $E_{p\mathrm{HP}} = mgY$ now becomes

$$E_{p\mathrm{HP}} = -mg\frac{L}{4}(3 + \cos\theta). \tag{16}$$

Lagrange's equation yields the following nonlinear equation of motion:

$$\ddot{\theta} - \frac{1}{2}\tan\left(\frac{\theta}{2}\right)\dot{\theta}^2 + \omega_0^2 \, 2\tan\left(\frac{\theta}{2}\right) = 0. \tag{17}$$

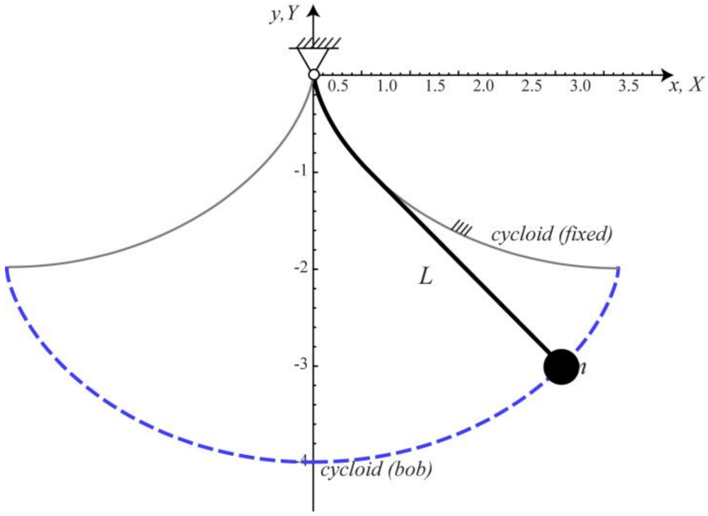

Figure 3. Huygens' pendulum

Time response obtained numerically from Eq. (17) is plotted in Fig. 4. for different values of the initial amplitude. It is seen that although the amplitude changes, the period remains constant.

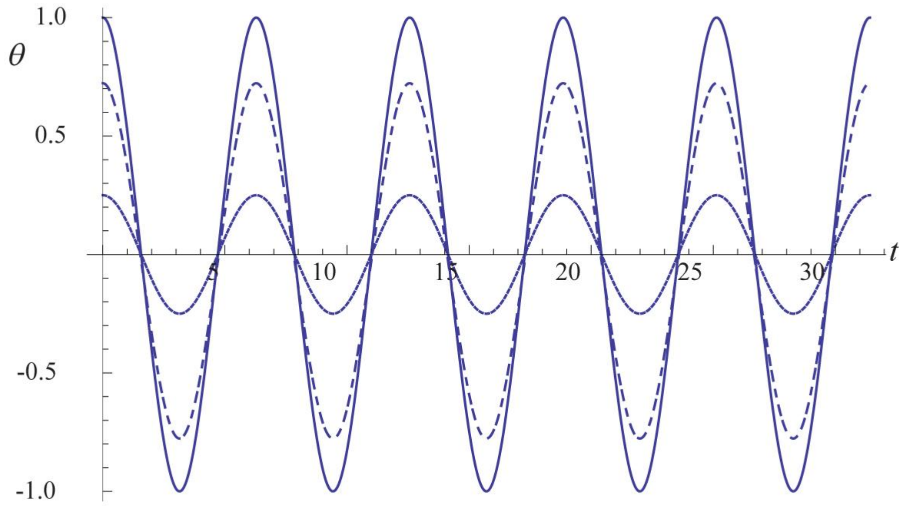

Figure 4. Time response obtained numerically from Eq. (17) for different values of the initial amplitude and for $\omega_0^2 = 1$

As Huygens' pendulum and the LP are both isochronous, one can establish the equivalence between the corresponding expressions for the kinetic energy (10) and (1a), obtaining $\dot{\varphi} = \dot{\theta}\cos(\theta/2)/2$. This is satisfied for $\varphi = \sin(\theta/2)$. By using this coordinate transformation, the expression for the potential energy of the HP (16) transform into the one of the LP (10). This transformation approach brings about the following question: 'Is it possible to generalize this type of transformation and establish the equivalence between a wider class of one degree-of-freedom nonlinear oscillators and a simple harmonic oscillator so that these nonlinear oscillators have a constant, amplitude-independent period?' The answer to this question is provided in the following section, based on some recent work by the author and her collaborators (Kovacic and Rand, 2013b; Kovacic and Rand, 2014; Kovacic and Mickens, 2015).

4. GENERALISATION FOR UNDAMPED CASES

If one assumes the kinetic energy of a nonlinear isochronous mechanical oscillator (NIO) in the form

$$E_{k\,\mathrm{NIO}} = \frac{1}{2} m f^2(\theta)\dot{\theta}^2, \tag{18}$$

and make it equal with the kinetic energy of the LP given by Eq. (1a), the following expression can be derived

$$\varphi = \frac{1}{L}\int_0^\theta f(\theta)d\theta. \tag{19}$$

Now, by substituting Eq. (18) into the potential energy of the LP given by Eq. (10), the form of the potential energy that will result in isochronous motion is obtained

$$E_{p\,\mathrm{NIO}} = \frac{mg}{2L}\left[\int_0^\theta f(\theta)d\theta\right]^2 + const. \tag{20}$$

Lagrange's equation yields the corresponding equation of motion:

$$\ddot{\theta} + \frac{f'(\theta)}{f(\theta)}\dot{\theta}^2 + \omega_0^2 \frac{\int_0^\theta f(\theta)d\theta}{f(\theta)} = 0, \tag{21}$$

where $f'(\theta) = df/d\theta$, and its periodic response around a trivial equilibrium will have the same period as the one of the LP, given by Eq. (12).

4.1 Example 1: Cubic system

As shown in Section 2, the CP has an amplitude-dependent period. Let us see how the system should be modified to have an isochronous response. This modification is carried out by keeping the restoring force of the CP unchanged. Thus, the last term on the left-hand side of Eq. (21) is made equal to the last one on the left-hand side of Eq. (8) with φ replaced by θ, i.e.

$$\omega_0^2 \frac{\int_0^\theta f(\theta)d\theta}{f(\theta)} = \omega_0^2\left(\theta - \frac{\theta^3}{6}\right). \tag{22}$$

This is satisfied for $f(\theta) = 6/(6-\theta^2)^{3/2}$. Equations (18) and (19) give now the corresponding kinetic and potential energy, while the equation of motion derived from Eq. (21) has the form

$$\ddot{\theta} + \frac{3\theta}{6-\theta^2}\dot{\theta}^2 + \omega_0^2\left(\theta - \frac{\theta^3}{6}\right) = 0. \tag{23}$$

Numerical solutions of this equation are plotted in Fig. 5 for different initial amplitudes and confirm that the motion is isochronous.

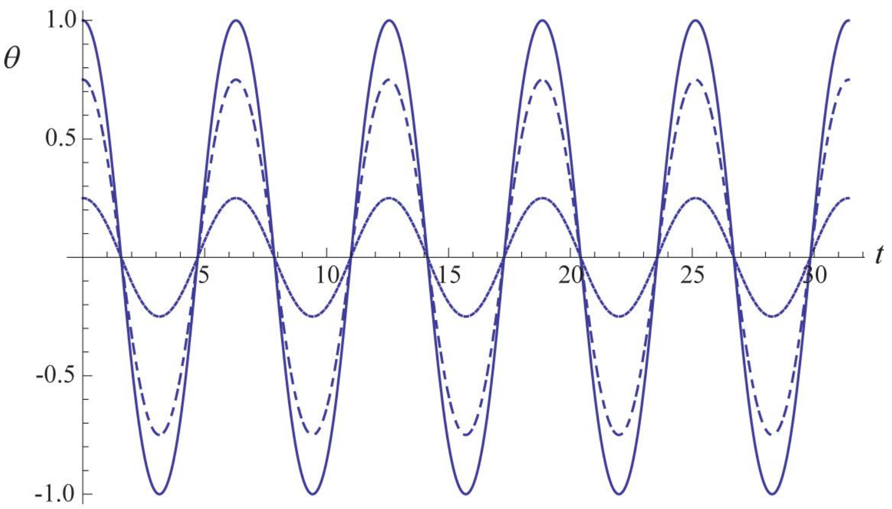

Figure 5. Time response obtained numerically from Eq. (23) for different values of the initial amplitude and for $\omega_0^2 = 1$

5. GENERALISATION FOR DAMPED CASES

The analogous approach can be used to find damped systems that have an isochronous period, but a decaying amplitude (Kovacic and Mickens, 2015). To that end, a viscously damped pendulum (DP) shown in Fig. 1b is considered (note that its length is labeled by $\overline{L}$). The dissipative function is

$$\Phi = \frac{1}{2}b\left(\overline{L}\,\dot{\varphi}\right)^2, \tag{24}$$

with the coefficient b being constant. The forms for the kinetic and potential energy are given by Eqs. (1a,b) with $L \equiv \overline{L}$, and the equation of motion that follows from $d(\partial E_k/\partial\dot{\varphi})/dt - \partial E_k/\partial\varphi + \partial E_p/\partial\varphi = -\partial\Phi/\partial\dot{\varphi}$ reads as

$$\ddot{\varphi} + 2\delta\dot{\varphi} + \overline{\omega}_0^2\varphi = 0, \tag{25}$$

where $2\delta = b/m$ and $\overline{\omega}_0^2 = g/\overline{L}$. It is known that its response has the so-called quasi-period defined by

$$T_{\mathrm{DP}} = \frac{2\pi}{\sqrt{\overline{\omega}_0^2 - \delta^2}}. \tag{26}$$

Making the requirement that this pendulum has the same period as the linear undamped one (12) equal to 2π, it concludes that $\overline{\omega}_0^2 = \omega_0^2 + \delta^2$, i.e. its length should be $\overline{L} = L/\left(1 + \delta^2 L/g\right)$, implying that $\overline{L} < L$. This pendulum will be named a tuned damped pendulum (TDP).

Let us now find the NIO that has the same period 2π. To that and, the kinetic energy of such nonlinear oscillator is assumed in the form (18) and made equal with the kinetic energy of the TDP. As a consequence, the expression for the coordinate transformation (19) and the potential energy (20) are derived with $L \equiv \overline{L}$. The dissipative function (24) becomes

$$\Phi_{\mathrm{NIO}} = \frac{b\left[f(\theta)\dot{\theta}\right]^2}{2}, \tag{27}$$

and is now displacement-dependent. Lagrange's equation yields the following equation of motion:

$$\ddot{\theta} + \frac{f'(\theta)}{f(\theta)}\dot{\theta}^2 + 2\delta\dot{\theta} + \left(\omega_0^2 + \delta^2\right)\frac{\int_0^\theta f(\theta)d\theta}{f(\theta)} = 0, \tag{28}$$

and its periodic solution around a trivial equilibrium will be isochronous with the period 2π and a decreasing amplitude.

5.1 Example 2: Modification of Eq. (17)

As an example, let us design a nonlinear isochronous oscillator as a modification of the example of Huygens' pendulum, but making it damped. Comparing Eqs. (27) and (21) and (17), one can directly derive the damped Eq. (17):

$$\ddot{\theta} - \frac{1}{2}\tan\left(\frac{\theta}{2}\right)\dot{\theta}^2 + 2\delta\dot{\theta} + 2\left(\omega_0^2 + \delta^2\right)\tan\left(\frac{\theta}{2}\right) = 0, \tag{29}$$

whose period is amplitude-independent and equal to 2π. This fact is also seen in Fig. 6, which was obtained by solving Eq. (28) numerically for different values of the initial amplitude. A decreasing amplitude is also noticeable. The equation of motion (29) stems from the kinetic energy given by (15), i.e. it corresponds to the case when the bob moves along a cycloid. The potential energy differs from Eq. (16) and is given by $E_p = E_{p\mathrm{HP}} + \Delta E_p$, where $\Delta E_p = -m\delta^2 L^2(3 + \cos\theta)/4 + m\delta^2 L^2\left(2 + \delta^2 L/g\right)/\left(1 + \delta^2 L/g\right)$. The dissipative function is $\Phi = b\left(L\cos(\theta/2)\dot{\theta}\right)^2/8$.

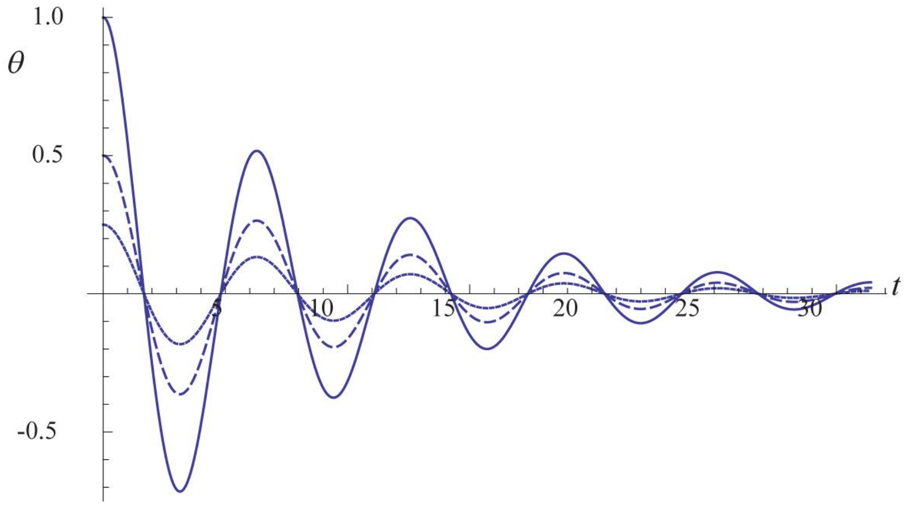

Figure 6. Time response obtained numerically from Eq. (29) for different values of the initial amplitude for $\omega_0^2 = 1$ and $\delta = 0.1$

5.2 Example 3: Modification of Eq. (23)

This example is to shown how to design a damped isochronous oscillator that has a restoring force of the same form as the CP. Comparing Eqs. (28) and (21) and (23), the corresponding equation of motion is found to be:

$$\ddot{\theta}+\frac{3\theta}{6-\theta^2}\dot{\theta}^2+2\delta\dot{\theta}+\left(\omega_0^2+\delta^2\right)\left(\theta-\frac{\theta^3}{6}\right)=0. \tag{30}$$

This equation can be derived for $E_k = 18m\dot{\theta}^2/\left(6-\theta^2\right)^3$, $E_p = 3m\left(\omega_0^2+\delta^2\right)/\left(6-\theta^2\right)$ and $\Phi = 18b\dot{\theta}^2/\left(6-\theta^2\right)^3$.

Numerical solutions of Eq. (30) are plotted in Fig. 6 for different initial amplitudes and confirm that the motion has a constant period for three different values of the initial amplitude.

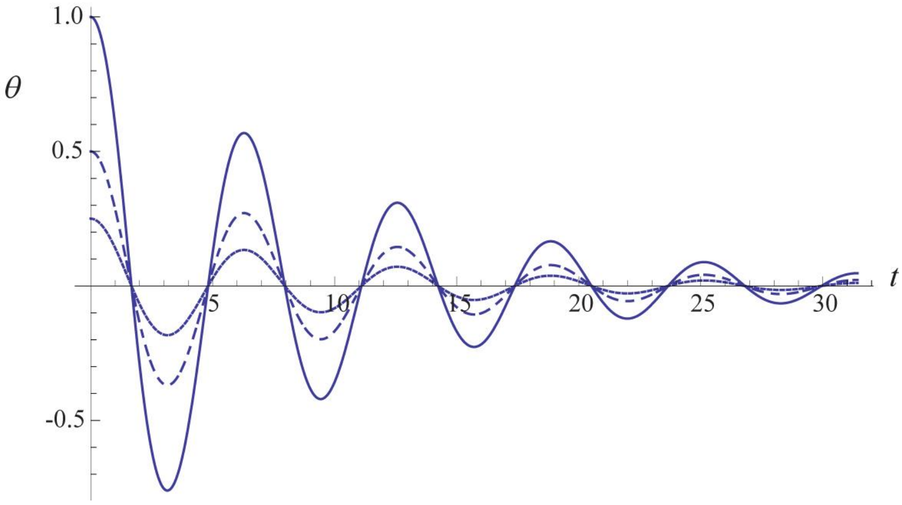

Figure 7. Time response obtained numerically from Eq. (30) for different values of the initial amplitude for $\omega_0^2 = 1$ and $\delta = 0.1$

6. CONCLUSIONS

A pendulum is known to have an amplitude-dependent period when performing larger oscillations, while for the case when it is linearised, its period is constant and amplitude-independent. Huygens found that when the pendulum wraps around a cycloid, the trajectory of a bob is also a cycloid, and its period is amplitude-independent. It has been shown in this study how one can establish a link between the kinetic and potential energy of Huygens' pendulum and a linear pendulum as well as how this transformation can be further generalised to a class of nonlinear oscillators with an amplitude-independent period. The approach has been used to design a nonlinear system whose restoring force correspond to the one of a cubic pendulum. Then, a damped linear pendulum has been adjusted to have a period equal to the one of the undamped linear pendulum. By using again a transformation approach, a general class of nonlinear damped oscillators is obtained, having the same amplitude-independent period, while their amplitude decreases in time. These results have also been illustrated by providing two examples. All the results obtained analytically have been confirmed numerically.

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