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# IS F = MA ALWAYS NEWTON'S SECOND LAW OR ALL THAT GLITTERS IS NOT GOLD?

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***Abstract.*** *In this work, we pay attention to the fact that mathematical expression $\mathbf{F} = m\mathbf{a}$ not always represents the Newton's second law, but, in some situations, it becomes the called d'Alembert's Principle. We also show that principle is, in fact, a corollary of Coriolis' Theorem, under certain assumptions.*

***Keywords:*** *Newton's $2^{nd}$ Law, d'Alembert's Principle, noninertial frame of reference, Coriolis' Theorem.*

## 1. INTRODUCTION

Some mechanics textbooks, in special those that deals with engineering dynamics (e.g., Meriam and Kraige (2012), and Hibbeler (2012)), present applications of the Newton's $2^{\text{nd}}$ law by mean of examples, after they have explained about of the Newton's laws. The most common example may be the conic pendulum problem formulated as follows:

> Consider a bob of mass $m$ attached a weightless, extensionless rope of length $l$. The rope makes an angle $\phi$ with the vertical. The bob's motion is along a circle of radio $r$ as shown in figure $(1)$ . Find the bob's speed $v$ and the rope's tension $T$.

The presented solution can be summarized as: in accordance with the Newton's $2^{\text{nd}}$ law of motion, $\mathbf{F} = m\mathbf{a}$, we have that the horizontal component of tension (centripetal force) is equal to product of the mass by centripetal acceleration: $T\sin\phi = ma = m = \dfrac{v^2}{r}$; furthermore, the vertical component of tension and the weight must be in equilibrium: $T\cos\phi = mg$. Therefore, $T = \dfrac{mg}{\cos\phi}$ and $v = \sqrt{rg\tan\phi}$.

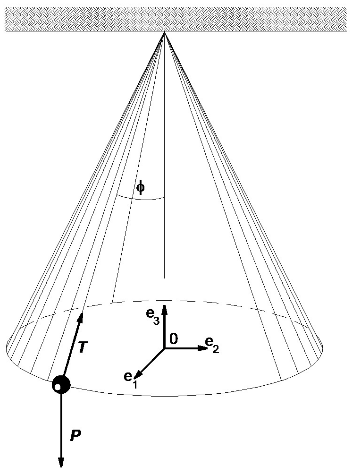


Figure 1. Conic pendulum movement

And it follows others similar examples.

Is really the presented solution a direct application of the Newton's $2^{\text{nd}}$ law? In this work, we will show the answer is no. In section (2) we deal with the axiomatic foundations of Classical Mechanics in order to have a clear understanding of what is happening to that problem. In section (3) we point out that the failure consists of using an inappropriate frame of reference. In section (4) we develop that problem in an inertial frame of reference. In section (5) we recall Coriolis' Theorem and present the solution to that problem in two different moving frames of reference, namely, cylindric and Frenet-Serret. Moreover we show that a particular form of Coriolis' Theorem corresponds to named d'Alembert's Principle.

## 2. THE THEORY

We start recalling the expression (equation) $\mathbf{F} = m\mathbf{a}$ arose for the first time in Euler's works, according to Truesdell (1968). Aside from this controversial issue in Mechanics History we remain focused only on applications of that expression which is known in the literature as Newton's 2$^{\text{nd}}$ law.

To employ properly the Newton's 2$^{\text{nd}}$ law it is necessary to understand the axiomatic foundations of Mechanics. Following Truesdell (1971) an axiomatization of a physic-mathematics theory begins, first of all, by a roll of quantities called *primitive quantities,* that is, they do not have definition, but can be measured or characterized by mathematical properties. Next, by definitions of other quantities in function of the primitives, called *derived quantities.* Afterwards, we set up *general axioms (laws, postulates)* which should relate those quantities in mathematical form. Finally, we set up *demonstrative theorems* which refer to a general theory or to particular cases.

In the vectorial mechanics we employ as primitive quantities: mass (positive scalar quantity), force (vectorial quantity) and frame of reference (space, time); and derived quantities: position vector, velocity and acceleration. Its mainly axioms are:

**Newton's 1$^{\text{st}}$ Law:** There is a frame of reference, called inertial frame of reference, such that a particle is at the rest or moves along a straight line with constant velocity (in that frame of reference) if only if the resultant of all applied force acts on the particle is null.

**Newton's 2$^{\text{nd}}$ Law:** In an inertial frame of reference, $\mathbf{F} = m\mathbf{a}$.

From the axiomatization has just presented it is clear that we can only apply to the Newton's 2$^{\text{nd}}$ law when the frame of reference chosen is an inertial one.

## 3. ANOTHER LOOK TO THE CONIC PENDULUM PROBLEM

Returning to the solution of the conic pendulum problem, a carefully observation shows us that the acceleration used in the Newton's 2$^{\text{nd}}$ law comes expressed by

$$\mathbf{a} = \frac{dv}{dt}\mathbf{e}_t \pm \frac{v^2}{r}\mathbf{e}_n \tag{1}$$

that is, the acceleration is actually expressed in a frame of reference whose basis employed is the Frenet-Serret's basis, $\{\mathbf{e}_t, \mathbf{e}_n, \mathbf{e}_b\}$. Such basis is not, in general, an inertial one. Hence the Newton's 2$^{\text{nd}}$ law can not be employed the way as the textbooks cited and many other do.

Thus, how can we solve this problem? One way consists of identifying an inertial frame of reference and solve the problem in it. This can always be done. Another way consists of adopting a noninertial frame of reference. In this case, we must fit the Newton's 2$^{\text{nd}}$ law to this kind of frames in order to get a solution. This fitting is done through Coriolis' Theorem and will be regarded in the section (5) of this present work.

## 4. CONIC PENDULUM PROBLEM SOLUTION IN AN INERTIAL FRAME OR REFERENCE

Let the inertial frame of reference $\{o, \beta\}$ be with the orthonormal basis $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ as shown in the figure (1). The position vector is given by $\mathbf{x}(t) = x_1(t)\mathbf{e}_1 + x_2(t)\mathbf{e}_2 + x_3(t)\mathbf{e}_3$. It then follows that the velocity and the acceleration are expressed by $\dfrac{d\mathbf{x}}{dt} = \dfrac{dx_1}{dt}\mathbf{e}_1 + \dfrac{dx_2}{dt}\mathbf{e}_2 + \dfrac{dx_3}{dt}\mathbf{e}_3$ and $\dfrac{d^2\mathbf{x}}{dt^2} = \dfrac{d^2x_1}{dt^2}\mathbf{e}_1 + \dfrac{d^2x_2}{dt^2}\mathbf{e}_2 + \dfrac{d^2x_3}{dt^2}\mathbf{e}_3$, respectively, in that frame of reference. The forces act on the particle are the weight and the constrained force at the hope. They are expressed mathematically in the inertial frame of reference chosen by $\mathbf{P} = -mg\mathbf{e}_3$ and $\mathbf{T} = -T\sin\phi\cos\theta\mathbf{e}_1 - T\sin\phi\sin\theta\mathbf{e}_2 + T\cos\phi\mathbf{e}_3$, respectively. Now we are under the condition required to apply to the Newton's 2$^{\text{nd}}$ law. Hence by inserting of that results into $\mathbf{F} = m\mathbf{a}$ and after a few algebraic manipulations we arrive at the differential ordinary equations system:

$$\left.\begin{aligned} m\frac{d^2x_1}{dt^2} &= -T\sin\phi\cos\theta \\ m\frac{d^2x_2}{dt^2} &= -T\sin\phi\sin\theta \\ m\frac{d^2x_3}{dt^2} &= T\cos\phi - mg \end{aligned}\right\} \tag{2}$$

Since the motion is plane, we have $x_3 = 0$, which leads immediately to the conclusion that $T = \dfrac{mg}{\cos\phi}$. Now as $\cos\theta = \dfrac{x_1}{r}$ and $\sin\theta = \dfrac{x_2}{r}$, where $r = \sqrt{x_1^2 + x_2^2}$, the system $(2)$ is reduced to the following uncoupled system:

$$\left.\begin{aligned} \frac{dv_1}{dt} &= -\frac{g\tan\phi}{r}x_1 \\ \frac{dv_2}{dt} &= -\frac{g\tan\phi}{r}x_2 \end{aligned}\right\} \tag{3}$$

where $v_1 = \dfrac{dx_1}{dt}$ and $v_2 = \dfrac{dx_2}{dt}$. Next, we proceed the change of variables: $v_1 = v_1(x_1(t))$ and $v_2 = v_2(x_2(t))$. Thus by the chain rule we get:

$$\left.\begin{aligned} \frac{dv_1}{dt} &= v_1\frac{dv_1}{dx_1} = -\frac{g\tan\phi}{r}x_1 \\ \frac{dv_2}{dt} &= v_2\frac{dv_2}{dx_2} = -\frac{g\tan\phi}{r}x_2 \end{aligned}\right\} \tag{4}$$

Performing the integration that system with initial conditions at the time $t = 0$: 1) the particle is at the position coordinates $x_1 = r$ and $x_2 = 0$, and 2) its velocity components are $v_1 = 0$ and $v_2 = \bar{v}$, we obtain:

$$v_1^2 = -\frac{g\tan\phi}{r}\left(x_1^2 - r^2\right) \tag{5}$$

$$v_2^2 - \bar{v}_2 = -\frac{g\tan\phi}{r}x_2^2 \tag{6}$$

Adding last two expressions we get:

$$v_1^2 + v_2^2 - \bar{v}^2 = 0 \tag{7}$$

Recalling $v = \sqrt{v_1^2 + v_2^2 + v_3^2}$ is the speed, we see it is constant in any point of trajectory and has value $\bar{v}$. To determine $\bar{v}$ in function of the problem data is enough to do $x_1 = 0$ and $x_2 = r$ in $(6)$ , since $v_2 = 0$. Therefore,

$$v = \bar{v} = \sqrt{rg\tan\phi} \tag{8}$$

## 5. CONIC PENDULUM PROBLEM SOLUTION IN A NONINERTIAL FRAMES OF REFERENCE

Before we set up the mainly results of this section we are going to introduce some explanation about the notation that will be employed. We are going to employ the matrix notation instead of the usual vectorial notation to represent vectors. This way, if $\beta = \{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ is an orthonormal basis in $\mathbb{R}^3$ (regarded as vectorial space) we put the column matrix $[z]_\beta = \begin{bmatrix} z_1 & z_2 & z_3 \end{bmatrix}^{\mathrm{T}}$ to represent the vector $\mathbf{z} = z_1\mathbf{e}_1 + z_2\mathbf{e}_2 + z_3\mathbf{e}_3$, where the symbol T represents the transposition operation. The vector product between vectors $\mathbf{z}$ and $\mathbf{w}$ is given by $[z]_\beta \times [w]_\beta = \begin{bmatrix} z_1 & z_2 & z_3 \end{bmatrix}^{\mathrm{T}} \times \begin{bmatrix} w_1 & w_2 & w_3 \end{bmatrix}^{\mathrm{T}} = \begin{bmatrix} z_2w_3 - z_3w_2 & z_3w_1 - z_1w_3 & z_1w_2 - z_2w_1 \end{bmatrix}^{\mathrm{T}}$. In addition, we put $[\dot{z}]_\beta = \begin{bmatrix} \dot{z}_1 & \dot{z}_2 & \dot{z}_3 \end{bmatrix}^{\mathrm{T}}$ and $[\ddot{z}]_\beta = \begin{bmatrix} \ddot{z}_1 & \ddot{z}_2 & \ddot{z}_3 \end{bmatrix}^{\mathrm{T}}$ to design the rates of variation in time of a vector $\mathbf{z}$, first and second, respectively.

In order to get to Coriolis' Theorem we need a once more postulate:

**Postulate 1** *Forces are frames independent. More precisely, let* $\mathbf{R} = \{o, \beta\}$ *and* $\mathbf{R}' = \{o', \beta'\}$ *be frames of reference. If* $\mathbf{F}$ *represent a force in* $\mathbf{R}$, *then*

$$\mathbf{F}' = Id\left(\mathbf{F}\right) \tag{9}$$

*where Id is identity operator. Or, in components*

$$[F]_{\beta'} = [H]_\beta^{\beta'}\,[F]_\beta \tag{10}$$

*where* $[H]_\beta^{\beta'}$ *is the matrix of identity operator (transition matrix).*

**Theorem 2 (Coriolis' Theorem)** *Let* $\mathbf{R} = \{o, \beta\}$ *and* $\mathbf{R}' = \{o', \beta'\}$ *be frames of reference, inertial and moving, respectively, where* $\beta = \{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ *and* $\beta' = \{\mathbf{e}_1'(t), \mathbf{e}_2'(t), \mathbf{e}_3'(t)\}$ *are orthonormal basis. If the resultant of all forces acting on a particle of mass* $m$ *is* $\mathbf{F}$, *hence*

$$m[a']_{\beta'} = [F]_{\beta'} - m[a_{o'}]_{\beta'} - m[\dot{\Omega}]_{\beta'} \times [x']_{\beta'} - m[\Omega]_{\beta'} \times \Big([\Omega]_{\beta'} \times [x']_{\beta'}\Big) - 2m[\Omega]_{\beta'} \times [v']_{\beta'} \tag{11}$$

*where the left-hand side reads mass times the acceleration of the particle relative to moving frame reference. And the right-hand the terms* $-m[a_{o'}]_{\beta'}$, $-m[\dot{\Omega}]_{\beta'} \times [x']_{\beta'}$, $-m[\Omega]_{\beta'} \times \left([\Omega]_{\beta'} \times [x']_{\beta'}\right)$ *and* $-2m[\Omega]_{\beta'} \times [v']_{\beta'}$ *are called generically of fictitious forces, namely: inertial force, Euler force, centrifugal force and Coriolis force, respectively. And where* $[F]_{\beta'}$ *is the representation of* $\mathbf{F}$ *in the moving frame of reference.*

A proof can be found in Greenwood (1988) or Woodhouse (1987).

### 5.1 Solution in Cylindrical Coordinates

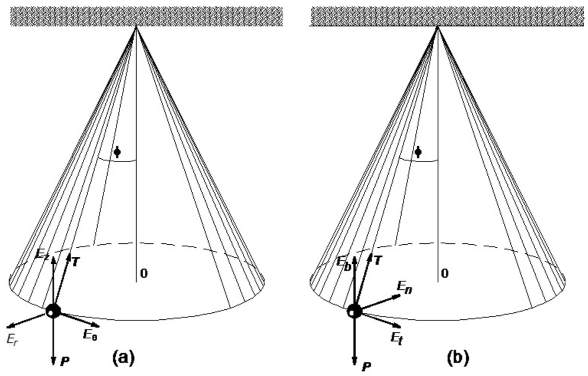


Figure 2. (a) Cyllindric basis. (b) Frenét-Serret basis

Next, we regard to a cylindrical coordinates system as shown in figure (2) (a) to describe the movement. In this case, the moving basis is given by $\beta' = \{\mathbf{e}_r(t), \mathbf{e}_\theta(t), \mathbf{e}_z(t)\}$. By convinience, it was drawn in the particle. In this system, the particle position vector comes expressed by $[x']_{\beta'} = \left[\begin{array}{ccc} r & 0 & 0 \end{array}\right]^{\rm T}$. It hence follows that its relative velocity and relative acceleration are both null. The angular velocity and angular acceleration are $[\Omega]_{\beta'} = \left[\begin{array}{ccc} 0 & 0 & \dot{\theta} \end{array}\right]^{\rm T}$ and $[\dot{\Omega}]_{\beta'} = \left[\begin{array}{ccc} 0 & 0 & \ddot{\theta} \end{array}\right]^{\rm T}$, respectively. On the other hand, the resultant of all forces that act on the particle can be expressed by $[F]_{\beta'} = \left[\begin{array}{ccc} -T\sin\phi & 0 & T\cos\phi - mg \end{array}\right]^{\rm T}$ in this moving frame of reference. Therefore the equation (11) is written as

$$\left.\begin{array}{rl} -T\sin\phi + mr\dot{\theta}^2 & = 0 \\ -mr\ddot{\theta} & = 0 \\ T\cos\phi - mg & = 0 \end{array}\right\} \tag{12}$$

Undoubtedly, we have the constrained force projection in the horizontal plane is in equilibrium with the centrifugal force: $-T\sin\phi + mr\dot{\theta}^2 = 0$. Since $\ddot{\theta} = 0$ we have uniform circular motion. Also, we find $T = \dfrac{mg}{\cos\phi}$. Eliminating $T$ from horizontal projection we have $r\dot{\theta}^2 = g\tan\phi$. And, whereas $r\dot{\theta}^2 = \dfrac{v^2}{r}$, we finally get $v = \sqrt{rg\tan\phi}$.

### 5.2 Solution in Frenet-Serret's Frame

**Corollary 3 (d'Alembert's Principle)** *Let* $\mathbf{R} = \{o, \beta\}$ *and* $\mathbf{R}' = \{o', \beta'\}$ *be frames of reference, inertial and moving, respectively, where* $\beta = \{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ *and* $\beta' = \{\mathbf{e}'_1(t), \mathbf{e}'_2(t), \mathbf{e}'_3(t)\}$ *are orthonormal basis. If the resultant of all forces acting on a particle of mass* $m$ *is* $\mathbf{F}$ *and if the vector position relative to moving frame of reference of the particle is null during the motion, then*

$$[F]_{\beta'} = m[a]_{\beta'} \tag{13}$$

*where* $[a]_{\beta'} \equiv [a_{o'}]_{\beta'}$ *is the acceleration of moving frame of reference and* $[F]_{\beta'}$ *is the expression of the force in that moving system.*

We note that (13) is similar to "$\mathbf{F} = m\mathbf{a}$", but is far from being considered the Newton's 2$^{\rm nd}$ law, since it refers to the moving frame of reference. The notation employed makes it quite clear.

We quite often find in the literature the expression (13) written in the following form

$$[F]_{\beta'} - m[a]_{\beta'} = [0]_{\beta'} \tag{14}$$

which written in vectorial notation becomes $\mathbf{F}-m\mathbf{a}=\mathbf{0}$. It is usually attributed to d'Alembert and it is called *d'Alembert's Principle* (Greenwood (1988), Lanczos (1986)).

The secret of employing $(13)$ or $(14)$ consists in observing that, among determined class of problem, is possible to make to coincide the moving frame of reference origin with a particle in every instant of time. Thus we can take advantage of this to find some problem unknowns such as acceleration components, constrained force, etc.

Returning to the conic pendulum problem we can write $(13)$ using the $\beta'$ moving frame of Frenet-Serret, figure $(2)$ (b), as

$$T\sin\phi\mathbf{e}_n+(T\cos\phi-mg)\mathbf{e}_b=m\left(\frac{dv}{dt}\mathbf{e}_t+\frac{v^2}{r}\mathbf{e}_n\right) \tag{15}$$

From vectorial identity we get

$$\left.\begin{array}{rcl} \dfrac{dv}{dt} & = & 0 \\ T\sin\phi & = & m\dfrac{v^2}{r} \\ T\cos\phi & = & mg \end{array}\right\} \tag{16}$$

and it then follows the results already known.

## 6. A BIT OF HISTORY

Returning the literature we find that there are two statements of principle usually assigned d'Alembert, according to Truesdell (1968). The first, we call vectorial version, comprise adding the inertia force (equal to the simetric product of the mass and acceleration of particle) in both members of Newton's $2^{\text{nd}}$ Law to get $\mathbf{F}-m\mathbf{a}=\mathbf{0}$. Certainly, this procedure does not involve any new principle in mechanics, it would be merely a rephrasing of Newton's $2^{\text{nd}}$ Law. Assigning d'Alembert this procedure is considereded by Hamel in insult to his intelligence (in Rosenberg (1977)). Evidently, we are not in front of a new principle, as it has been demonstrated in the Corollary $(3)$. Another interpretation coming from the literature is that d'Alembert's Principle allows to reduce an dynamics problem to an statics one, called *dynamics equilibrium*. According to Dugas (1955) this reading does not complies with the d'Alembert thoughts who had clear notion about motion and equilibrium. In spite of Dugas (1955) this reading sounds true by the Corollary $(3)$, but we must be criterious since there are hypothesis to be met in order to benefit from that corollary. The second, we call analytic version, consists in "decomposing the motion in two parts, one being "natural" and the other due to the presence of the constraints", Truesdell (1968). d'Alembert's Principles asserts that the forces corresponding to the accelerations due to the constraints force form a system in static equilibrium. For a modern reading of that analytic version we refer the reader to Tavares Jr., H. M. and Sampaio, R. (1993).

The employment of non inertial frames of reference is not new in Mechanics. It was used by Huyghens in *Horologium Oscilatarium* in 1659 (Dugas (1955)). Neither the idea of relative equilibrium, since in 1665 the Italian astronomer Giovanni Borelli stated that the centrifugal force upon Galilean Moons, in their orbital motion, was exactly in equilibrium with the attractive force of Jupiter (in Fowles and Cassidy (1999)).

Following the chronological order, d'Alembert's Principle appeared in his book *Traté de Dynamique* in 1743, while Coriolis' Theorem appeared in *Mémoire sur les équations du mouvement relatif de systèmes de corps* in J. École Polytech., vol. 15, cahier 24, 142-154, 1835. Hence to prevail the interpretation given in the Corollary $(3)$ we can conclude that Huyghens, d'Alembert's and Borelli foresaw that the dynamics of a problem could be addressed in moving frames in a time where frames of reference were not fully understood.

## 7. CONCLUSION

In this work, we solved the conic pendulum problem in three different ways. Namely, we solved it: first, in an the inertial frame of reference; second, in the cylindrical moving frame of reference; and, finally, regarding a Frenet-Serret's moving frame of reference. In all of them we payed attention to the role played by the frame of reference. In several engineering dynamics textbooks such considerations are both omitted or tacitly mentioned. Hence, presenting $m\frac{v^2}{r}=T\sin\phi$ as the solution of the conic pendulum problem is to present in advance some simplifications and hide important details.

We believe that the misuse of the Newton's $2^{\text{nd}}$ law is due to missing information that it only applies in an inertial frame of reference. Consequently, every solution of a mechanics problem should begin by identifying a clear and well defined inertial frame of reference in order to avoid some mistakes of this kind.

We found that d'Alembert's Principle (vectorial version) is not a principle in Mechanics but a corollary of Coriolis' Theorem. The expression $(13)$ tells us that applied forces on the particle are in equilibrium with fictitious force. Therefore the interpretation that referred principle reduces from a dynamics problem to a statics problem is certainly held.

Finally, we observe that it can be advantageous to carry out the movement analysis in the moving frame of reference which moves with the particle together.

## 8. ACKNOWLEDGEMENTS

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