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# ANALYTICAL ANALYSIS OF A PARAMETRIC WIRE STRAND MODEL SUBJECTED TO TENSILE LOAD

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***Abstract.*** This paper proposes an analytical approach to determine the mechanical response of cables, under tensile load, using the spatial beam theory. Several authors have focused on solving this problem using numerical tools. However, there is a lack of analytical solutions that could be used as benchmarks to check the numerical models or optimize the cable layout before starting complex numerical analysis. In the present work, the parametric spatial description were derived for one wire strands, considering cable surrounding a core, since the response of all wires are similar. The main goal is to solve the differential equilibrium equations of the spatial beam theory to represent the response of the wire. Two cases were analyzed (helix angle equals to 0 and 75º), obtaining good results when compared to the FEM model.
***Keywords:*** *Wire, Spatial Beam Theory, Differential Geometry, Parameterization.*

## 1. INTRODUCTION

Cables are structures that can resist a great force, especially tensions, having high strength-to-weight ratio (Stanova *et al*., 2011). These components are used in several civil engineering applications, such as hangers for bridges, mooring ships and offshore platforms (Ghroreishi *et al*., 2006). The simplest geometry is the 1+6, were there is one core surrounded for six other helical wires, or strands (Feyrer, 2007). Here one of those strands is analyzed, since the response of one wire can be simulated and used to represent the whole cable.

Cables have been the focus of many studies. Nowadays, most of them use the finite elements method or experimental procedures to evaluate such structures. Although good results are being obtained with these approaches, there are few expressions to compare and validate the results. Also, experiments are limited in the literature, restricting the cases that can be compared. Using FEM is not only expensive, but demands time and involves a high complexity. Experimental works have similar problems.

The theories available to be used generally are based on some assumption. The most known theories are proposed by Costelo, 1990, Jolicour and Cardou, (1996), and Machida and Durelli, (1973). Experimental results as Utting and Jones, (1987), can be found in the literature as well as FEM models developed by Nawrocki and Labrosse, (2000).

In order to overcome the lack of analytical expressions to be used as benchmark, the equilibrium equations of (Love, 1944) were solved. The corresponding boundary conditions were applied and two cases were discussed. The forces and moments diagrams were validated with the numerical model.

## 2. PARAMETRIZATION

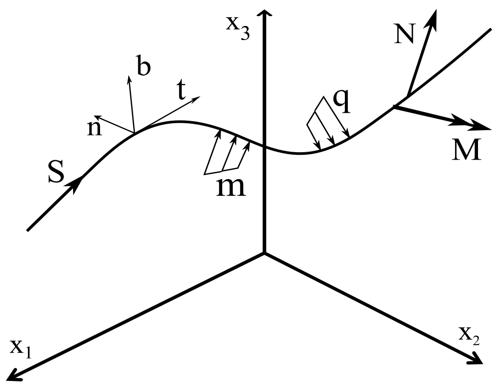


Figure 1. Space curve with the triad and the loads applied

Considering a space curve, that can be described through its center line in Fig. 1, the position vector, which depends on the length along (S), can be described by

$$\boldsymbol{r}(S) = x_1(S)\boldsymbol{e_1} + x_2(S)\boldsymbol{e_2} + x_3(S)\boldsymbol{e_3}, \tag{1}$$

where $x_1$, $x_2$ and $x_3$ are the components in each principal direction and ***e*** the direction versor. Based on this description, three basic vectors can be derived considering only the dependence on the length (Østergaard *et al.*, 2011). These components form the so called Frenet-Serret triad. According to Ramsey (1988), the formulas can be written as

$$\frac{d}{dS}\begin{bmatrix}\boldsymbol{t}\\\boldsymbol{n}\\\boldsymbol{b}\end{bmatrix}=\begin{bmatrix}0&\kappa&0\\-\kappa&0&\tau\\0&-\tau&0\end{bmatrix}\begin{bmatrix}\boldsymbol{t}\\\boldsymbol{n}\\\boldsymbol{b}\end{bmatrix}, \tag{2}$$

where ***t***, ***n*** and ***b*** denotes the tangent, normal and binormal vectors, respectively, and $\kappa$ and $\tau$ represents the curvature and torsion of the space curve. It is possible to obtain the first component, ***t***, as the derivative of the position vector. The three components are orthogonal to each other, therefore, the second term, n, is the solution of Eq. (2) and the third as the cross product of the previously terms.

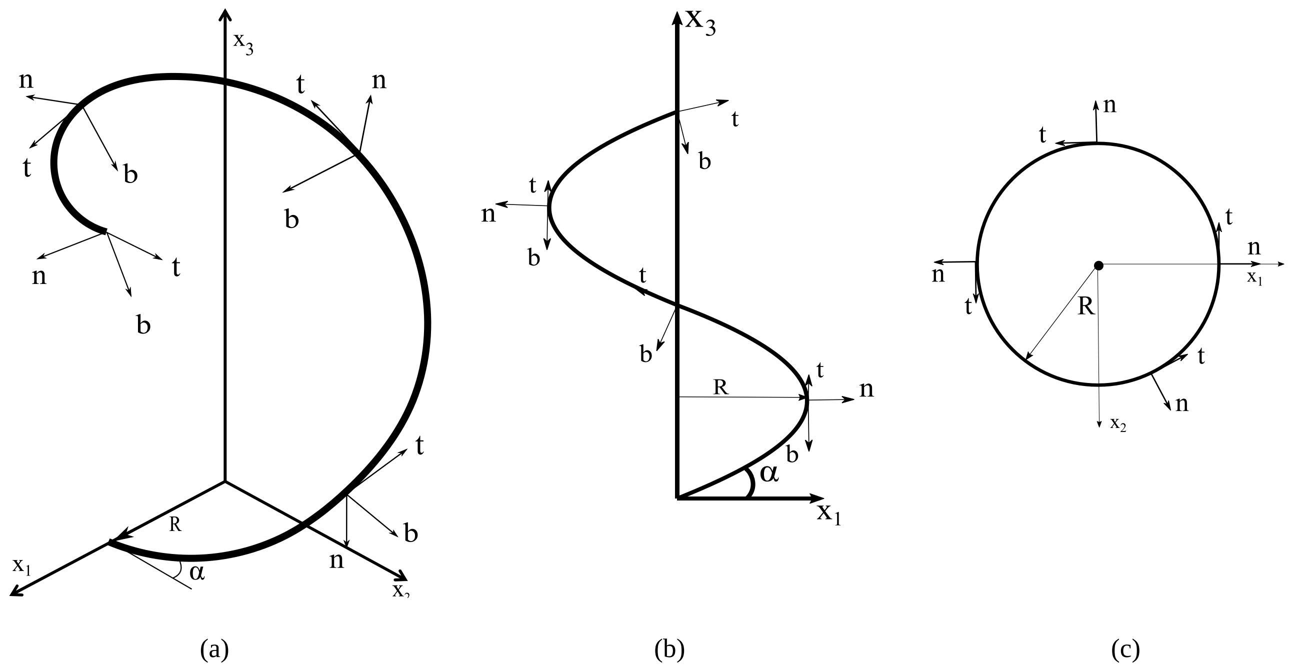


Figure 2. (a) Parameterized helix with the triad, superior (b) and front (c) views.

According to Elata, et al., (2003), the cable can be parameterized as

$$x_1 = Rcos\left(\frac{Scos(\alpha)}{R}\right),$$
$$x_2 = Rsin\left(\frac{Scos(\alpha)}{R}\right), \tag{3}$$
$$x_3 = Ssin(\alpha),$$

where R is the rod radius and α is the helix angle. Using this parameterization, and a symbolic computational system software (Maple®), the helix is defined in Fig. 2, together with the triad. It is represented only one turn of the wire, the rope is a repetition of this The orthogonality is present as indicated by the vectors moving around the curve.

As the components of the position vector are now defined, it is possible to obtain the components of Eq. (2). For this purpose Eq. (3) is used in Eq. (1) and then in Eq. (2).

$$\boldsymbol{t}(S) = \begin{bmatrix} -sin\left(\frac{Scos(\alpha)}{R}\right)cos(\alpha) \\ cos\left(\frac{Scos(\alpha)}{R}\right)cos(\alpha) \\ sin(\alpha) \end{bmatrix},$$

$$\boldsymbol{n}(S) = \begin{bmatrix} \frac{-cos\left(\frac{Scos(\alpha)}{R}\right)cos(\alpha)^2}{R\kappa} \\ \frac{-sin\left(\frac{Scos(\alpha)}{R}\right)cos(\alpha)^2}{R\kappa} \\ 0 \end{bmatrix}, \tag{4}$$

$$\boldsymbol{b}(S) = \begin{bmatrix} \frac{sin\left(\frac{Scos(\alpha)}{R}\right)sin(\alpha)cos(\alpha)^2}{R\kappa} \\ \frac{-cos\left(\frac{Scos(\alpha)}{R}\right)sin(\alpha)cos(\alpha)^2}{R\kappa} \\ \frac{cos(\alpha)^3}{R\kappa} \end{bmatrix},$$

the parameters $\kappa$ (curvature) and $\tau$ (torsion) can also be defined. The first express the rate of change of the tangent as a point moves along the curve, and the other the twist of the same point along $S$ (Kraus, 1967). By the same process that led to Eq. (4), the well know couple is written as

$$\kappa = \frac{-cos^2(\alpha)}{R};$$
$$\tau = \frac{sin(\alpha)cos(\alpha)}{R}. \tag{5}$$

## 3. BEAM THEORY

Considering the loads acting on the generic curved beam, shown in Fig. 1, and the triad, where **N** is a concentrated force, **M** represents a moment, **q** and **m** are distributed force and moment, respectively.

Using the triad to decompose the loads, in order to find the resultants on the local reference system, by the transfer matrix

$$\boldsymbol{T} = \begin{bmatrix} \boldsymbol{t} \\ \boldsymbol{n} \\ \boldsymbol{b} \end{bmatrix}. \tag{6}$$

It is possible to use an infinitesimal portion of the cable to determine the equilibrium from these components. Knowing that the distributed loads must be integrated and that in this small portion of material the differentiation of the force must be zero, using the chain rule one can demonstrate, in terms of components, that Eq. (7) is valid,

$$\frac{dN_1}{ds}t + \frac{dt}{ds}N_1 + \frac{dN_2}{ds}n + \frac{dn}{ds}N_2 + \frac{dN_3}{ds}b + \frac{db}{ds}N_3 + q_1 t + q_2 n + q_3 b = 0. \tag{7}$$

As know, from Eq. (2), the Frenet-Serret formulas can be used into Eq. (7) generating the system of equilibrium equations,

$$\frac{dN_1}{dS} - \kappa N_2 + q_1 = 0,$$
$$\frac{dN_2}{dS} + \kappa N_1 - \tau N_3 + q_2 = 0, \tag{8}$$

$$\frac{dN_3}{dS} + \tau N_2 + q_3 = 0.$$

For the equilibrium of moments the procedure to be adopted is similar, the only difference is the consideration that the forces also produce moments (due to the distance **r**). The moment system is presented as

$$\begin{aligned}
&\frac{dM_1}{dS} - \kappa M_2 + m_1 = 0, \\
&\frac{dM_2}{dS} + \kappa M_1 - \tau M_3 - N_3 + m_2 = 0, \\
&\frac{dM_2}{dS} + \tau M_2 + N_2 + m_3 = 0.
\end{aligned} \tag{9}$$

The spatial beam theory, developed by Love (1944), leads to the same linearized systems, presented here as Eq. (8) and (9), considering the kinematic response of the rod.

## 4. CABLE MODELING

Using the parameterization (Eq. (3)), the 5 parameters of the curve, presented in Eq. (4) and (5), are determined and the wire is defined. According to Costelo (1990) the cable can be considering to be a curved rod. The equilibrium equations of spatial beams (Love, 1944) were solved and the boundary conditions used to determine the constants of integration.

The solution of the system, Eq. (8) and (9), produced results capable of describing the cable behavior. The six components $(N_1, N_2, N_3, M_1, M_2, M_3)$ obtained contain 6 integral constants that must be solved before analyzing the results. As expected it is possible to notice that the forces can be solved independently of the moments, however the opposite is not true, $M_1, M_2, M_3$ cannot be obtained without the first 3 constants.

In order to simplify the model only one wire is studied, due to the similar behavior of the other 5 surrounding cables, so the response of one wire can be used to study the complete cable whit all wires.

. The strand is considered to be clamped and the forces and moments resultant, from this condition, are a function of the length and, moreover, this position determines the final value of the vector position.

Knowing the geometric description of the cable, such as radius and helix angle, all the parameters are defined as numerical values or dependent only of the position S at the curve. Adopting the same procedure used to decompound the loads, the six boundary loads are aligned to the local system. In other words, the cross product between the forces, or moments, and the rotation matrix is performed. The resultant vector is

$$\boldsymbol{F}(S) = \begin{bmatrix} -sin\left(\frac{Scos(\alpha)}{R}\right) cos(\alpha) V_x + cos\left(\frac{Scos(\alpha)}{R}\right) cos(\alpha) V_y + sin(\alpha) V_z \\ \frac{-cos\left(\frac{Scos(\alpha)}{R}\right) cos(\alpha)^2}{R\kappa} V_x + \frac{-sin\left(\frac{Scos(\alpha)}{R}\right) cos(\alpha)^2}{R\kappa} V_y \\ \frac{sin\left(\frac{Scos(\alpha)}{R}\right) sin(\alpha) cos(\alpha)^2}{R\kappa} V_x + \frac{-cos\left(\frac{Scos(\alpha)}{R}\right) sin(\alpha) cos(\alpha)^2}{R\kappa} V_y + \frac{cos(\alpha)^3}{R\kappa} V_z \end{bmatrix}, \tag{10}$$

where $V_x$, $V_y$ and $V_z$ are the three components of the force. It is possible to observe that the second term has no influence of the component in the $x_2$ direction. In view of the helix angle, the triad (***t***, ***n***, ***b***) is also rotated in reference of the global system, therefore, there is a component of $V_z$ in the $x_1$ direction, if not, there should be only a component in $x_3$ due to the parallelism of the local and global systems. This relation was already expected. The boundary conditions for the moments can be obtained adopting the same procedure.

Applying the correct boundary condition to the clamped end ($S$=0) and setting the vector force to produce tension on the wire, the following results were generated.

## 5. RESULTS

After the determination of the six constants the analytical expressions were evaluated. Initially a plane ring of 270º was compared to the numerical response given by a commercial software (ANSYS®). Then the rod was simulated and, once again, the results compared with FEM.

For the first case the helix angle was defined as zero. Consequently, the parameter $\tau$ is also zero and that $\kappa$ is the inverse of the beam radius. This means that the system of six equations, presented as Eq. (8) and (9), are reduced to a simpler case of a curved beam in a $x_1x_2$ plane, as shown in Fig. 4. The force was set as 10 N in the negative $x_3$ direction and the radius *2 m*.

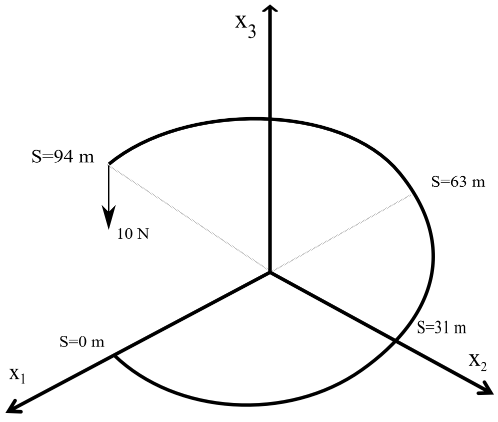


Figure 3. Curved beam with the load applied.

The results for the force are presented in Fig. 4. It shows the result force, in the global system, where it is possible to notice the constant value corresponding to the applied load. Both methodologies generate the same result. For the moments diagram, Fig. 5, four regions can be highlighted. Initially, where $S$=0, the corresponding moments are exactly the boundary condition, which is the radius times the force for both directions. The second point, $S$=31.41 m or at 90º, two moments are zero ($M_y$ and $M_z$), because there is no distance to generate the moment, and $M_x$ is one diameter distant from the load. Where $S$=62.83 m the moments have the same modulus of the clamped end, however $M_z$ has opposite sign. For $S$=94.24 m, or the end of the rod, all the moments are zero, this is where the load is being applied. There is no moment in $x_3$, what was expected, since the force is always parallel to the axis $x_3$. It is important to realize that the highest moment does not occur in the clamp, but where the distance is bigger (equals to twice the radius).

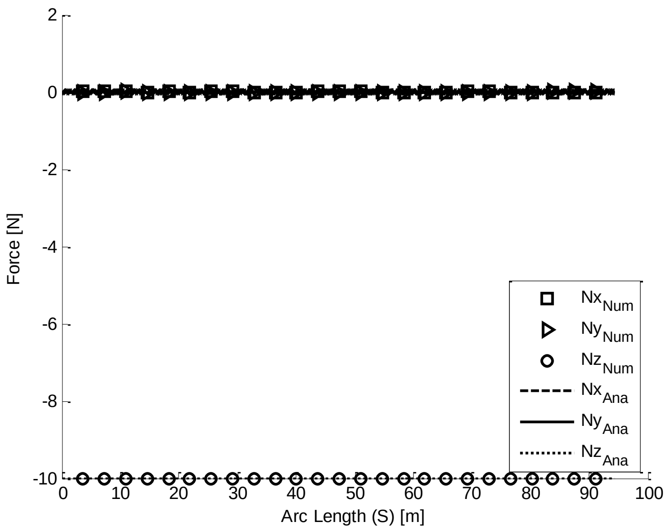


Figure 4. Forces for the first case, α=0, with Num indicating the numerical result and Ana the analytic response.

The second case analyzed, Fig. 6, is the rod itself, setting the helix angle to 75˚ and the radius at 21.553 m. Figure 7 shows the force through the S coordinate. Once again it is possible to notice that the value is constant and have the same answer for both methodologies. The moments are presented in Fig. 9 for the tension force of 10 N. The same four points can be emphasized. At $S$=0, where the boundary condition is applied, the moments are all zero, just like at the end where is the load. Since the force is applied at a point exactly above the clamp there is no distance to generate moment.

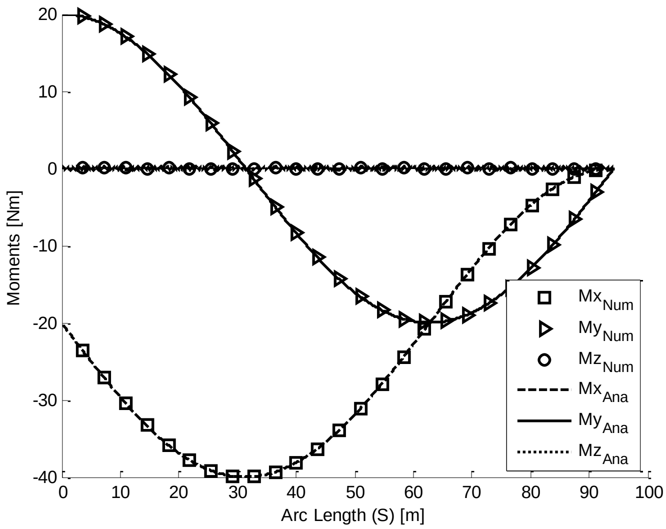


Figure 5. Moments diagram for the ring with a load in the $x_3$ direction

For *S*=122 m and *S*=365 m, the moments have the same modulus (although at *S*=365 m $M_x$ has different sign), which correspond to the moment from the distance of one radius ($M_x$ and $M_z$). The maximum value for the moment is at the opposite side of the force application, where the distance is equivalent of two times the radius ($M_z$). Both $M_y$ and $M_x$ are zero, due to the geometry of the cable.

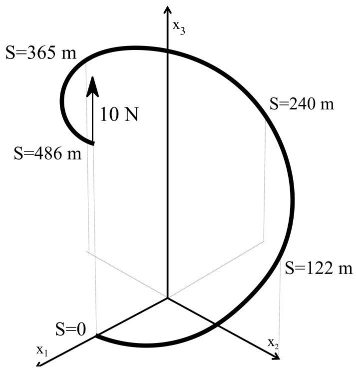


Figure 6 – Second case analyzed, α=75º

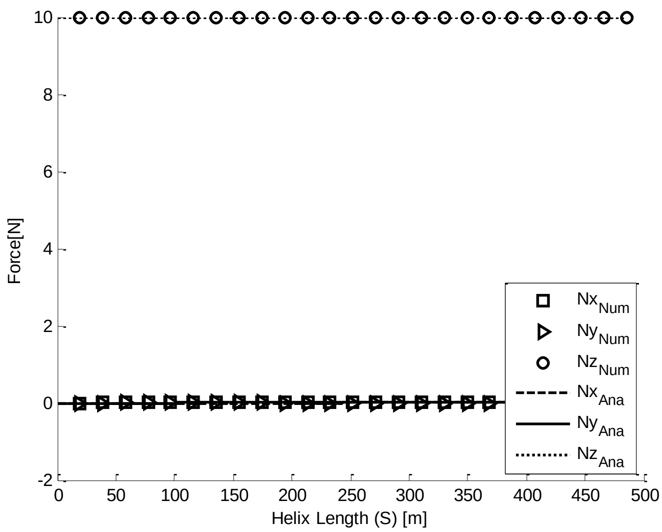


Figure 7 – Forces for the second case.

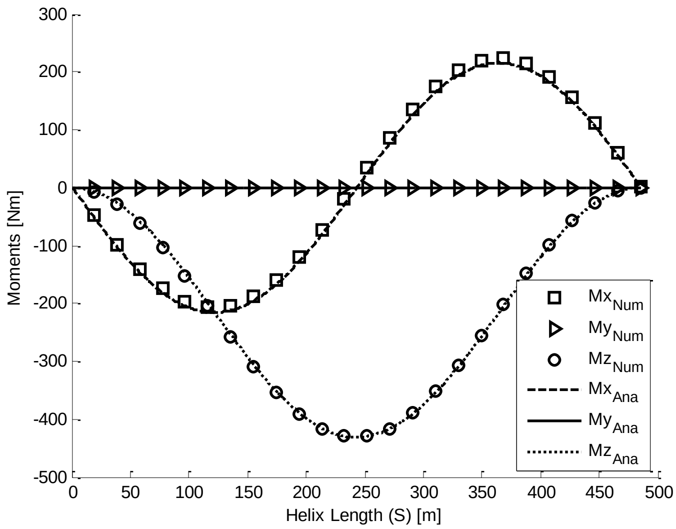


Figure 8. Moments for the second case.

The results for the second case present a difference that reaches 4% from analytical and numerical solutions. This difference is not considered significant and, therefore, is a proof that the method is valid and can be used for further analysis. All the results were expected and represent well the cable behavior. For the same boundary conditions of the second analysis, the length was increased to 10 pitches in order to test the periodicity of moments, what is estimated to occur in the rod. Figure 9 presents this behavior.

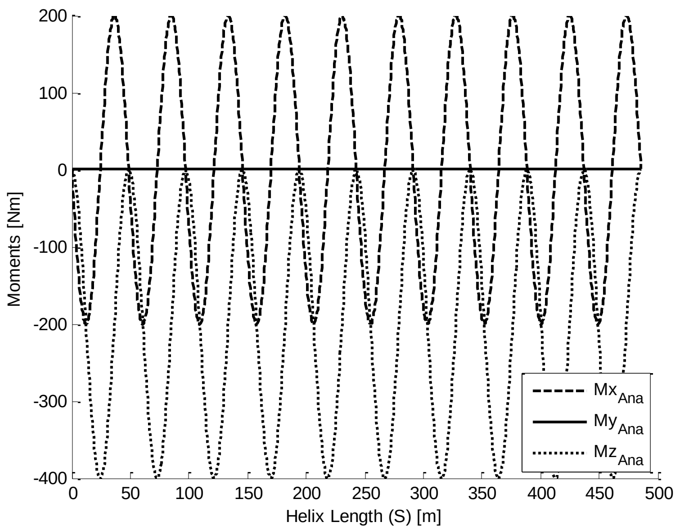


Figure 9. Long cable analyzed with the second case parameters.

## 6. CONCLUSIONS

The governing differential equations for spatial beam theory were analytically solved to determine the cable behavior. The parameterization of a helix was used and only one wire was modeled. Results were obtained for all resultant forces in both local and global coordinate. With the triad it was able to decompose the loads in the local system.

Two cases were analyzed with the methodology developed. For the plane ring ($\alpha = 0$) the analytical expressions were compared to the numerical solution. The results were identical for forces and moments, showing that the three dimensional equations could be reduce into a two dimension problem.

For the second case, where the wire was subjected to tension, the results were compared and the FEM model presented good agreement with the analytical solution. The periodicity was also investigated and the moments proved to be repeated along the cable. The methodology applied is, therefore, validated and can be used for further analysis, particularly using superposition of the equations developed here to describe complete cables.

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