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# Analysis of the influence of construction parameters in optimizing the ecological function of an irreversible Otto cycle

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***Abstract:*** *In this article it´s presented an analysis of an irreversible Otto cycle with objective to optimize the net power obtained through ecological function. The considered cycle operates between two thermal reservoirs with infinite heat capacity rates, it's has internal irreversibility's from the non-isentropic compression and expansion processes, irreversibility's arising of the thermal resistance in the heat change and the heat leakage from high temperature reservoir to low temperature reservoir. A graphical analysis of the value of the ecological function, net power thermal efficiency and entropy generation rate is presented. The optimal values of compression rate are presented for each analyzed criteria. Through graphical analysis it is possible to present the relation between compression ratios for ecological function and other criteria, besides, the ratio of power, efficiency and entropy generation rate due to ecological function and other criteria. The results presented can be useful in the better understanding of the effects of the irreversibility's in the ecological optimization of the Otto cycle.*

***Keywords:*** *Otto Cycle, compression rate, ecological function, irreversibility's.*

## 1. INTRODUCTION

Called heat engines, whose working fluid is due to the burning of a fuel-air mixture, the internal combustion engines transformed the modern society, being of great importance for the world economy and power production. Because of its importance and increasingly strict environmental laws, currently studies related to internal combustion engines are focused on the quality of the produced energy. Thus, current research is not only directed to the engine and began to focus on the relationship between the engine and the environment that surrounds it.

The optimization of internal combustion engines have emerged in the work of Chen (1994), which considered an internal combustion engine operating between two thermal energy reservoirs at different temperatures with the heat transfer process being performed by heat exchangers between thermal energy reservoirs and the engine. The irreversibility considered are due to the thermal resistance of the heat exchangers. This research has its roots in the work of Curzon and Ahlborn (1975), where it was shown that for internal combustion engines operating between two thermal reservoirs (endorreversible) and in maximum power condition, maximum thermal efficiency are close to the values obtained by expression $\eta = 1 - \sqrt{T_L / T_H}$, where $T_L$ is the low temperature reservoir and $T_H$ is the high temperature reservoir.

Using a thermodynamic objective function, called ecological function, Angulo-Brown (1991) conducted an optimization of an endorreversible Carnot cycle, becoming known as ecological optimization. In this type of optimization, the goal is the best compromise between the net work produced by an engine $W$ and this engine interactions with the environment that surrounds it, being expressed by $E = W - T_L S_G$, where $S_G$ is the generation cycle of entropy. In a later study, Yan (1991) suggested that the ecological criteria proposed by Brown (1991) would be more suitable if the temperature of the low temperature reservoir $T_L$ was equal to the reference temperature environment $T_0$. Using this objective function, Cheng and Chen (1998) ecologically optimized an endorreversible Brayton cycle. The authors presented a thermodynamic modeling of an irreversible Otto cycle, subsequently optimized by the concept of ecological function.

This paper proposes the modeling of the Otto cycle using some constructive parameters such as compression ratio. The results are shown in graphic form indicating optimal values of the compression ratio in connection with the compression ratio obtained at the maximum power condition. This analysis allows a better understanding of ecological function, and especially points out optimum constructive parameters for an engine to operate at its maximum ecological function.

## 2. MATHEMATICAL MODELLING

Fig. 1 shows a schematic diagram of the Rankine cycle which will be analyzed, where $\dot{Q}_{H,C}$ is the rate heat effectively transferred from the reservoir of high temperature to the cycle, $\dot{Q}_{L,C}$ is the rate heat effectively transferred from the cycle to the reservoir of low temperature, while $\dot{Q}_I$ is the rate heat transferred directly from high temperature to the low temperature reservoir. This term is also known as "heat leakage". Finally, $\dot{W}$ is the net power of the cycle.

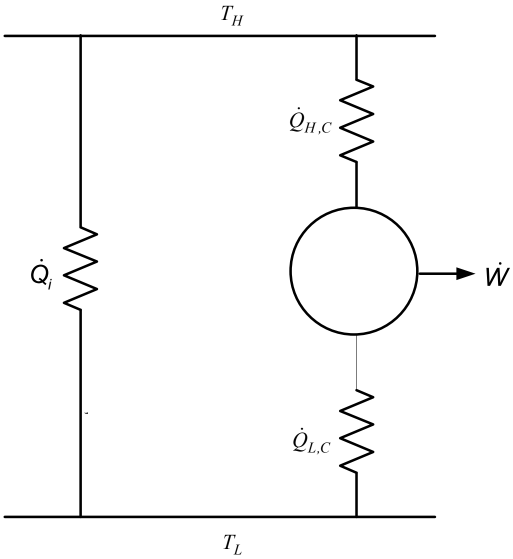



Figure 1 - Schematic diagram of the Otto cycle.

Fig. 2 shows the temperature-entropy cycle diagram showing the internal irreversibility, i.e., non-isentropic compression and expansion processes:

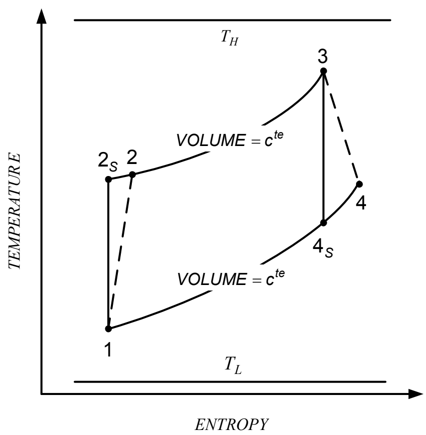



Figure 2 - Temperature-entropy diagram for the irreversible Otto cycle.

Heat transfer rates absorbed and rejected by the fluid can be modeled as heat exchangers and then written as follows:

$$\dot{Q}_H = \dot{Q}_{H,C} + \dot{Q}_I \tag{1}$$

$$\dot{Q}_L = \dot{Q}_{L,C} + \dot{Q}_I \tag{2}$$

with:

$$\dot{Q}_{H,C} = (UA)_H \frac{(T_H - T_3) - (T_H - T_2)}{\ln[(T_H - T_3)/(T_H - T_2)]} = \dot{C}_v \varepsilon_H (T_H - T_2) = \dot{C}_v (T_3 - T_2) \quad (3)$$

$$\dot{Q}_{L,C} = (UA)_L \frac{(T_4 - T_L) - (T_1 - T_L)}{\ln[(T_4 - T_L)/(T_1 - T_L)]} = \dot{C}_v \varepsilon_L (T_4 - T_L) = \dot{C}_v (T_4 - T_1) \quad (4)$$

where $T_H$ and $T_L$ are the temperatures of the hot and cold currents in the exchangers, respectively, $\dot{C}_v$ is the thermal capacity rate of the fluids at constant volume, $U_H A_H$ and $U_L A_L$ is the result between the global coefficient of the heat transfer and the heat exchange area and the effectiveness $\varepsilon_H$ and $\varepsilon_F$ are given accordingly to the number of heat transfer units $NTU_H$ and $NTU_L$ of the heat exchangers, $\varepsilon = 1 - e^{-NTU}$ .

$$T_3 = T_2 + (T_H - T_2)\varepsilon_H \quad (5)$$

$$T_1 = T_4 - (T_4 - T_L)\varepsilon_L \quad (6)$$

The heat transfer rate between the reservoirs can be calculated as, Bejan (1988):

$$\dot{Q}_I = \dot{C}_I (T_H - T_L) \quad (7)$$

Using the first law of termodynamics and Eqs. (3) and (4), the net power can be determinated by two expressions:

$$\dot{W} = \dot{C}_v \varepsilon_H (T_H - T_2) + \dot{C}_v \varepsilon_L (T_4 - T_L) \quad (8)$$

$$\dot{W} = \dot{C}_v (T_3 - T_2) + \dot{C}_v (T_4 - T_1) \quad (9)$$

Isolating $T_4$ in the Eq. (8), it is obtained:

$$T_4 = \left[ \frac{-\dot{W} + \varepsilon_H (T_H - T_2)}{\varepsilon_L} \right] + T_L \quad (10)$$

Considering a reversible Otto cycle, the following relation between cycle temperatures can be written:

$$\frac{T_3}{T_{2S}} = \frac{T_{4S}}{T_1} \quad (11)$$

Once the cycle considered in this work is an irreversible cycle, one can write a relationship between actual temperatures and the isentropic temperatures using the definitions of the isentropic efficiencies of compression and expansion, namely:

$$\eta_C = \frac{T_{2S} - T_1}{T_2 - T_1} \quad (12)$$

$$\eta_E = \frac{T_3 - T_4}{T_3 - T_{4S}} \quad (13)$$

The compression ratio is one of the basic parameters for design and construction of internal combustion engines. Thus, for a better understanding of this parameter, the following expressions are used to define the compression ratio for a reversible cycle:

$$\frac{T_{2S}}{T_1} = r^{k-1} \quad (14)$$

$$\frac{T_3}{T_{4S}} = r^{k-1} \tag{15}$$

By substituting the Eqs. (12) and (13) in Eqs. (14) and (15), we obtain:

$$T_1 = \frac{\eta_C T_2}{-1+\eta_C + r^{k-1}} \tag{15}$$

$$T_4 = \left(\frac{T_3}{r^{k-1}} - T_3\right)\eta_C + T_3 \tag{16}$$

Using Eqs. (9), (15) and (16) we obtain an expression for the net power as a function of $T_2$ and $r$:

$$\dot{W} = -\frac{\left\langle \left\{ \left[ (\varepsilon_H - 1)T_2 - \varepsilon_H T_H \right] \cdot r^{k-1} + \eta_E \left[ -1+\eta_C \right] \left[ (\varepsilon_{H-1})T_2 - \varepsilon_H T_H \right] \right\} \dot{C}_v \left( r^{k-1} - 1 \right) \right\rangle}{r^{k-1}\left(-1+\eta_C + r^{k-1}\right)} \tag{17}$$

Substituting Eqs. (5), (6), (10) and (17) and grouping the terms in $T_2$, yields the following expression:

$$AT_2^2 + BT_2 + C = 0$$

where:

$$A = \frac{a_1}{y^2} + \frac{\frac{a_2}{x} + a_3}{y} + a_4$$

$$B = \frac{b_1}{y^2} + \frac{b_2}{y} + b_3$$

$$C = \frac{c_1}{y^2} + \frac{c_2}{y} + c_3$$

with:

$$y = \eta_c - 1 + x$$

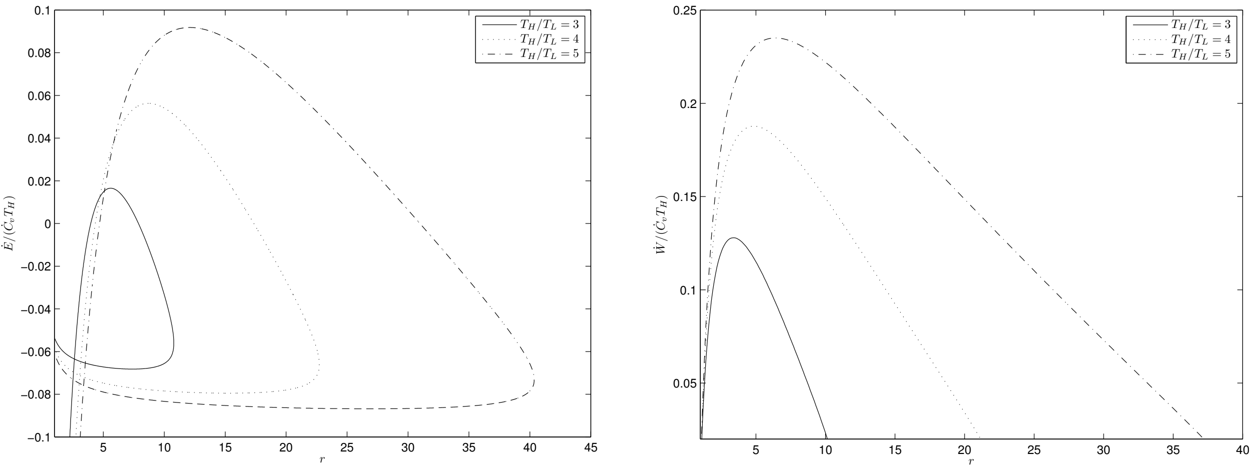
$$x = r^{k-1}$$

The coefficients $a_1, a_2, a_3, a_4, b_1, b_2, b_3, c_1, c_2$ e $c_3$ are complex and can be viewed in the appendix.

## 3. RESULTS AND DISCUSSION

Fig. 3(a) shows the behavior of the dimensionless ecological function as a function of compression ratio for different values of the ratio $T_H/T_L$. It is found that there are points where the ecological function is negative. At these points, power values are smaller than the entropy generation. It is observed that there is a point where the ecological function is maximum, being related to an optimal compression ratio. Thus, for a better ecological performance, it is desirable to operate the engine with a compression ratio close to that. As the ratio values $T_H/T_L$. increase, the values of ecological function increases substantially with the optimal compression ratio. When the compression ratio values are increased, the ecological function initially increases rapidly to its maximum point, and from this point the decrease of ecological function occurs more slowly.

Fig. 3(b) shows the behavior of the dimensionless net power as a function of compression ratio for different values of the ratio $T_H/T_L$. It is noted that for different values of $T_H/T_L$ there is a single maximum point for the dimensionless net power, being related to an optimum compression ratio point. Furthermore, with the increase of the $T_H/T_L$ there is an increase in both the maximum dimensionless net power as the optimal compression ratio. The behavior of this curve can also be seen in the work of Chen, *et al.* (2006).




(a) (b)

Figure 3. (a) Dimensionless ecological function as a function of the compression ratio (a) Dimensionless net power as a function of the compression ratio, being $NTU_H = NTU_L = 3;\ \eta_c = \eta_e = 0.95;\ \dot{C}_I/\dot{C}_v = 0.005;\ k = 1.4.$

Comparing the optimal values of compression ratio obtained in Figs. 3 (a) and 3 (b), there is an average ratio of 1.8 for each temperature ratio $T_H/T_L$, i.e., the compression ratio in the maximum dimensionless ecological function condition is about 1.8 times higher than the ratio compression in the dimensionless maximum net power condition. On the other hand, using the optimal compression ratio values obtained in Fig. 3 (a) in Figure 3 (b), it is noted that the dimensionless net power has lower values than the dimensionless maximum net power. In this case it is observed an approximate ratio of 0.85, i.e., the dimensionless net power obtained in dimensionless maximum ecological function condition is about 85% of the dimensionless net power obtained in dimensionless maximum net power condition.

Fig. 4 shows the behavior of the thermal efficiency as a function of compression ratio for different values of the ratio $T_H/T_L$. There is a similarity between the curve shown in Fig. 4 and Fig. 3 (b). With the increase of the $T_H/T_L$ there is an increase in both thermal efficiency and optimal compression ratio. An increase in compression ratio causes an increase of the thermal efficiency to the point of maximum thermal efficiency. After this point, an increase in the compression ratio values causes a decrease in the values of thermal efficiency. Performing a similar analysis performed in compression ratio for dimensionless net power, but for thermal efficiency, it is found that the ratio of the compression ratios due to dimensionless maximum ecological function and maximum thermal efficiency is 0.80. The thermal efficiency due to dimensionless maximum net power is about 98% of the maximum thermal efficiency.

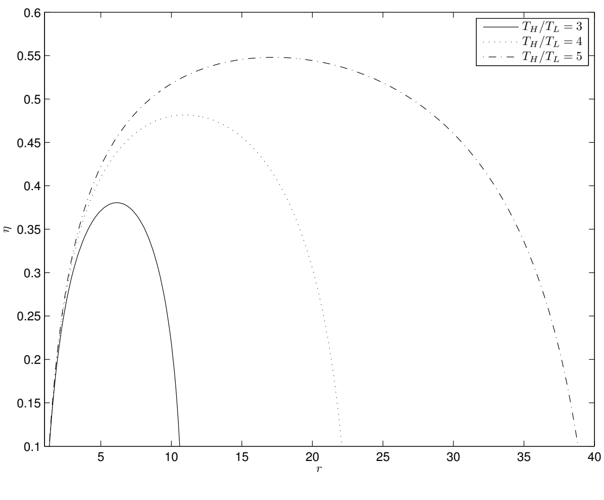



Figure 4. Thermal efficiency as a function of the compression ratio, being $NTU_H = NTU_L = 3;\ \eta_c = \eta_e = 0.95;$ $\dot{C}_I/\dot{C}_v = 0.005;\ k = 1.4.$

Fig. 5 shows the behavior of the dimensionless entropy generation rate for different values of the ratio $T_H/T_L$. Analyzing the values of dimensionless entropy generation, it appears that an increased compression ratio causes a decrease of the values of dimensionless entropy generation to the point where there is no more power output. In other words, the point of minimum entropy generation coincides with the point where the power output is equal to zero. The increases in $T_H/T_L$ ratio cause an increase in the dimensionless entropy generation and continues to shift the values of dimensionless minimum entropy generation to the right, i.e., the compression ratio values are higher as $T_H/T_L$ increases.

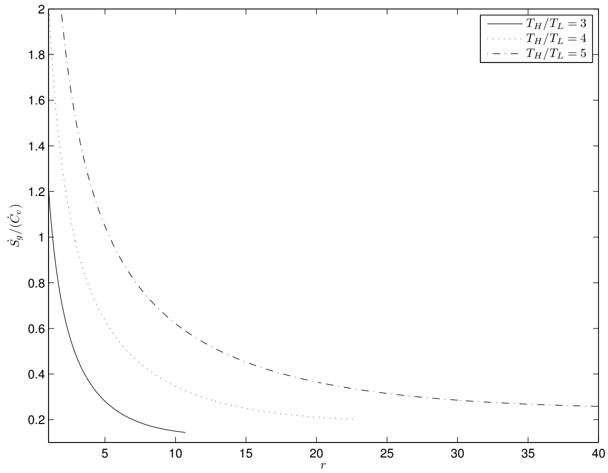



Figure 5. Dimensionless entropy generation rate as a function of the compression ratio, being $NTU_H = NTU_L = 3$; $\eta_c = \eta_e = 0.95$; $\dot{C}_I / \dot{C}_v = 0.005$; $k = 1.4$.

## 4. CONCLUSIONS

This work presented an analysis of the influence of construction parameters in optimizing the ecological function of an irreversible Otto cycle. The cycle was modeled between two thermal energy reservoirs, a high temperature reservoir and a low temperature reservoir. The irreversibilities of this cycle are derived from the temperature difference between the thermal reservoir and the working fluid, the heat leak rate, that is transferred directly from the high temperature reservoir to the low temperature reservoir without power production and irreversibilities resulting from non-isoentropic compression and expansion processes, being expressed by isoentropic efficiencies. A modeling that contemplates compression ratio is proposed.

Through a graphical analysis it was possible to achieve optimal compression ratio values for different criteria, such as maximum power, maximum thermal efficiency, maximum entropy generating and maximum ecological function. Mathematical modeling engines often have encompassed the compression ratio, however, there are few studies on the ecological optimization that includes construction parameters. In a graphical analysis showed that the optimal compression ratio due to ecological function is relatively higher than the optimal compression ratio due to maximum power. When comparing the same optimum compression ratios due to ecological function and thermal efficiency it was found that they are very close. Finally, when comparing the optimal compression ratios due to ecological function and the generation of entropy, it was observed lower values of this ratio.

It is expected that the results of this study serve for a better understanding of the ecological function, since the previously published work does not have construction parameters. Thus, the addition of this parameter seeks to bring ecological optimizations for less idealized modeling.

## 5. ACKNOWLEDGEMENTS

To Federal Institute of Paraná and Universidade Estadual Paulista Júlio de Mesquita Filho.

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## 7. RESPONSIBILITY NOTICE

The authors are the only responsible for the printed material included in this paper.

## 8. APPENDIX

$$a_1 = \frac{2\left\{x[\varepsilon_H - 1]^2\left[\left(-\frac{3}{2}+\eta_c\right)x - \frac{1}{2}\eta_c + \frac{1}{2}\right]\eta_e^2 - x[\varepsilon_H - 1][\eta_c - 1 + x]\eta_e - \frac{1}{2}\eta_c^2\left(\eta_c - 1 + r^{k-1}\right)^2[\eta_c - 1][\varepsilon_L - 1]\right\}}{\eta_e \varepsilon_H x^2}$$

$$a_2 = \frac{\left(\{1+[\eta_e - 1]\eta_c\}\{\varepsilon_H - 1\}\varepsilon_L^2 + \{[(-\eta_e + 3)\varepsilon_H - 2 + \eta_e]\eta_c - 3\varepsilon_H + 1\}\varepsilon_L - 2\varepsilon_H\{\eta_c - 1\}\right)(\varepsilon_H - 1)(\eta_c - 1)}{x\varepsilon_L^2}$$

$$a_3 = \frac{[(\varepsilon_H - 1)\eta_e + \eta_c]\{[\varepsilon_H - 1][2 + (\varepsilon_L - 2)\eta_c][\varepsilon_L - 1]\eta_e + [(-\varepsilon_L + 1)\varepsilon_L^2 + (3\varepsilon_H - 4)\varepsilon_L - 2\varepsilon_H + 2]\eta_c + (\varepsilon_L - 1)(\varepsilon_L - 2)(\varepsilon_L - 1)\}}{\eta_e \varepsilon_L^2}$$

$$a_4 = \frac{(\varepsilon_H - 1)(\varepsilon_L - 1)[(-\eta_c + 1 + \eta_c\varepsilon_L)(\varepsilon_H - 1)\eta_e - (\eta_c - 1)(\varepsilon_L - 1)\varepsilon_H + (-1 + 2\eta_c)\varepsilon_L - \eta_c + 1](\eta_e - 1)}{\eta_e \varepsilon_L^2}$$

$$b_1 = -\frac{4T_H\left\{\left[\left(\eta_c - \frac{3}{2}\right)x - \frac{1}{2}\eta_c + \frac{1}{2}\right][\varepsilon_H - 1]\eta_e - \frac{1}{2}x\right\}(\eta_c - 1)(\varepsilon_L - 1)y}{x^2\varepsilon_L^2}$$

$$b_2 = \frac{2\eta_e\varepsilon_H T_H + 8\eta_e\varepsilon_H^2 T_H\eta_c + T_L\varepsilon_L^2 x\eta_e - 2\varepsilon_H T_H\eta_c^2 x - 2T_L\eta_c^2 x\varepsilon_L + T_L\eta_c^2\varepsilon_L^2 x + 2\eta_e\varepsilon_H T_H\eta_c^2 - 2T_L\varepsilon_H\eta_c\varepsilon_L x\eta_e}{\eta_e x\varepsilon_L^2}$$

$$+\frac{-2T_L\varepsilon_H\varepsilon_L\eta_e + 4\varepsilon_H^2 T_H\eta_c\varepsilon_L^2\eta_e + 2T_L\eta_c^2\varepsilon_L\eta_e + 4\eta_e^2\varepsilon_H^2 T_H\varepsilon_L x + \varepsilon_H T_H\eta_c\varepsilon_L^2 x - 3T_H\varepsilon_H\varepsilon_L\eta_c^2 x\eta_e - 2\eta_c^2\varepsilon_H^2\varepsilon_L^2 T_H\eta_e}{\eta_e x\varepsilon_L^2}$$

$$+\frac{6\varepsilon_L T_H\eta_c\varepsilon_H^2 x\eta_e - 5\varepsilon_L T_H\eta_c\varepsilon_H x\eta_e + 4\eta_e^2\varepsilon_H T_H x - 4\eta_e^2\varepsilon_H^2 T_H x + 2T_L\eta_c\varepsilon_L x + 6\varepsilon_H^2\eta_e T_H\varepsilon_L - 4T_H\eta_c\varepsilon_H^2 x\eta_e}{\eta_e x\varepsilon_L^2}$$

$$+\frac{-2\varepsilon_L T_H\eta_c\varepsilon_H\eta_e^2 + 2\varepsilon_L T_H\varepsilon_H\eta_e^2\eta_c^2 - 2T_H\eta_e\varepsilon_H^2\varepsilon_L^2\eta_c^2 - 2T_H\varepsilon_H\varepsilon_L^2\eta_c^2\eta_e^2 - 2T_H\varepsilon_L\varepsilon_H^2\eta_c^2\eta_e^2 - 6T_H\eta_c\varepsilon_L x\varepsilon_H^2\eta_e^2 + 9\varepsilon_L T_H\eta_c\varepsilon_H\eta_e}{\eta_e x\varepsilon_L^2}$$

$$+\frac{9\varepsilon_L T_H\eta_c\varepsilon_H\eta_e + 2\varepsilon_H^2\eta_e^2\varepsilon_L^2\eta_c^2 T_H + 2\varepsilon_H^2\eta_e^2 T_H\eta_c\varepsilon_L + 6\varepsilon_H^2\eta_c^2 T_H\varepsilon_L\eta_e + 6\eta_e^2\varepsilon_H T_H\eta_c\varepsilon_L x + 2T_H\eta_c\varepsilon_H x - 5\varepsilon_H T_H\eta_c^2\varepsilon_L\eta_e}{\eta_e x\varepsilon_L^2}$$

$$+\frac{-3\varepsilon_H T_H\eta_e\varepsilon_L\eta_c^2 - 3\varepsilon_H T_H\eta_c\varepsilon_L x + 4T_L\eta_c\eta_e\varepsilon_L\varepsilon_H - 4T_H\eta_c\eta_e^2 x\varepsilon_H + 4\eta_e^2\varepsilon_H^2 T_H\eta_c x - 12\varepsilon_H^2 T_H\eta_c\eta_e\varepsilon + 2\eta_e^2\varepsilon_L^2\varepsilon_H T_H\eta_c}{\eta_e x\varepsilon_L^2}$$

$$+\frac{2\eta_c^2\varepsilon_L\varepsilon_H T_L\eta_e - 2T_L\varepsilon_H\eta_c\varepsilon_L^2\eta_e + 2\varepsilon_H T_H\eta_c x\eta_e - 4\varepsilon_H T_H\varepsilon_L\eta_e - 4\eta_e^2\varepsilon_H T_H\varepsilon_L x + 2\eta_c^2\varepsilon_L^2\varepsilon_H T_H\eta_e + 2\eta_c^2\varepsilon_H T_H x\eta_e}{\eta_e x\varepsilon_L^2}$$

$$+\frac{\eta_c^2\varepsilon_L^2\varepsilon_H T_L\eta_e - 4\varepsilon_L^2\varepsilon_H T_H\eta_e\eta_c - 4\varepsilon_H T_H\eta_e\eta_c - 4\eta_c^2\varepsilon_H^2 T_H\eta_e - 4\varepsilon_H^2 T_H\eta_e - \varepsilon_L^2\varepsilon_H T_L\eta_e x - 2\varepsilon_L^2\varepsilon_H T_H\eta_e x + 2\varepsilon_L^2\varepsilon_H^2 T_H\eta_e x}{\eta_e x\varepsilon_L^2}$$

$$+\frac{-\varepsilon_L^2\eta_c T_L\eta_e x-2\varepsilon_L T_L\eta_e x-\varepsilon_L^2 T_L\eta_e-x\varepsilon_H\varepsilon_L^2 T_H\eta_c^2+3x\varepsilon_H\varepsilon_L T_H\eta_c^2+2x\varepsilon_H\varepsilon_L^2 T_H\eta_e\eta_c+\varepsilon_L^2 T_L\eta_e x\eta_c\varepsilon_H+2\varepsilon_H^2\varepsilon_L^2 T_H\eta_e^2 x\eta_c}{\eta_e x\varepsilon_L^2}$$

$$+\frac{-2\varepsilon_H\varepsilon_L^2 T_H\eta_e^2 x\eta_c-2\varepsilon_H^2\varepsilon_L^2 T_H\eta_e x\eta_c+\varepsilon_H T_H\eta_e x\eta_c^2\varepsilon_L^2-4\varepsilon_H T_H x\eta_e+2\varepsilon_H\varepsilon_L^2 T_H\eta_e+4\varepsilon_H^2 T_H x\eta_e+2T_L\eta_c\varepsilon_L^2\eta_e+T_L\varepsilon_H\varepsilon_L^2\eta_e}{\eta_e x\varepsilon_L^2}$$

$$+\frac{-T_L\eta_c\varepsilon_L^2 x+2T_L\varepsilon_H\eta_e-\eta_e T_L\eta_c^2\varepsilon_L^2+2T_L\varepsilon_H\varepsilon_L x\eta_e-6\varepsilon_H^2 T_H\varepsilon_L x\eta_e+6\varepsilon_L\varepsilon_H T_H x\eta_e-2\varepsilon_L^2\varepsilon_H^2 T_H\eta_e+2T_L\eta_e\eta_c x\varepsilon_L}{\eta_e x\varepsilon_L^2}$$

$$b_3=\frac{4\eta_e\varepsilon_H T_H+2\eta_c\varepsilon_H T_H+4\eta_e\eta_c\varepsilon_H^2 T_H+2\varepsilon_L^2 T_H-2\varepsilon_H T_H-4\varepsilon_H^2\varepsilon_L T_H-2\varepsilon_L^2\varepsilon_H T_L+2\varepsilon_H^2\varepsilon_L^2 T_H+4\varepsilon_L\varepsilon_H T_H+2\varepsilon_L\varepsilon_H T_L}{\eta_e\varepsilon_L^2}$$

$$+\frac{-2\varepsilon_H\varepsilon_L^2 T_H-2T_L\eta_e\eta_c\varepsilon_H-2T_L\eta_e\varepsilon_H\varepsilon_L+4\varepsilon_H^2\varepsilon_L^2 T_H\eta_e\eta_c+6\varepsilon_H^2\varepsilon_L T_H\eta_e-4\eta_e^2 T_H\eta_c\varepsilon_H\varepsilon_L-2\eta_e^2\varepsilon_H^2\varepsilon_L^2 T_H\eta_c}{\eta_e\varepsilon_L^2}$$

$$+\frac{9\varepsilon_H\varepsilon_L\eta_e\eta_c T_H+4\eta_e^2\varepsilon_H^2\varepsilon_L\eta_c T_H+2T_L\varepsilon_H\varepsilon_L\eta_e\eta_c-8\varepsilon_H^2\varepsilon_L T_H\eta_e\eta_c+2\eta_e^2\varepsilon_L^2\varepsilon_H\eta_c T_H-2T_L\varepsilon_H\varepsilon_L^2\eta_e\eta_c-6\varepsilon_H\varepsilon_L T_H\eta_e}{\eta_e\varepsilon_L^2}$$

$$+\frac{-5\varepsilon_L^2\varepsilon_H\eta_c\eta_e T_H-3T_H\eta_c\varepsilon_L^2-2\varepsilon_L T_H-2\eta_c\varepsilon_H\varepsilon_L T_L+2\eta_c T_L\varepsilon_L^2\varepsilon_H-2\varepsilon_H^2\varepsilon_L^2\eta_c T_H+4\eta_c\varepsilon_H^2\varepsilon_L T_H-4\eta_c\eta_e\varepsilon_H T_H-4\eta_e T_H\varepsilon_H^2}{\eta_e\varepsilon_L^2}$$

$$+\frac{-2\eta_c T_H\varepsilon_H^2-T_L\varepsilon_L^2\eta_e-5\varepsilon_H\varepsilon_L\eta_c T_H+2\eta_e^2\eta_c T_H\varepsilon_H-2\eta_e^2\varepsilon_H^2\eta_c T_H+2\eta_e\varepsilon_H T_H\varepsilon_L^2+2\eta_c\eta_e T_L\varepsilon_L^2+\eta_e\varepsilon_H T_L\varepsilon_L^2+2T_L\varepsilon_L\eta_e}{\eta_e\varepsilon_L^2}$$

$$+\frac{-2\varepsilon_H^2\varepsilon_L^2\eta_e T_H-2\eta_e^2\varepsilon_H^2 T_H\varepsilon_L+3\varepsilon_H T_H\eta_c\varepsilon_L^2-2\eta_e^2\varepsilon_H T_H+2\eta_e^2\varepsilon_H^2 T_H+2\varepsilon_L T_L\eta_c+2\eta_e^2\varepsilon_H\varepsilon_L T_H+2\varepsilon_H^2 T_H}{\eta_e\varepsilon_L^2}$$

$$c_1=\frac{2(\varepsilon_H-1)T_H^2\varepsilon_H^2\left[\left(x-\frac{1}{2}\right)\eta_c-\frac{3}{2}x+\frac{1}{2}\right](\eta_c-1)y\eta_e}{x^2\varepsilon_L^2}$$

$$c_2=-\frac{\varepsilon_H T_H}{x\varepsilon_L^2}\left(-2\varepsilon_H T_H-3\eta_e\eta_c\varepsilon_H\varepsilon_L xT_H-2\eta_c\varepsilon_L xT_L+\eta_e\eta_c\varepsilon_H\varepsilon_L^2 xT_H+\eta_c\varepsilon_L^2 xT_L+2\eta_e\eta_c\varepsilon_H xT_H-2\eta_c\varepsilon_L^2 T_L\right)$$

$$-\frac{\varepsilon_H T_H}{x\varepsilon_L^2}\left(\eta_c^2\varepsilon_L^2 T_H\varepsilon_H\eta_e-\varepsilon_L^2 T_H\varepsilon_H\eta_e\eta_c+\eta_c^2\varepsilon_L^2 T_L+2\eta_e\varepsilon_H\varepsilon_L xT_H+2\varepsilon_L xT_L+3\varepsilon_H\varepsilon_L\eta_c xT_H-\varepsilon_L^2 T_H\varepsilon_H\eta_c x-T_L\varepsilon_L^2 x\right)$$

$$-\frac{\varepsilon_H T_H}{x\varepsilon_L^2}\left(+4T_L\eta_c\varepsilon_L-\eta_c^2\varepsilon_L T_H\varepsilon_H\eta_e+\eta_c\varepsilon_L T_H\varepsilon_H\eta_e-2T_L\eta_c^2\varepsilon_L-2T_L\varepsilon_L-6\varepsilon_L T_H\varepsilon_H\eta_c+3\varepsilon_L T_H\varepsilon_H\eta_c^2-2\eta_e\varepsilon_H T_H x\right)$$

$$-\frac{\varepsilon_H T_H}{x\varepsilon_L^2}\left(+T_L\varepsilon_H^2-2\varepsilon_H T_H\eta_c x-\varepsilon_H T_H\eta_c^2\varepsilon_L^2+2\varepsilon_H\varepsilon_L^2 T_H\eta_c-3\varepsilon_H\varepsilon_L xT_H+\varepsilon_H\varepsilon_L^2 xT_H+4\varepsilon_H T_H\eta_c-2\varepsilon_H T_H\eta_c^2\right)$$

$$-\frac{\varepsilon_H T_H}{x\varepsilon_L^2}\left(-2\varepsilon_H T_H\varepsilon_L-\varepsilon_H T_H\varepsilon_L^2+2\varepsilon_H T_H x\right)$$

$$c_3=\frac{\{\varepsilon_L T_L(\eta_c-1)+[(\eta_e\eta_c+1-\eta_c)\varepsilon_L-(\eta_e-1)(\eta_c-1)]\varepsilon_H T_H\}[\varepsilon_H(\eta_e-1)(\varepsilon_H-1)T_H+T_L\varepsilon_L]}{\eta_e\varepsilon_L^2}$$