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ALGEBRAIC DERIVATIVE ESTIMATION AND APPLICATIONS IN ADAPTIVE CONTROL OF MAGNETIC LEVITATION SYSTEMS

Matheus Schwalb Moraes
Paulo Sergio Pereira da Silva
University of São Paulo
matmoraes@usp.br, paulo@lac.usp.br

Abstract. *The algebraic derivative estimation method is used to provide a fast online state estimation in magnetic levitation control. Since magnetic levitation plants are highly nonlinear and unstable, nonlinear adaptive control is used to handle uncertainties on physical parameters. The state estimation is used together with two adaptive control schemes (filtered Least Squares parameter estimation and Lyapunov-based adaptive control law). The proposed approaches aim the improvement of tracking performance, which is the main concern in the early stage of levitation, where a undesired overshoot can stick the magnet with the levitated object. Experimental results were obtained in a laboratory prototype showing the excellent performance of the proposed adaptive control schemes when combined with the algebraic estimator. The two control approaches are compared and some properties of the algebraic estimation method in closed loop with a feedback law are discussed.*

Keywords: *derivative estimation, adaptive control, nonlinear control, parameter estimation, magnetic levitation...*

1. INTRODUCTION

The control of magnetically suspended objects has received a great attention in the recent years, mainly due to advances on the magnetic bearing technology. The advantages of these bearings compared to traditional solutions are: absence of mechanical wear and friction, active vibration control, unbalance compensation, lubricant-free operation, etc. Typically, proportional-integral-differential (PID) controllers are used for the regulation of the system on a nominal levitation gap, under the assumption of a perfectly known gain parameter.

In the classical approach, the uncertainty of the input gain is compensated via integral action in the control loop, causing undesired overshoots when the initial gap is much bigger than the nominal gap, since the PID control is usually projected for the nominal gap. This work provides experimental results of adaptive control using the algebraic derivative estimation technique for a fast state estimation, aiming the tracking performance of the system through the whole process of levitation.

2. ALGEBRAIC DERIVATIVE ESTIMATION

The algebraic derivative estimation method is based on the identification of the coefficients of an arbitrary order truncated Taylor expansion. For this purpose, we assume that the signal $\tilde{y}(t)$ is an analytic function and can be approximated by a truncated Taylor expansion calculated on $t = 0$:

$$\tilde{y}(t) \simeq y(t) = \sum_{j=1}^{k} \frac{f^{(n)}(0)}{n!} t^n \tag{1}$$

The algebraic identification procedure was first presented in Fliess and Sira-Ramirez (2004) and consists in several algebraic manipulations on the operational domain, which has the same properties of the Laplace transform. Finally, the resulting expression is transformed back to time domain. In our paper, the estimation filters will be discretized as sliding-window FIR filters, which can be simply implemented on real-time applications.

To illustrate the method, the identification procedure will be shown for the first order derivative estimation, using a second order truncated Taylor expansion. A general formulation for arbitrary order derivative estimators can be found in Mboup *et al.* (2007), Zehetner *et al.* (2007) and Mboup *et al.* (2009).

2.1 First Order Derivative Estimator

Consider the second order approximation of the noisy signal $\tilde{y}(t)$ arround $t = 0$:

$$y(t) = a_0 + a_1 t + a_2 \frac{t^2}{2} \tag{2}$$

Applying the Laplace Transform:

$$Y(s) = \frac{a_0}{s} + \frac{a_1}{s^2} + \frac{a_2}{s^3} \tag{3}$$

Multiplying by s^3 to isolate a_2:

$$s^3 Y(s) = s^2 a_0 + s a_1 + a_2 \tag{4}$$

Differentiating with respect to s, in order to eliminate a_2:

$$3s^2 Y(s) + s^3 \frac{d}{ds} Y(s) = 2s a_0 + a_1 \tag{5}$$

Multiplying by $\frac{1}{s}$, isolating a_0:

$$3sY(s) + s^2 \frac{d}{ds} Y(s) = 2a_0 + \frac{a_1}{s} \tag{6}$$

This way, a_0 can be eliminated multiplying both sides of the equality by the differential operator $\frac{d}{ds}$:

$$3Y(s) + 5s \frac{d}{ds} Y(s) + s^2 \frac{d^2}{ds^2} Y(s) = -a_1 \frac{1}{s^2} \tag{7}$$

Multiplying both sides by $\frac{1}{s^3}$ for the elimination of time derivatives and filtering, letting every factor involving $Y(s)$ be integrated at least one time[1]:

$$3\frac{1}{s^3} Y(s) + 5\frac{1}{s^2}\frac{d}{ds} Y(s) + \frac{1}{s}\frac{d^2}{ds^2} Y(s) = -a_1 \frac{1}{s^5} \tag{8}$$

The operations from Eq. (4) to Eq. (7) can be resumed to the application of the algebraic operator $\frac{1}{s^3}\frac{d}{ds}\frac{1}{s}\frac{d}{ds}s^3$ to Eq. (3). This algebraic operator is called an "annihilator" for a_1.

The right part of Eq. (8) is transformed to time domain using the relation:

$$\frac{c}{s^\alpha} \;,\; \alpha \geq 1 \;,\; c \in \mathbb{C} \;\longleftrightarrow\; c\frac{t^{\alpha-1}}{(\alpha-1)!} \tag{9}$$

Remembering that the differentiation of $Y(s)$ with respect to s is related to a multiplication by $-t$ in time domain and left multiplying $Y(s)$ by $\frac{1}{s^\alpha}$, $\alpha \geq 1$ corresponds to iterated integrals, the Cauchy formula for repeated integration is used to provide a closed-form expression for the operator $\frac{1}{s^\alpha}\frac{d^n}{ds^n}$ in a single integral:

$$\frac{1}{s^\alpha}\frac{d^n}{ds^n} Y(s) \;\longleftrightarrow\; \frac{(-1)^n}{(\alpha-1)!}\int_0^t (t-\tau)^{\alpha-1}\tau^n y(\tau)\,\mathrm{d}\tau \tag{10}$$

Transforming Eq.(8) back to time domain using Eq.(9) and Eq.(10):

$$\frac{3}{2}\int_0^t (t-\tau)^2 y(\tau)\,\mathrm{d}\tau - 5\int_0^t (t-\tau)\tau y(\tau)\,\mathrm{d}\tau + \int_0^t \tau^2 y(\tau)\,\mathrm{d}\tau = -a_1\frac{t^4}{4!} \tag{11}$$

Rearranging the expression, the estimation of a_1 is obtained:

$$a_1 = \dot{y}(t) = -\frac{12}{t^4}\int_0^t (3t^2 - 16t\tau + 15\tau^2) y(\tau)\,\mathrm{d}\tau \tag{12}$$

The first order derivative estimation is obtained substituting $y(\tau)$ by the noisy observation $\tilde{y}(\tau)$ in the above expression.

[1] In this context, the iterated integrals are low-pass filters which attenuate noises.

2.2 Second Order Derivative Estimator

Applying the annihilator $\frac{1}{s^3}\frac{d^2}{ds^2}s^2$ on the second order approximation of the signal Eq. (3) and transforming back to time domain:

$$a_2 = \ddot{y}(t) = \frac{60}{t^5}\int_0^t (t^2 - 6t\tau + 6\tau^2)y(\tau)\,\mathrm{d}\tau \tag{13}$$

The second order derivative estimation is obtained substituting $y(\tau)$ by the noisy observation $\tilde{y}(\tau)$ in the above expression. A step-by-step formulation for this estimator can be found in Moraes (2015).

3. DISCRETE IMPLEMENTATION OF THE ALGEBRAIC DERIVATIVE ESTIMATORS

In the general case, the j-order derivative estimator when all factors involving $Y(s)$ are integrated at least one time can be summarized as:

$$a_j = \int_0^t P(t,\tau)y(\tau)\,\mathrm{d}\tau \tag{14}$$

The above expression provides an estimator for the parameter a_j that holds only in the vicinity of $t = 0$, since the estimation is based on the Taylor series expansion around $t = 0$. In a practical implementation, one is usually interested in estimations for $t \gg 0$. This fact motivates an adaptation of the method, integrating backwards in a small sliding window T to provide a causal derivative estimation at an arbitrary time instant t. After some changes of variables:

$$a_j = y^{(j)}(t) = (-1)^j \int_0^T P(T,\tau)y(t-\tau)\,\mathrm{d}\tau \tag{15}$$

This expression can be digitally implemented using the trapezoidal integration method, resulting in a digital FIR filter. For this purpose, we consider $T = T_sN$, where T is the estimation window, $(N+1)$ is the number of samples in the window and T_s is a fixed sample period:

$$a_j = y^{(j)}(t) \simeq (-1)^j \sum_{k=1}^{N+1} \alpha_k P(T,\tau_k)y(t-\tau_k) \tag{16}$$

To reduce the computational effort, the discretization of $P(T,\tau_k)$ is computed a priori. At the same time, we take into account the coefficients α_k derived from the trapezoidal integration:

$$\begin{aligned}
\hat{P}_{\alpha_k}(T,\tau_k) &= \begin{bmatrix}\alpha_1 P(T,\tau_1) & \alpha_2 P(T,\tau_2) & \dots & \alpha_{N+1}P(T,\tau_{N+1})\end{bmatrix}\\
\tau_k &= (k-1)T_s\\
\alpha_k &= \frac{T_s}{2}\,,\; k=1 \;or\; k=N+1\\
\alpha_k &= T_s\,,\; k=2,3,...,N
\end{aligned} \tag{17}$$

The buffer vector $\hat{y}(t-\tau_k)$ containing the last $N+1$ samples is vertically arranged. At each sample clock, the hole vector is shifted. The last value is discarded and the top position receives the newest value $y(t)$:

$$\hat{y}(t-\tau_k) = \begin{bmatrix} y(t) \\ y(t-T_s) \\ \vdots \\ y(t-T_sN) \end{bmatrix} \tag{18}$$

Finally, $\hat{P}_{\alpha_k}(T,\tau_k)$ and $\hat{y}(t-\tau_k)$ are multiplied at each sample time, providing a real time estimation of a_j:

$$a_j(t) = y^{(j)}(t) \simeq (-1)^j \sum_{k=1}^{N+1} \alpha_k P(T,\tau_k)y(t-\tau_k) = (-1)^j \hat{P}_{\alpha_k}(T,\tau_k)\hat{y}(t-\tau_k) \tag{19}$$

It's important to notice that the choice of the estimation window T establishes a trade off between noise immunity and precision. While smaller values of T account for a fast filter response and a better "local modeling" of the signal by Eq. (1), bigger values improve noise immunity with better high-frequency filtering properties.

At the same time that these filters can be easily implemented, the computing complexity grows linearly with the number of samples, since $N+1$ multiplications and N sums are required at each sample time to perform the estimation. Considering that nowadays we have access to fast and cheap microcontrollers, the computational effort is not a limiting factor for the implementation.

4. SYSTEM DYNAMICS

Consider the following dynamic model of the magnetic levitation system shown in Fig. 1, where F_{mag} is de magnetic force:

$$m\ddot{x} = mg - F_{mag} \tag{20}$$

$$F_{mag} = \frac{K_{mag} i^2}{x^2} \tag{21}$$

m is the levitated mass, g denotes the gravity acceleration, x is the levitation gap, i is the eletric current through the coil and K_{mag} is the magnetic parameter of the system.

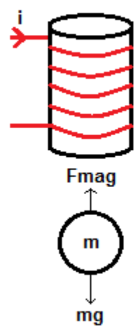

Figure 1. Magnetic Levitation System

5. TRACKING CONTROL LAW FOR THE NOMINAL SYSTEM

Consider the following change of variables:

$$\theta = -\frac{m}{K_{mag}} \qquad u = \frac{i^2}{x^2} \tag{22}$$

The system dynamics can be written:

$$\theta(\ddot{x} - g) = u \tag{23}$$

If the parameter θ is exactly known, one can consider the following tracking control law:

$$\begin{aligned} u &= \theta(v - g) \\ v &= \ddot{x}_d - \lambda^2 \tilde{x} - 2\lambda\dot{\tilde{x}} \end{aligned} \tag{24}$$

x_d is the desired trajectory, $\tilde{x} = (x - x_d)$ is the tracking error and $\lambda \in \mathbb{R}^+$. While u can be considered the linearization control, v is the stabilization part related to pole placement. The resulting asymptotically stable linear error dynamics is shown below, leading $\tilde{x} \to 0$. This formulation results in two negative closed-loop poles equal to $-\lambda$.

$$\ddot{\tilde{x}} + 2\lambda\dot{\tilde{x}} + \lambda^2\tilde{x} = 0 \tag{25}$$

The nominal control is projected for a perfectly known input gain θ. In practice, this parameter has a uncertainty margin and is slowly time-varying. This issue causes an imperfect linearization of the system, leading to poor tracking performance or even the instability of the closed loop system.

6. LYAPUNOV-BASED ADAPTIVE CONTROL

Instead of using the unknown parameter θ, use an initial estimation $\hat{\theta}$ of the input gain in the nominal control law of Eq. (24). This choice results in the following dynamics, where $w = (v - g)$:

$$\theta(\ddot{x} - g) = \hat{\theta}(v - g) = \hat{\theta}w \tag{26}$$

Subtracting θw and considering that $\tilde{\theta} = (\hat{\theta} - \theta)$ is the estimation error:

$$\theta(\ddot{x} - g - w) = \tilde{\theta}w \tag{27}$$

Performing the change of variables:

$$\psi = \dot{\tilde{x}} + \lambda\tilde{x} \tag{28}$$

$$(\ddot{x} - g - w) = (\ddot{x} - g - \ddot{x}_d + g + 2\lambda\dot{\tilde{x}} + \lambda^2\tilde{x}) = \dot{\psi} + \lambda\psi \tag{29}$$

Which leads:

$$\theta(\dot{\psi} + \lambda\psi) = \tilde{\theta}w \tag{30}$$

Regarding that θ is a negative constant, one can consider the following positive definite Lyapunov function:

$$V = \frac{1}{2}\left(-\theta\psi^2 + \frac{1}{\gamma}\tilde{\theta}^2\right) \tag{31}$$

$$\dot{V} = -\theta\psi\dot{\psi} + \frac{1}{\gamma}\tilde{\theta}\dot{\hat{\theta}} \;=\; -\psi(\tilde{\theta}w - \psi\theta\lambda) + \frac{1}{\gamma}\tilde{\theta}\dot{\hat{\theta}} \;=\; \psi^2\theta\lambda + \tilde{\theta}\left(-\psi w + \frac{1}{\gamma}\dot{\hat{\theta}}\right) \tag{32}$$

Considering the following parameter update law:

$$\dot{\hat{\theta}} = \gamma\psi w \tag{33}$$

Substituting Eq. (33) in Eq. (32) is easy to see that $\dot{V} = \psi^2\theta\lambda \leq 0$. It can be shown that $\dot{V} \to 0$. Thus, $\psi \to 0$ which is equivalent of $\dot{\tilde{x}} \to -\lambda\tilde{x}$. This way, the closed loop system tends to a first order stable dynamics, resulting in $\tilde{x} \to 0$ (tracking error converges to zero).

Since the magnetic levitation system is very unstable, a high gain is needed to guarantee a fast parameter convergence and minimize the tracking error overshoot. After some point, increasing the gain is no longer interesting because of noise amplification.

7. LEAST SQUARES ADAPTIVE CONTROL

First, consider the nominal system dynamic:

$$\theta(\ddot{x} - g) = u \tag{34}$$

The acceleration error can be written:

$$e = \ddot{x} - \left(g + \frac{u}{\theta}\right) = \ddot{x} - g + \beta u \tag{35}$$

Where $\beta = -\frac{1}{\theta}$. Performing the minimization of the quadratic error over a sliding time window $[t - h_e, t]$:

$$J = \int_{t-h_e}^{t} e^2 \,\mathrm{d}t = \int_{t-h_e}^{t} \left[(\ddot{x}-g)^2 + 2(\ddot{x}-g)\beta u + \beta^2 u^2\right] \,\mathrm{d}t \tag{36}$$

$$\frac{\partial J}{\partial \beta} = \int_{t-h_e}^{t} \left[2(\ddot{x}-g)u + 2\beta u^2\right] \,\mathrm{d}t = 0 \tag{37}$$

$$-\int_{t-h_e}^{t} (\ddot{x}-g)u \,\mathrm{d}t = \int_{t-h_e}^{t} \beta u^2 \,\mathrm{d}t \tag{38}$$

Considering that β is constant in the small time window $[t-h_e, t]$:

$$\hat{\beta} = \frac{\int_{t-h_e}^{t} (g-\ddot{x})u \,\mathrm{d}t}{\int_{t-h_e}^{t} u^2 \,\mathrm{d}t} \tag{39}$$

The above expression provides a fast parameter estimation when $\int_{t-h_e}^{t} u^2 \,\mathrm{d}t \neq 0$. To initialize the levitation, an initial magnetic parameter estimation K_{mag_0} is provided to the controller. This estimation is used in the feedback control law until least squares estimation (LSE) becomes non-singular. After this point, the least squares estimation (LSE) is used for feedback providing an online estimation $\hat{\theta} = -\frac{1}{\hat{\beta}}$ of the real parameter θ in the nominal tracking control law of Eq. (24).

Since u and x are noisy signals, the numerator and denominator of Eq. (39) are simultaneously filtered by some low pass filter. This filtering takes advantage of the rational form of the LSE, since the quotient will not be affected by the filters. For this purpose, a simple first order low pass filter $Q(s) = \frac{1}{0.05s+1}$ is used in this work. On the following notation, $\xi_f(t) = Q(s)\xi(t)$ where $\xi_f(t)$ is defined as the output of $Q(s)$ when an input $\xi(t)$ is applied to the filter. Hence the estimation of β may be rewritten as:

$$\hat{\beta}_f = \frac{Q(s) \int_{t-h_e}^{t} (g-\ddot{x})u \,\mathrm{d}t}{Q(s) \int_{t-h_e}^{t} u^2 \,\mathrm{d}t} \tag{40}$$

8. EXPERIMENTAL SETUP AND RESULTS

Figure 2. Coil + Levitated Mass

The experimental setup used for control tests include a National Instruments PCI-6259 acquisition board working together with Matlab Realtime + Simulink Model. The power electronics receive the control signal and generate the electrical current in the coil, which is rolled on a ferromagnetic core. The levitation gap is measured through a optic sensor and the measure is sent back to the acquisition board.

All the experiments were realized with a 10kHz sample frequency, an initial magnetic parameter estimation $K_{mag_0} = 1.6 \times 10^{-4} \frac{Nm^2}{A^2}$, control poles -40 defined by $\lambda = 40$, levitated mass $m = 0.240Kg$ and gravity acceleration $g = 9.81 \frac{m}{s^2}$. In the Lyapunov-based adaptive control the adaptive gain used is $\gamma = 8 \times 10^4$. In the least squares approach, the estimation window is $h_e = 0.08s$. All the first and second order derivatives were provided by the estimators presented in Section 2, with $T = 0.01$ for first derivative estimation and $T = 0.02$ for second derivative estimation. The desired trajectory x_d

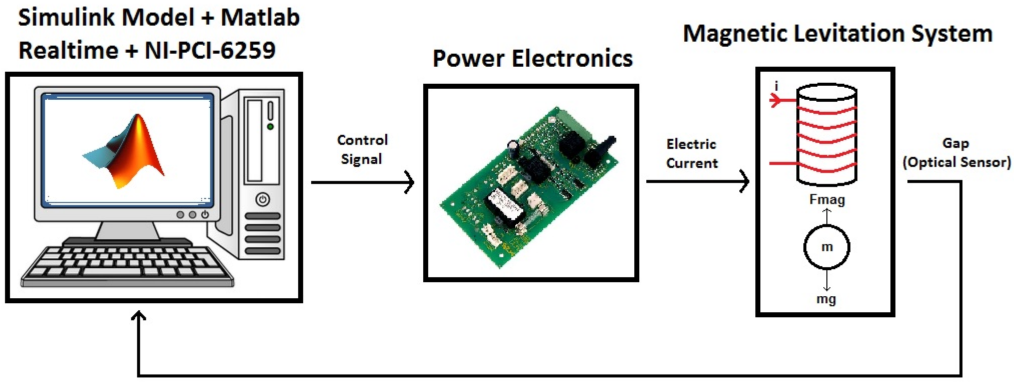

Figure 3. Experimental Setup Diagram

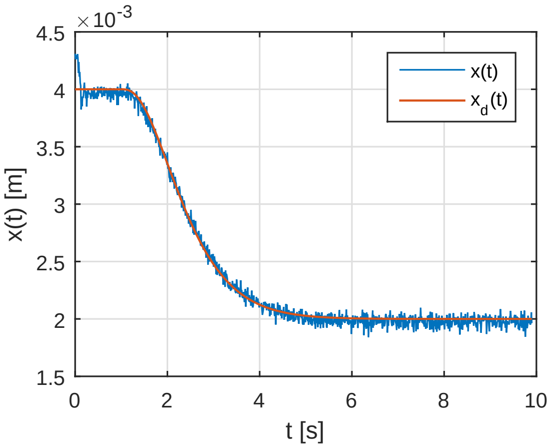

Figure 4. Lyapunov Adaptive: Levitation Gap

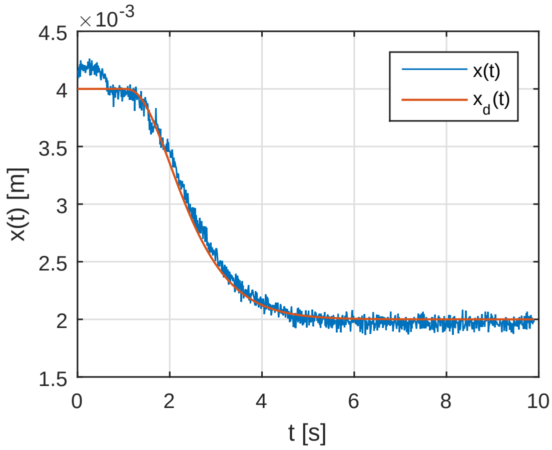

Figure 5. Least Squares: Levitation Gap

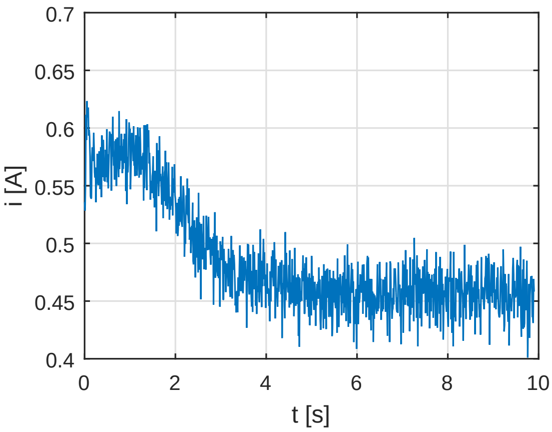

Figure 6. Lyapunov Adaptive: Control Effort

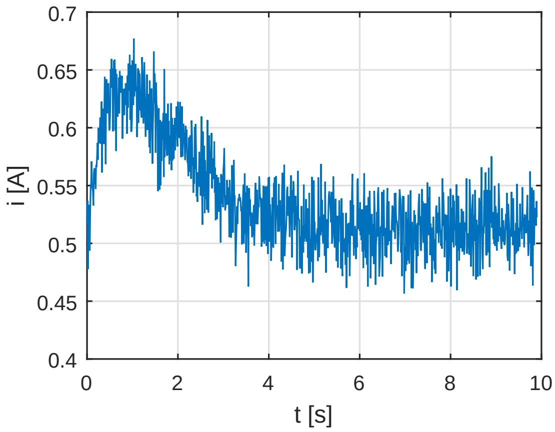

Figure 7. Least Squares: Control Effort

was generated by a step on $t = 1s$ filtered by the third order filter $\frac{8}{(s+2)^3}$ to ensure that the desired trajectory is of class $\mathbf{C}^2$.

Both control schemes presented excellent performance while the Lyapunov-Based control showed a slightly better response during the levitation transient. The magnetic parameters converge to values near to each other. This difference can be explained by the temperature dependence of the magnetic parameter, since the experiments were not made at the same day and the coil becomes warm when several experiments are made in a short period of time. This different K_{mag} value also affects the current, which has higher values for lower values of K_{mag}.

It is interesting to notice that the least squares adaptive control relies totally on the identification of the magnetic parameter to provide a good tracking. A nominal feedback law including an integral action could be used to improve robustness and tracking performance. This approach was not included in this article, in order to show that LSE alone can

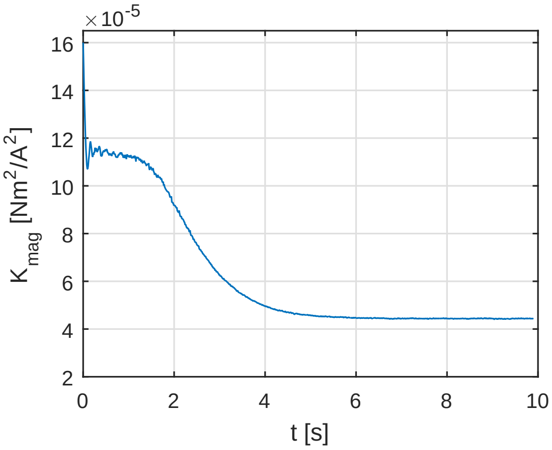

Figure 8. Lyapunov Adaptive: Magnetic Parameter

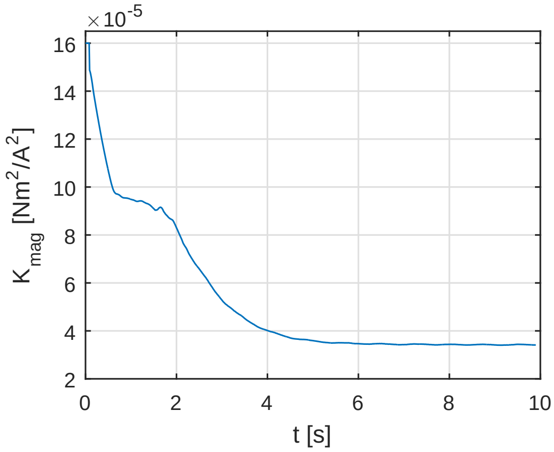

Figure 9. Least Squares: Magnetic Parameter

provide good estimation and tracking.

It is a hard task to do a honest comparison between Lyapunov-based and LSE control due to different tuning parameters of each scheme. While the Lyapunov-Based uses the adaptive gain γ, the LSE has the estimation window h_e and also uses the second derivative estimation in a window T.

The Lyapunov-based control combines a specific control law together with a specific adaptation law towards the elimination of the tracking error. On the other side, the least squares adaptive control relies on the direct estimation of the magnetic parameter. Since this estimation is independent of the control law, it can be used together with different control schemes. The major drawback is the need for the estimation of $\ddot{x}$, which gives an advantage to Lyapunov-based control since it uses only the estimation of $\dot{x}$.

9. CONCLUSION

The experimental results show an excellent tracking performance of the proposed control approaches. At the same time, the algebraic derivative estimator proved to be a feasible alternative to the classic state observers, providing a fast and reliable estimation with a good closed loop performance without the need of a model-based observer. The generic formulation for the discrete-time estimators allows a straightforward implementation together with several control techniques.

The use of the algebraic derivative estimator as a state observer has also been successfully tested with other control approaches, such as feedback linearization and sliding mode control. Due to lack of space, these controllers were not presented in this paper.

10. ACKNOWLEDGEMENTS

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