23rd ABCM International Congress of Mechanical Engineering
December 6-11, 2015, Rio de Janeiro, RJ, Brazil

# ACCURATE STEADY-STATE SOLUTIONS FOR UNSTABLE PROBLEMS

**Renan de S. Teixeira**
Divisão de Metrologia em Dinâmica de Fluidos - Dinam, Instituto Nacional de Metrologia, Qualidade e Tecnologia - INMETRO, Av. Nossa Senhora das Graças, 50, Prédio 06, 25250-020, Mantiquira-Xerém, Duque de Caxias - RJ - Brasil
rsteixeira-pronametro@inmetro.gov.br

**Leonardo S. de B. Alves**
Departamento de Engenharia Mecânica - TEM, Universidade Federal Fluminense - UFF, Rua Passo da Pátria, 156, Bloco E, Sala 216, 24210-240, São Domingos, Niterói - RJ - Brazil
leonardo.alves@mec.uff.br

***Abstract.*** *Several fluid mechanics problems require accurate reference solutions for their analysis and computational studies. Thermal and hydrodynamic stability analysis are such examples, where the reference solution should be a representative base flow. Numerical simulations of unstable problems are yet another example where accurate reference solutions are important to minimizing the unwanted oscillations due ill-posed initial and boundary conditions. Steady-state provide the best alternative to be used as reference solutions. Selective frequency damping (SFD) was developed to provide such steady-state solutions. It works by introducing a source term in the governing equations that forces the time marching scheme to converge towards a reference solution, which is the filtered version of the unsteady solution marching in time. Since this source term disappears at steady-state, the reference solution becomes the steady-state in this limit. However, this technique has been applied to globally unstable problems only. Recently, physical-time damping (PTD) was developed to generate steady-states for any type of unstable flows. The method works with the dissipative properties of the Euler implicit scheme. But this method is not able to introduce enough dissipation overcome absolute instabilities with large temporal growth rates. This is caused by nonlinear effects, which limit the dissipative properties of the implicit Euler scheme. The present paper describes a robust technique to generate steady-states for unstable problems, called Minimal Gain Marching (MGM) schemes. This method is based on constructing time marching schemes that achieve the minimal gain of the linear stability spectrum at the smallest possible time step. This paper extends previous studies, showing that such a technique works quite well for several different flows with different stability characteristics.*

***Keywords:*** *Initial and Boundary Conditions, Steady-State Solutions, Numerical Stability Analysis and Marching Schemes.*

## 1. INTRODUCTION

Several numerical studies require accurate reference solutions in many engineering branches. Initial and boundary conditions are excellent examples, where the use of the reference solutions are significant. Unwanted oscillations were developed in flow field when both approximate and similar solutions were used as initial conditions in the unsteady numerical simulation of absolutely unstable mixing-layers (Lardjane *et al.*, 2004). However, the similar solutions provide oscillations amplitude that is much smaller and their frequency content has a good literature agreement. This phenomena occurs in the calculations of the temporal disturbance growth rates, which can generate relative errors as high as 18% because of the approximate initial conditions, even though high order schemes are employed (Germanos *et al.*, 2009). Another significative problem caused by inaccurate initial condition is the presence of large perturbations in early times in the numerical studies of flows around two-dimensional bodies (Bijl *et al.*, 2002; Wang and Mavriplis, 2007). Therefore, it is need to run large simulation times for these numerical waves to be vanished of the simulated flow field before accurate results can be obtained. Absorbing layers (Blaschak and Kriegsmann, 1988) or buffers zones (Bodony, 2006) are a type of boundary condition, which introduce a source term in conservation equation that force the numerical solution to converge towards a reference solution near the artificial boundary. Hence, accurate reference solution are quite useful and it can improve the boundary condition effectiveness. This boundary condition approach is relevant in aeroacoustics (Colonius and Lele, 2004) and receptivity (Saric *et al.*, 2003) studies. In this areas the problems are worse due the magnitude difference between the perturbation of interest and the vorticity waves that feed them. The use of steady-states as reference solutions for both initial condition and absorbing layers minimized these problems in the simulation of acoustic field of airfoils (Collis and Lele, 1999; Barone and Lele, 2002). The same procedure was adopted to minimize numerical errors in receptivity (Barone and Lele, 2005) and startup vortex (de S. Teixeira and de B. Alves, 2011) studies of compressible mixing-layers. Consequently, a new method that is able to generate steady-states would yield the most accurate reference solutions.

Hydrodynamic stability analysis is another example where accurate reference solutions are importante, because these solutions should be a representative profile of the base flow in this context. The mechanism to verify the stability of a given flow field is based on the perturbation of the base flow, where we mesure the growth or decay of introduced disturbance. Actually, the base flow would be an undisturbed solution of the governing equations, although earlier investigations employed profiles that fit well experimental data (Michalke, 1984). However, incorrect predictions might occur, since numerical errors in the base flow become forcing terms in the disturbance governing equations. An alternative procedure is to transform this difficulty into an asset, where the generated force term is reallocated as source term into the governing equations. Then, the approximate solution from the original governing equation becomes an exact solution of the modified governing equation. This scheme is named Method of Manufactured Solution and it is used for code verification (Steinberg and Roache, 1985; Roache, 2002). Recently, the stability analysis and the method of manufactured solution have been blended, which direct numerical simulations of the modified governing equations is used to perform a linear stability analysis of a time-averaged solution (Jones *et al.*, 2010). The main problem of this approach is that numerical residue becomes an integral part of model, which reduces its capability to predict the correct flow physics. Then, the preferred base flows are the similarity solutions calculated by simplified models (Theofilis, 2003). Although this type of solutions may not be enough to obtain accuracy analysis of complex flow fields. Opposite trends for the range of unstable frequencies in the linear stability analysis of transverse jets using approximate (de B. Alves *et al.*, 2008) and similar (Kelly and de B. Alves, 2008) profiles as base flow are compelling evidence that an inaccurate base flow will lead to qualitatively incorrect predictions (Bagheri *et al.*, 2009). Hence, the need to develop methods capable of generating accurate base flows is urgent.

Actually, there are few methods for the construction of accurate steady-states solutions. The most traditional method is the Newton-type method coupled with continuation techniques (Tuckerman and Barkley, 2000). These methods have several convergence difficulties for globally unstable problems, they are quite sensitive to initial guess and require heavy computational resources for large systems. An alternative approach was developed by Åkervik *et al.* (2006), named Selective Frequency Damping. The method consists to introduce a source term in the governing equations that forces the unsteady solution to converge towards a filtered reference solution, where a filtered version of the unsteady solution is marched in time with an additional filter equation. This source term vanishes at steady-state and the reference solution is obtained in this limit. Moreover, the method is easy to implement and needs the adjustment of two parameters. However, this approach has been employed for specific type of unstable flows. Only globally unstable flows were investigated with selective frequency damping technique, i. e., flows with a single dominant self-excitation frequency (Benoît, 2008; Nichols and Schmid, 2008; Bagheri *et al.*, 2009; Sipp *et al.*, 2010; Mack and Schmid, 2010; Chandler *et al.*, 2012; Ilak *et al.*, 2012). Furthermore, the technique has severe convergence difficulties when the problem is susceptible to stationary instabilities as well as instabilities with a broad frequency spectrum. Recently, the Minimal Gain Marching scheme was developed as an alternative method to generate steady-states solutions for any unstable problem (de S. Teixeira and de B. Alves, 2013; Teixeira, 2014; de S. Teixeira and de B. Alves, 2014). This new class of schemes is based on the fundamental of numerical analysis, where the convergence of given dynamical system will occur if the numerical analysis gain of the marching scheme is sufficiently small. Therefore, the coefficients of any time marching scheme can be modified to achieve the optimal minimal gain for the convergence of unstable conditions. The only additional constraint needed the time marching scheme must be at least first order accurate. This paper describes an investigations of the behavior of the minimal gain marching scheme for steady-state unstable problem compared with selective frequency damping technique.

## 2. MATHEMATICAL MODEL

The nonlinear viscous Burgers equation can be written as

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = \mu\frac{\partial^2 u}{\partial x^2} \tag{1}$$

where, $u$ is the velocity, the independent variables $t$ time and $x$ space and $\mu$ dynamic viscosity. The equation (1) was modified to describe a simple example of an unstable problem. A linear $u-$dependent force term was introduced into the equation (1). Hence, we obtain

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = \mu\frac{\partial^2 u}{\partial x^2} + R(u - u_0). \tag{2}$$

The boundary conditions are

$$u(0,t) = u_0 + A_m \sin(\omega t) \tag{3}$$

$$\left.\frac{\partial u}{\partial t}\right|_{x=L} + u^n(L,t)\left.\frac{\partial u}{\partial x}\right|_{x=L}^{n} = \mu\left.\frac{\partial^2 u}{\partial x^2}\right|_{x=L}^{n} + R(u^n(L,t) - u_0) \tag{4}$$

were $L$ is the length, $\omega$ the disturbance frequency and $A_m$ amplitude.

Obviously, $u = u_0$ is a trivial solution of the equation (2). A linear stability analysis of this base flow was investigated. First, the equation (2) was perturbed with

$$u = u_0 + \epsilon \hat{u}, \tag{5}$$

where $\epsilon$ is the disturbance amplitude such that $\epsilon \ll 1$. Therefore, linearising the equation and substituting normal modes $\psi(x,t) = \hat{\psi}\exp^{i(kx-\omega t)}$ where $k = k_r + ik_i$ and $\omega = \omega_r + i\omega_i$ are complex numbers. Thus, we obtain the follow dispersion relation

$$D(k,\omega) = -i\frac{(\omega - u_0 k)}{\mu} + k^2 - \frac{R}{\mu} \tag{6}$$

With $k$ real and $\omega$ complex we have the temporal instability, while if the parameters were assumed as $\omega$ real and $k$ complex we obtain a convective instability. Hence, we obtain some stability conclusions that the problem is

- 
  - Stable with $R < \dfrac{k^2}{\mu}$;
  - Marginally stable with $R = \dfrac{k^2}{\mu}$;
  - Convectively unstable with $R > \dfrac{k^2}{\mu}$;
- 
  - Stable with $R < 0$.
  - Convectively unstable with $0 \leq R < \dfrac{u_0^2}{4\mu}$;
  - Absolutely unstable with $R \geq \dfrac{u_0^2}{4\mu}$;

in both unstable the more unstable mode is steady-state, i. e., in the temporal case the more unstable mode is the zero frequency and the spatial case is the zero wave number (Barletta and de B. Alves, 2014).

### 2.1 Numerical Method

The Runge-Kutta IMEX scheme is used to solve the Burgers equation (2). This method evaluates the stiff term explicitly, while the others are calculated with a implicit scheme. These methods have been used to present a midpoint between implicit and explicit methods (Ascher *et al.*, 1997). The Runge-Kutta IMEX scheme is described as

$$\begin{aligned} \mathbf{k}_i &= \mathbf{Q}^n + \Delta t \sum_{j=1}^{i-1} \tilde{a}_{i,j}\mathbf{f}(t_0 + \tilde{c}_j\Delta t, \mathbf{k}_j) + \Delta t \sum_{j=1}^{\nu} a_{i,j}\mathbf{g}(t_0 + c_j\Delta t, \mathbf{k}_j) \\ \mathbf{Q}^{n+1} &= \mathbf{Q}^n + \Delta t \sum_{j=1}^{\nu} \tilde{\omega}_i\mathbf{f}(t_0 + \tilde{c}_j\Delta t, \mathbf{k}_j) + \Delta t \sum_{j=1}^{\nu} \omega_i\mathbf{g}(t_0 + c_j\Delta t, \mathbf{k}_j) \end{aligned} \tag{7}$$

where $\nu$ is the intermediate stages, the vector $\mathbf{f}$ is the stiff term and vector $\mathbf{g}$ is associated with non-stiff term of the equation. The intermediate stages coefficients $\mathbf{k}_i$ are described in two Butcher tables, one of the them for explicit term and the other implicit. where the matrix $\tilde{A}$ and $A$ are composed by the coefficients $\tilde{a}_{i,j}$ and $a_{i,j}$, respectively.

$$\begin{array}{c|c} \tilde{c} & \tilde{A} \\ \hline & \tilde{\omega}^T \end{array} \qquad\qquad \begin{array}{c|c} c & A \\ \hline & \omega^T \end{array}$$

Table 1. Butcher Table for explicit term (left) and implicit (rigth).

The coefficients used in this work are presented in the Table (2). This is a third order Runge-Kutta IMEX scheme where $\gamma = 1 - \dfrac{1}{\sqrt{2}}$.

The spatial terms were discretized with $5^a$ order upwind for inviscid term and $4^a$ order central difference for viscous term.

## 3. DISCUSSION AND RESULTS

Initially, a numerical verification is presented in Figure (1) (left). The decay slope of the numerical error in this figure shows the method order implemented. The solid lines are the exact slop. As shown in the graph the code seems right. The

$$
\begin{array}{c|ccc}
0 & 0 & 0 & 0 \\
1 & 1 & 0 & 0 \\
1/2 & 1/4 & 1/4 & 0 \\
\hline
 & 1/6 & 1/6 & 2/3
\end{array}
\qquad\qquad
\begin{array}{c|ccc}
\gamma & \gamma & 0 & 0 \\
1-\gamma & 1-2\gamma & \gamma & 0 \\
1/2 & 1/2-\gamma & 0 & \gamma \\
\hline
 & 1/6 & 1/6 & 2/3
\end{array}
$$

Table 2. Butcher Table of the $3^a$ order Runge-Kutta IMEX

Figure (1) (right) shows the convective instability behavior in the problem. As can be observed, the disturbance amplitude has a spatial exponential growth. The flow was perturbed on the inlet with a frequency $\omega_r = 5.98$ obtained from stability analysis for a wavenumber $k_r = 6$, the control parameter $R = 0.5$ and amplitude $10^{-06}$ with a sinusoidal function. The base flow velocity is $u_0 = 1.0m/s$ and dynamic viscosity $\mu = 10^{-02}N \cdot s/m^2$ were used in all simulations.

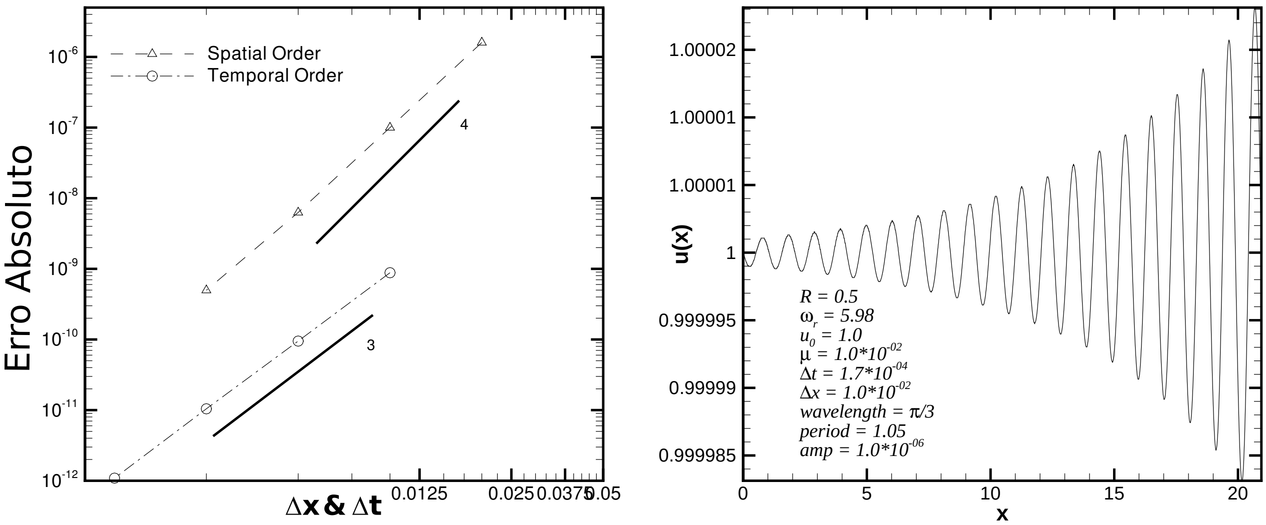



Figure 1. Spatial and temporal order verification of the Burger's problem (left). Convective behavior of the excited problem (right).

A hydrodynamic stability analysis verification was investigated. The wavenumber and spatial growth rate disturbance ware calculated with numerical results as the case shown in Figure (1) (right) and compared with the stability analysis. The Table (3) shows the error fluctuation in the growth rate and wavenumber for different inlet frequencies. We can observe that we increase the growth rate disturbance the error is increasing, but the error in wavenumber is well controlled. Indeed, the spatial growth is predict with good accuracy.

The stability analysis of this problem described in the last section indicates the more unstable mode is the zero frequency and zero wavenumber. The numerical results show that numerical error is slightly larger in inlet due spatial discretization near the boundaries. This error inevitably excites this steady-state mode. If numerical domain is sufficiently large, this zero mode eventually dominate the flow and forms a shock wave. Thus, the presented in Figure(1)(right) and Table (3) was done with small values of $R$ just to delay the onset of the steady-state mode and allow to analyze the oscillating modes introduced in the flow entrance. Figure (2) shows these two different modes in the flow where the flow was excited with the frequency $\omega_r = 24.62$ and the control parameter $R = 7.0$. It is clear that the zero wavenumber mode eventually acquires an amplitude greater than the excited oscillating mode and dominates the flow. Moreover, the dashed line shows that the origin of the more unstable disturbance is the order of the $10^{-11}$ and is in the inlet. This behavior limits the study to the problem of small domains.

The merging of zero frequency disturbance that appears in the flow independent of any inlet excitement it was studied the convergence of the MGM (minimal gain marching scheme) and SFD (selective frequency damping scheme) without external excitation of oscillatory modes. To facilitate this study, flow input was disturbed by a pulse amplitude $10^{-6}$. After amplification of the stationary disturbance to a range of about $10^{-1}$, i. e., just before the formation of the shock wave, the steady-state technique were activated and convergence toward $u_0$ be achieved with a tolerance $10^{-8}$ . Figure (3) (left) shows that the number of iterations of the SFD scheme was equal to $78,361$ iterations. This occur because this method is based on disturbance frequency. As the frequency is zero the method has difficult to converge. Figure (3) (right) shows the number of iterations in the convergence process to MGM scheme. The great case gets to be more than 4 times

| $R$ | $\omega_R$ | $k_x$ | Numerical Experiment | Stability Analysis (AEL) | \|Relative Error\| |
|---|---|---|---|---|---|
| 0.5 | 1.98151 | $k_r$ | 1.994270 | 2.000000000 | 0.00286500 |
| | | $k_i$ | 0.461343 | 0.462135694 | 0.00171528 |
| | 3.97270 | $k_r$ | 4.000000 | 4.000000000 | 0.00000025 |
| | | $k_i$ | 0.341157 | 0.341163928 | 0.00002030 |
| | 5.98317 | $k_r$ | 6.000000 | 6.000000000 | 0.00000025 |
| | | $k_i$ | 0.140192 | 0.140196550 | 0.00003245 |
| 1.0 | 2.94489 | $k_r$ | 2.991790 | 3.000000000 | 0.00273667 |
| | | $k_i$ | 0.919865 | 0.918435230 | 0.00155675 |
| | 5.99227 | $k_r$ | 6.000010 | 6.000000000 | 0.00000166 |
| | | $k_i$ | 0.644412 | 0.644149200 | 0.00040798 |
| | 8.96573 | $k_r$ | 9.000010 | 9.000000000 | 0.00000111 |
| | | $k_i$ | 0.190335 | 0.190362370 | 0.00014377 |
| 2.0 | 6.78531 | $k_r$ | 6.998570 | 7.000000000 | 0.00020428 |
| | | $k_i$ | 1.528810 | 1.533516730 | 0.00306924 |
| | 9.79795 | $k_r$ | 10.00020 | 10.00000000 | 0.00002000 |
| | | $k_i$ | 1.009930 | 1.010205140 | 0.00027236 |
| | 10.8248 | $k_r$ | 11.00010 | 11.00000000 | 0.00000909 |
| | | $k_i$ | 0.796090 | 0.796341590 | 0.00031593 |

Table 3. Table of the stability analysis Burgers problem verification.

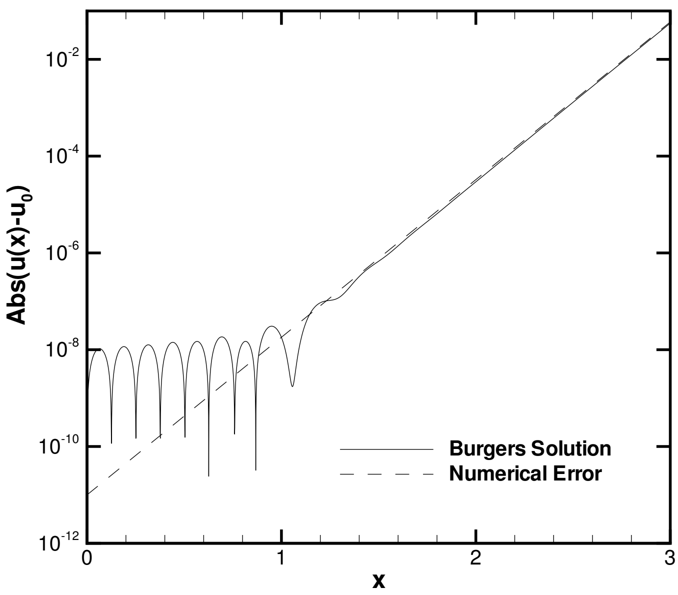
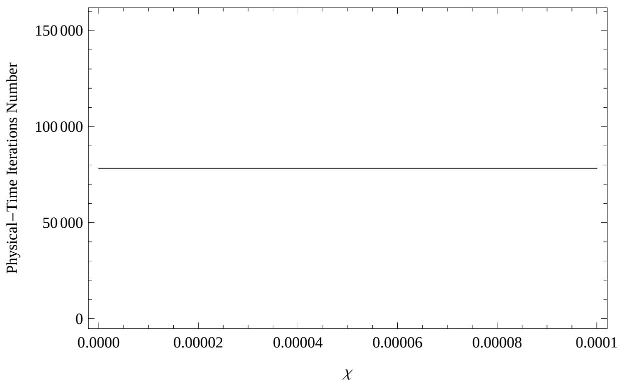



Figure 2. Steady-State frequency error behavior with $R = 7.0$ and $\omega_r = 24.62$

faster than the SFD. Furthermore, it is possible to accelerate the convergence of the traditional PTD, which uses implicit Euler applying PTD with MGMs, as shown in curve $\theta_1 = 0.5$ e $\theta_2 = 2.0$, since this scheme can work with slightly larger steps in time.



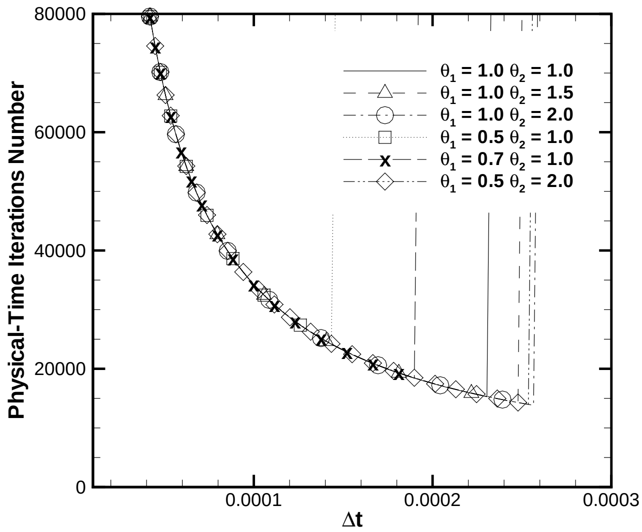




Figure 3. Iterations number of the convergence with SFD (esquerda) and MGM (direita).

A last study was performed to Burgers problem using the solution shown in Figure (1) (right) as an initial condition to evaluate the damping rate of the techniques studied in the oscillatory excitation case, as shown in Figure (4). This figure describes the evolution of the disturbance within a fixed time after the solution reaches a limit cycle behavior. As can be seen, with the step time for each optimal combination of parameters of MGM technique obtained over $50\%$ of the magnitude of disturbance input after 5 wavelengths downstream of the entry. Since the SFD technique failed to reduce the amplitude of the disturbance to any combination of its control parameters, showing its limitations for convectively unstable problems.

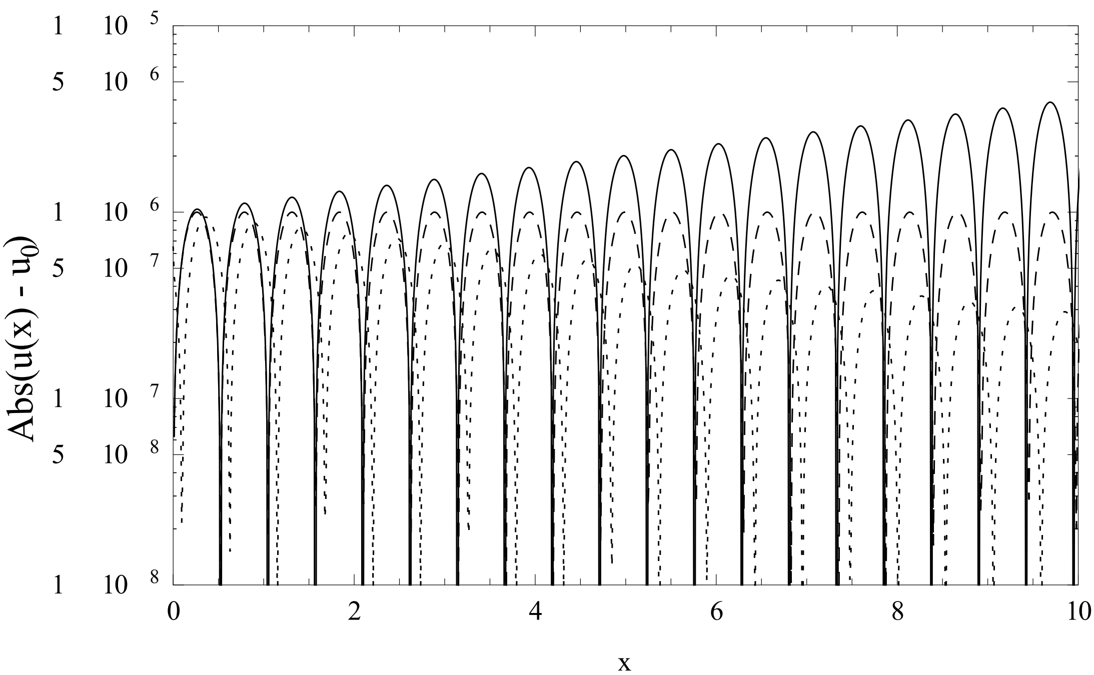



Figure 4. Maximum damping for MGM (dotted line) and SFD (dashed line) compared with the initial condition (solid line).

## 4. CONCLUSIONS

The unstable Burgers problem showed how the MGM scheme (minimal gain marching) is more efficient than others in the literature. A linear stability analysis of the problem show that the zero frequency is more unstable. Hence, the disturbance excitation with this frequency has the higher growth rate. Thus, simulating the problem with convective instability, the numerical noise excites the frequency of greater growth rate. Therefore, the flow is contaminated near the inlet, limiting the simulation domain. However, we made two important tests to show the efficiency of MGM scheme. The first consisted of the damping of this unstable frequency disturbance. Again, the robustness of the method was proven MGM. Results for the MGM scheme had excellent agreement with the results of the numerical stability analysis. The new scheme was more efficient than the PTD and SFD schemes. In the second test, a positive frequency perturbation is imposed on the flow entry. MGM diagram showing that initiates a damping process, reducing the magnitude of the disturbance along the main flow direction, as opposed to the SFD. The results of this test showed a more robust and efficient methodology of the new front to the existing method in the literature.

## 5. ACKNOWLEDGEMENTS

The authors would like to acknowledge the financial support of CAPES Brazil, Instituto Militar de Engenharia (IME), Instituto Nacional de Metrologia Qualidade e Tecnologia (Inmetro) and Universidade Federal Fluminense (UFF).

## 6. REFERENCES

The list of references must be introduced as a new section, located at the end of the paper. The first line of each reference must be aligned at left. All the other lines must be indented by 0.5 cm from the left margin. All references included in the reference list must have been mentioned in the text.

References must be listed in alphabetical order, according to the last name of the first author. See the following examples:

Åkervik, E., Brandt, L., Henningson, D.S., Hœpffner, J., Marxen, O. and Schlatter, P., 2006. "Steady solutions of the navier-stokes equations by selective frequency damping". *Physics of Fluids*, Vol. 18, No. 6, 068102. doi:http://dx.doi.org/10.1063/1.2211705. URL http://scitation.aip.org/content/aip/journal/pof2/18/6/10.1063/1.2211705.

Ascher, U.M., Ruuth, S.J. and Spiteri, R.J., 1997. "Implicit-explicit Runge-Kutta methods for time-dependent partial differential equations". *Applied Numerical Mathematics*, Vol. 25, pp. 151–167.

Bagheri, S., Schlatter, P., Schmid, P.J. and Henningson, D.S., 2009. "Global stability of a jet in crossflow". *Journal of Fluid Mechanics*, Vol. 624, pp. 33–44. doi:doi:10.1017/S0022112009006053.

Barletta, A. and de B. Alves, L.S., 2014. "Transition to absolute instability for (not so) dummies". *Cornell University Library*.

Barone, M.F. and Lele, S.K., 2002. "A numerical technique for trailing edge acoustic scattering problems". In *AIAA Conference Paper*. 0226, pp. 1 – 12.

Barone, M.F. and Lele, S.K., 2005. "Receptivity of the compressible mixing layer". *Journal of Fluid Mechanics*, Vol. 540, pp. 301–335. ISSN 1469-7645. doi:10.1017/S0022112005005884. URL http://journals.cambridge.org/article_S0022112005005884.

Benoît, P., 2008. "Local and global instabilities in the wake of a sphere". *Journal of Fluid Mechanics*, Vol. 603, pp. 39–61. ISSN 1469-7645. doi:10.1017/S0022112008000736.

Bijl, H., Carpenter, M.H., Vatsa, V.N. and Kennedy, C.A., 2002. "Implicit time integration schemes for the unsteady compressible Navier-Stokes equations: Laminar flow". *Journal of Computational Physics*, Vol. 179, pp. 313–329.

Blaschak, J.G. and Kriegsmann, G.A., 1988. "A comparative study of absorbing boundary conditions". *Journal of Computational Physics*, Vol. 77, pp. 109–139.

Bodony, D.J., 2006. "An analysis of sponge zones for computational fluid mechanics". *Journal of Computational Physics*, Vol. 212, No. 2, pp. 681–702.

Chandler, G.J., Juniper, M.P., Nichols, J.W. and Schmid, P.J., 2012. "Adjoint algorithms for the navier-stokes equations in the low mach number limit". *Journal of Computational Physics*, Vol. 231, No. 4, pp. 1900 – 1916. ISSN 0021-9991.

Collis, S.S. and Lele, S.K., 1999. "Receptivity to surface roughness near a swept leading edge". *Journal of Fluid Mechanics*, Vol. 380, pp. 141–168. ISSN 1469-7645. doi:10.1017/S0022112098003449. URL http://journals.cambridge.org/article_S0022112098003449.

Colonius, T. and Lele, S.K., 2004. "Computational aeroacoustics: Progress on nonlinear problems of sound generation". *Progress in Aerospace Sciences*, Vol. 40, pp. 345–416.

de B. Alves, L.S., Kelly, R.E. and Karagozian, A.R., 2008. "Transverse jet shear layer instabilities. part ii: Linear analysis for large jet-to-crossflow velocity ratios". *Journal of Fluid Mechanics*, Vol. 602, pp. 383–401.

de S. Teixeira, R. and de B. Alves, L.S., 2011. "Self-excited compressible planar mixing layer at arbitrary mach number: Numerical or phisical phenomena?" In *21th International Congress of Mechanical Engineering*. Natal, RN, Brazil.

de S. Teixeira, R. and de B. Alves, L.S., 2013. "Minimal gain time marching schemes for the construction of accurate steady-states". In *Instability and Control of Massively Separated Flows(ICOMASEF)*. Prato, Italy.

de S. Teixeira, R. and de B. Alves, L.S., 2014. "Minimal gain marching schemes for unstable problems with multiple frequencies". In *15th Brazilian Congress of Thermal Sciences and Engineering*. Belém, Brazil.

Germanos, R.A.C., de Souza, L.F. and de Medeiros, M.A.F., 2009. "Numerical investigation of the three-dimensional secondary instabilities in the time-developing compressible mixing layer". *Journal of the Brazilian Society of Mechanical Sciences and Engineering*, Vol. 31, No. 2, pp. 125–136.

Ilak, M., Schlatter, P., Bagheri, S. and Henningson, D.S., 2012. "Bifurcation and stability analysis of a jet in crossflow: onset of global instability at a low velocity ratio". *Journal of Fluid Mechanics*, Vol. 696, pp. 94–121. doi: doi:10.1017/jfm.2012.10.

Jones, L.E., Sandberg, R.D. and Sandham, N.D., 2010. "Stability and receptivity characteristics of a laminar separation bubble on an aerofoil". *Journal of Fluid Mechanics*, Vol. 648, pp. 257–296.

Kelly, R.E. and de B. Alves, L.S., 2008. "A uniformly valid asymptotic solution for the transverse jet and its linear stability analysis". *Philosophical Transactions of the Royal Society of London. Series A: Mathematical and Physical Sciences*, Vol. 366, pp. 2729–2744.

Lardjane, N., Fedioun, I. and Gokalp, I., 2004. "Accurate initial conditions for the direct numerical simulation of temporal compressible binary shear layers with high density ratio". *Computers & Fluids*, Vol. 33, pp. 549–576.

Mack, C.J. and Schmid, P.J., 2010. "Direct numerical study of hypersonic flow about a swept parabolic body". *Computers & Fluids*, Vol. 39, No. 10, pp. 1932 – 1943. ISSN 0045-7930.

Michalke, A., 1984. "Survey on jet instability theory". *Progress in Aerospace Science*, Vol. 21, pp. 159–199.

Nichols, J.W. and Schmid, P.J., 2008. "The effect of a lifted flame on the stability of round fuel jets". *Journal of Fluid Mechanics*, Vol. 609, pp. 275–284. ISSN 1469-7645. doi:10.1017/S0022112008002528.

Roache, P.J., 2002. "Code verification by the method of manufactured solutions". *ASME Journal of Fluids Engineering*, Vol. 114, No. 1, pp. 4–10.

Saric, W.S., Reed, H.L. and White, E.B., 2003. "Stability and transition of three-dimensional boundary-layers". *Annual Review of Fluid Dynamics*, Vol. 35, p. 413Ű440.

Sipp, D., Barbagallo, A., O. Marquet, O. and Meliga, P., 2010. "Dynamics and control of global instabilities in open-flows: A linearized approach". *Applied Mechanics Reviews*, Vol. 63, p. 26.

Steinberg, S. and Roache, P.J., 1985. "Symbolic manipulation and computational fluid dynamics". *Journal of Computational Physics*, Vol. 57, No. 2, pp. 251–284.

Teixeira, R.S., 2014. *Esquemas de marcha com ganho mínimo para o desenvolvimento de soluções em regime permanente para problemas instáveis*. Ph.D. thesis, Instituto Militar de Engenharia, Rio de Janeiro.

Theofilis, V., 2003. "Advances in global linear instability analysis of nonparallel and three-dimensional flows". *Progress in Aerospace Sciences*, Vol. 39, pp. 249–315.

Tuckerman, L.S. and Barkley, D., 2000. "Bifurcation analysis for timesteppers". In *Numerical Methods for Bifurcation Problems and Large-Scale Dynamical Systems*, Springer New York, Vol. 119 of *The IMA Volumes in Mathematics and its Applications*, pp. 453–466. ISBN 978-1-4612-7044-7. doi:10.1007/978-1-4612-1208-9. URL `http://dx.doi.org/10.1007/978-1-4612-1208-9`.

Wang, L. and Mavriplis, D.J., 2007. "Implicit solution of the unsteady Euler equations for high-order accurate discontinuous Galerkin discretizations". *Journal of Computational Physics*, Vol. 225, No. 2, pp. 1994–2015.

## 7. RESPONSIBILITY NOTICE

The following text, properly adapted to the number of authors, must be included in the last section of the paper: The author(s) is (are) the only responsible for the printed material included in this paper.