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MODELLING AND DYNAMIC ANALYSIS OF BELT DRIVE SYSTEM BY HYBRID FLEXIBLE MULTIBODY METHOD

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Abstract. *Serpentine belt drive systems are widely used in automotive and industrial applications for torque and power transmission. In such systems, variations in belt tension due to dynamic load of accessories lead to premature wear and reduced component life. Predict the transmission dynamic behavior is critical in design stage, and requires comprehensive modelling of the system including pulleys, auto-tensioner, belt and dynamic excitation sources. This paper aims to model the belt drive system using hybrid flexible multibody approach, treating pulleys and auto-tensioner as rigid elements and belt as flexible ones. It was used the absolute nodal coordinate formulation (ANCF), nonlinear finite element method that describes the system through gradients and absolute nodal position coordinates. Assumptions of small deformations and displacements, usually assumed in the linear finite element theory, are automatically eliminated in this method, which takes into account the continuum mechanics nonlinear deformation tensor. For purposes of experimental validation, a test bench was specially designed, built and instrumented, allowing analysis of the transmission system made up of two pulleys and belt, or three pulleys (one auto-tensioner) and belt, over different configurations. The system parameters, such as stiffness and inertia properties, were experimentally identified and used in numerical simulations. Experimental tests carried out in the two pulleys and belt system configuration showed satisfactory correlation with the results of theoretical modal analysis, upon the application of wide levels of belt tension.*

Keywords: *absolute nodal coordinate formulation, flexible multibody dynamic system, serpentine drive system, nonlinear finite element method.*

1. INTRODUCTION

Belt drive systems are widely used for torque and power transmission in machine elements. Among these systems, those constituted by a belt and two fixed center pulleys, with manual adjustment tension level, are still common in many applications. Several machines such as air compressors, washing machines, turboblowers, robots and industrial processing equipment, are driven by this type of transmission.

Such systems can exhibit complex dynamic behavior due nonlinearities of geometric or material nature. The dynamic loads applied on the system generally result in rotational vibration of pulleys and transversal of belt spans adjacent to them. Understand the dynamic effects and stability conditions have been research object in many works.

In relation to the proposed mathematical models, two stand out. One of them seeks to model the system as purely rotational, where the belt have negligible inertia and acts as springs that couples the pulleys, treated as discrete elements. In this approach, it is neglected the span's transverse vibration and their influence on the dynamics of the system (Shangguan and Zeng, 2013). The second one models the system considering the inertia effects and span's transverse vibration, as well as dependence of the natural frequencies with belt's axial velocity. This hybrid analytical model, composed of discrete elements (pulleys) and continuous (belt), has high complexity both from theoretical as computational view point (Beikmann *et al.*, 1996; Ding and Zu, 2013; Martins, 2013). The main contributions in this area have been to incorporate the bending stiffness effects (replacing the string models by analytical beam ones), the presence of auto-tensioner devices, including dry and viscous friction in the system, and belt's viscoelasticity.

More recent studies have sought to give more comprehensive and numerical approach to the problem, opening a new field of promising research. Leamy and Wasfy (2005) modeled a transmission system by classical finite element method, including interaction of belt-pulley contact, yet not considered in the models and treated independently. In this approach, sliding effects, loss of contact and stick-slip can be naturally observed. The main problems reported were the non-inclusion of nonlinear effects by the proposed model and the need to mesh discretization in a very large number of elements, adversely affecting computational performance (Kerkkänen *et al.*, 2006).

Shabana (1997, 2013) developed a new non-linear finite element formulation based on absolute nodal coordinates for applications in flexible multibody. In this method, nodal displacements are directly described in global reference system, and nodal inclinations, represented by gradients, do not impose any restriction on angular deformations. Recent research has revealed that the proposed formulation is very promising and efficient to be applied in belt drive systems (Cepon *et al.*, 2009a, 2011).

This paper presents the application of absolute nodal coordinates formulation (ANCF) in the hybrid modeling of a belt drive system with fixed center pulleys. The spans are modeled as flexible elements, while the pulleys as rigid discrete ones. Details of boundary conditions are discussed, as well as the theoretical modal analysis procedure. The theoretical results are supported by experiments conducted on test bench.

2. THEORETICAL MODEL

In belt drive systems is known the strong dependence of the transverse mode's natural frequencies with the axial tension level applied. In such systems, the rotational modes origined by dynamic loads can excite the transverse ones due to the coupling of degrees of freedom, which can generate large lateral displacements of the flexible elements and thus enhance nonlinear effects in the response.

The model proposed by Berzeri and Shabana (2000) is an higher-order beam element based on continuum mechanics, in which the strain tensor is related to the displacement field taking into account the non-linear terms (Green-Lagrange tensor). In this way, large displacements and axial/transverse strain can be captured by this element, relaxing the assumption of infinitesimal displacement and deformations previously assumed in classical finite element method. Another important feature of this element consists in the coupling of longitudinal with transversal degrees of freedom, allowing the movement in transverse or angular direction to be reflected in the longitudinal direction, and vice versa. The model then naturally incorporates the possible effects of geometric nonlinearity in the system. The convergence of the method is guaranteed with a reduced number of elements, allowing smaller mesh discretization and more computational efficiency. With all these features, the presented method has broad potential to be applied to transmission systems.

Recent research shows the use of the absolute nodal coordinates formulation in many applications, especially those involving nonlinear effects (flexible physical pendulum and rotating beams, interaction between rigid bodies sliding on extensible cables, deformable mechanisms, among others) (Gerstmayr *et al.*, 2013; Hussein *et al.*, 2010).

To support the assumptions to be used in the proposed model, uniaxial tensile tests were performed in micro-V belt (8PK model, EPDM material) using testing machine, in order to get the deformation that occurs in typical transmission systems. A segment belt with L=0.146 m was extracted and stretched quasi-statically to reach maximum force of 900 N. The displacement obtained was 0.001 m, resulting in a deformation of 0.68%. Thus, the assumption of small deformations to the longitudinal strain is assumed. Since the transverse deformations of spans can be large, it is not adopted any simplifying assumption for the transversal model.

2.1 ANCF beam model

In the absolute nodal coordinates formulation the inertial forces result in a simple expression, generating constant mass matrices and easily manipulated in numerical integrators. As the description is made directly in the global system, the centrifugal forces naturally vanish and there is no need for use of coordinate transformation. However, the elastic forces have quite complex expressions, even in the simplest cases and when linearity assumptions are adopted.

The Figure 1 illustrates a typical ANCF beam element with the respective degrees of freedom. The position vector of an arbitrary point of the beam can be described using a global function shape - that includes the rigid body modes, and absolute nodal coordinates (Berzeri and Shabana, 2000):

$$\mathbf{r} = \mathbf{S}(x)\mathbf{e} \tag{1}$$

The global shape function is given by

$$\mathbf{S} = [s_1[\mathbf{I}] \;\; s_2[\mathbf{I}] \;\; s_3[\mathbf{I}] \;\; s_4[\mathbf{I}]] \tag{2}$$

where $[\mathbf{I}]$ is the identity matrix of order 2. The interpolation polynomials of third order are defined as

$$s_1 = 1 - 3\xi^2 + 2\xi^3, \; s_2 = L(\xi - 2\xi^2 + \xi^3), \; s_3 = 3\xi^2 - 2\xi^3, \; s_4 = L(\xi^3 - \xi^2) \tag{3}$$

wherein $\xi = x/L$ and L the length of the beam in undeformed state. The variable x identifies the position of an arbitrary point on the neutral axis. The vector of element's nodal coordinates is defined by

$$\mathbf{e} = [e_1 \;\; e_2 \;\; e_3 \;\; e_4 \;\; e_5 \;\; e_6 \;\; e_7 \;\; e_8]^{\mathrm{T}} \tag{4}$$

In this vector, the nodal displacements are

$$e_1 = r_1|_{x=0}, \;\; e_2 = r_2|_{x=0}, \;\; e_5 = r_1|_{x=L}, \;\; e_6 = r_2|_{x=L} \tag{5}$$

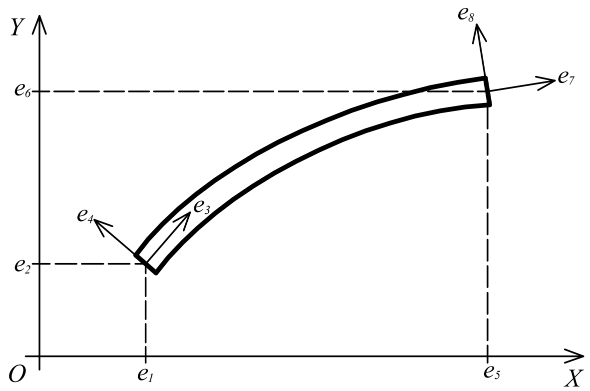

Figure 1. Absolute coordinates of a beam.

And the absolute inclinations of nodes, represented by gradients

$$e_3 = \left.\frac{\partial r_1}{\partial x}\right|_{x=0}, \quad e_4 = \left.\frac{\partial r_2}{\partial x}\right|_{x=0}, \quad e_7 = \left.\frac{\partial r_1}{\partial x}\right|_{x=L}, \quad e_8 = \left.\frac{\partial r_2}{\partial x}\right|_{x=L} \tag{6}$$

Differentiating the position vector with respect to time, the kinetic energy of the finite element is expressed as

$$T = \frac{1}{2}\dot{\mathbf{e}}^{\mathrm{T}}\mathbf{M}_a\dot{\mathbf{e}}, \ \mathbf{M}_a = \int_V \rho \mathbf{S}^{\mathrm{T}}\mathbf{S}dV \tag{7}$$

where ρ is the mass density and $\mathbf{M}_a$ is the constant mass matrix element.
The generalized external forces, including body and surface forces are defined using the principle of virtual work

$$\delta W_e = \mathbf{Q}_e^{\mathrm{T}}\delta\mathbf{e} \tag{8}$$

The element's elastic forces vector, that arise from longitudinal and transverse strain, can be determined using the total strain energy U as

$$\mathbf{Q}_k = \left(\frac{\partial U}{\partial \mathbf{e}}\right)^{\mathrm{T}} \tag{9}$$

The equation of motion of the finite element can then be determined from

$$\mathbf{M}_a\ddot{\mathbf{e}} + \mathbf{Q}_k = \mathbf{Q}_e \tag{10}$$

2.1.1 Longitudinal and transverse force models

To determine the vector of elastic forces, the expressions of strain energy both longitudinal and transverse need to be defined. At a given point x inside the beam, an infinitesimal segment dx in undeformed state becomes the length ds when deformed. This length is given by

$$ds = \sqrt{\mathbf{r}'^{\mathrm{T}}\mathbf{r}'}dx, \ \ \mathbf{r}' = \frac{d\mathbf{r}}{dx} \tag{11}$$

Using the definition of the Green-Lagrange strain measure, the longitudinal strain ϵ_l can be expressed as

$$ds^2 - dx^2 = 2dx\epsilon_l dx, \ \epsilon_l = \frac{1}{2}(\mathbf{r}'^{\mathrm{T}}\mathbf{r}' - 1) \tag{12}$$

Therefore, considering elastic beam material with elastic modulus E and cross-sectional area A, the longitudinal strain energy is calculated from

$$U_l = \frac{1}{2}\int_0^L EA\epsilon_l^2 dx \tag{13}$$

Using the Eq. (9), Eq. (12) and Eq. (13), the vector of longitudinal elastic forces is given by

$$\mathbf{Q}_l = \left(\frac{\partial U_l}{\partial \mathbf{e}}\right)^{\mathrm{T}} = \int_0^L EA\epsilon_l \mathbf{S}'^{\mathrm{T}}\mathbf{S}'\mathbf{e}\, dx = \mathbf{K}_l\mathbf{e} \tag{14}$$

where $\mathbf{K}_l$ is the stiffness matrix associated with axial strain. It should be noted that this matrix is strongly non-linear, dependent on all the element's degrees of freedom.

The transverse strain energy is calculated from the beam element curvature. The bending moment is given by the expression $M = EI\kappa$, with I being the area moment of inertia. The curvature κ can be determined from the global position vector

$$\kappa = \left|\frac{d^2\mathbf{r}}{ds^2}\right| = \frac{|\mathbf{r}' \times \mathbf{r}''|}{|\mathbf{r}'|^3} \tag{15}$$

Thus, the strain energy due to bending is given by

$$U_t = \frac{1}{2}\int_0^L EI\kappa^2 dx \tag{16}$$

By making use of Eq. (9), Eq. (15) and Eq. (16), the vector of transverse elastic forces may be determined as

$$\mathbf{Q}_t = \left(\frac{\partial U_t}{\partial \mathbf{e}}\right)^{\mathrm{T}} = \int_0^L EI\mathbf{S}''^{\mathrm{T}}\mathbf{S}''\mathbf{e}\, dx = \mathbf{K}_t\mathbf{e} \tag{17}$$

with $\mathbf{K}_t$ the stiffness matrix associated to the transverse strain. In this case, the condition previously established of small longitudinal strain in the belt is used and ds approximate by dx.

Finally, the total vector of elastic forces $\mathbf{Q}_k$ of the Eq. (10) is given by the sum of the Eq. (14) and Eq. (17).

2.2 Belt drive system

The mechanical model of the belt drive system is shown in Fig. 2. The reference systems B_1 and B_2 are fixed in the driver and driven pulleys, respectively. The B_3 and B_3 systems are fixed to the ends of spans. To incorporate the effect of the bearing's flexibility in the horizontal direction, a spring with structural rigidity k_t is added in the driven pulley, representing the equivalent stiffness of both bearings.

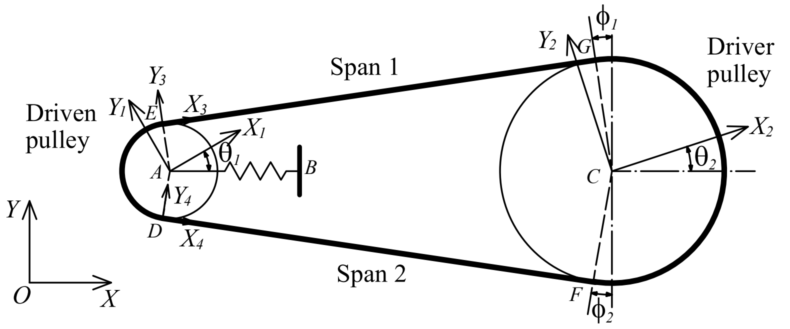

Figure 2. Belt drive system.

Two closed loop equations are required to calculate the initial configuration. The geometric constraint equations are written as

$$\text{Loop 1}: \; \mathbf{r}_{OA} + \mathbf{r}_{AE} + \mathbf{r}_{EG} = \mathbf{r}_{OC} + \mathbf{r}_{CG} \qquad \text{Loop 2}: \; \mathbf{r}_{OA} + \mathbf{r}_{AD} + \mathbf{r}_{DF} = \mathbf{r}_{OC} + \mathbf{r}_{CF} \tag{18}$$

The position vectors of Eq. (18) are obtained with the aid of appropriate local systems B_3 and B_4 and their coordinate transformation matrices (Shabana, 2013).

The belt length can be calculated from

$$L = X_1 + X_2 + R_1(\pi - \phi_1 + \phi_2) + R_2(\pi + \phi_1 - \phi_2) \tag{19}$$

where X_1 and X_2 are the span's lengths and R_1 and R_2 the radii of driver and driven pulleys, respectively.

Given the pulleys coordinates X_C, Y_C and Y_A, the undeformed configuration of the system can be determined by performing $L = L_0$ subject to constraint equations (Eq. (18)), where L_0 is the undeformed belt length. The nonlinear algebraic equations resulting can be solved by Newton-Raphson method, providing as output the span lengths X_1 and X_2, the inclination angles ϕ_1 and ϕ_2 and the position X_a of the driven pulley.

Assuming that local system B_1 and B_2 are positioned in the center of mass of the pulleys, the respective mass matrices are diagonal and expressed as

$$\mathbf{M}_1 = \begin{bmatrix} m_1 & 0 & 0 \\ 0 & m_1 & 0 \\ 0 & 0 & J_1 \end{bmatrix} \qquad \mathbf{M}_2 = \begin{bmatrix} m_2 & 0 & 0 \\ 0 & m_2 & 0 \\ 0 & 0 & J_2 \end{bmatrix} \tag{20}$$

where m and J are the mass and mass inertia moment of the pulleys, respectively. The stiffness matrix associated to driven pulley, due to the equivalent flexibility bearing, is represented by

$$\mathbf{K}_1 = \begin{bmatrix} k_{t1} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{21}$$

Making use of Eq. (7) and Eq. (18), the global system's mass matrix is determined using the classical concept of assemblage in finite elements

$$\mathbf{M} = \begin{bmatrix} \mathbf{M}_1 & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{M}_2 & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{M}_{a3} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{M}_{a4} \end{bmatrix} \tag{22}$$

wherein $\mathbf{M}_{a3}$ and $\mathbf{M}_{a4}$ the mass matrices of the belt spans 1 and 2.
Similarly, from Eq. (14), Eq. (17) and Eq. (21), the global stiffness matrix is given by

$$\mathbf{K} = \begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{K}_3 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{K}_4 \end{bmatrix} \tag{23}$$

where $\mathbf{K}_3$ and $\mathbf{K}_4$ are the stiffness matrices of the belt spans 1 and 2.
The weight forces of elements are calculated with the aid of Eq. (8). Thus, the global equation of motion of the transmission system is given by

$$\mathbf{M}\ddot{\mathbf{q}} + \mathbf{K}\mathbf{q} = \mathbf{Q}_{ext} \tag{24}$$

In this equation, the vector of generalized coordinates is defined as

$$\mathbf{q} = [x_A \; y_A \; \theta_1 \; x_C \; y_C \; \theta_2 \; \mathbf{e}_3^{\mathrm{T}} \; \mathbf{e}_4^{\mathrm{T}}]^{\mathrm{T}} \tag{25}$$

where $\mathbf{e}_3$ and $\mathbf{e}_4$ represent vectors with the degrees of freedom of the flexible spans defined in Eq. (4), varying according to the mesh discretization. The vector $\mathbf{Q}_{ext}$ represents all the external forces acting on the system.

The constrains between the degrees of freedom need to be properly established and conveniently grouped into a vector $\mathbf{C}(\mathbf{q}, t)$. It is assumed that the spans are connected to the pulleys by means of joints, providing bi-articulated beam end conditions, according to the work of Cepon *et al.* (2009b). Thus, the absolute nodal coordinates of the node 1 are constrained to move along the axis X_3 around point E due to the rotation of the driven pulley, by the amount $R_1\theta_1$. The node 2 of the element is also restricted to linear movement along the axis X_3 in point G, but by the amount $R_2\theta_2$. Similarly, the constrains of the span 2 are written to both nodes. The restrictions prescribed for the gradients are defined considering the span orientation.

From the restrictions imposed on the system, it is always possible to separate a set of dependent coordinates from the generalized coordinates. Using the vector $\mathbf{C}(\mathbf{q}, t)$, a transformation matrix $\mathbf{B}_i(\mathbf{q}, t)$ can be obtained to eliminate the dependent coordinates, resulting in equations of motion where only appear the independent ones, that can be integrated numerically to obtain the time response of the system (Shabana, 2013).

The static configration is determined by eliminating the inertial forces in Eq. (24) and solving the system for independent generalized coordinates. In this step, a non-linear solver must be used due to non-linearities geometric present.

The modal analysis procedure can be performed from the linearization of the equations of motion around the equilibrium configuration, by using the theory of tangent stiffness matrix (Bonet and Wood, 2008; Shabana, 2010). Through $\mathbf{F} = \mathbf{Q}_{ext} - \mathbf{K}\mathbf{q}$, the stiffness matrix $\mathbf{K}_T$ is determined as

$$\mathbf{K}_T = -\frac{\partial \mathbf{F}}{\partial \mathrm{e}} \tag{26}$$

wherein $\mathbf{K}_T$ a constant matrix associated with independent coordinates and can be used together with the reduced mass matrix to determine the modal parameters of the system.

3. TEST BENCH

A test bench (Fig. 3a) was built for verification and validation of the model. On this test bench, the driver pulley bearing is fixed and the driven pulley is mounted on a base with linear adjust of horizontal position. An adjustable support also allows the incorporation of the auto-tensioner device in future studies.

The test bench is equipped with torque, speed and deformation transducers, the latter for the measurement of dynamic reactions acting on the driven pulley bearing. To measure the force acting on the belt in static condition, it was built two jaws connected to a instrumented bar with strain gauges. The device joins two previously cut ends of the belt, allowing the quantification of internal tension.

The pulley's inertia properties were obtained experimentally by three-wire pendulum method. The transverse rigidity modulus belt EI was estimated from theoretical expression provided by Kong and Parker (2004). The equivalent stiffness bearing was measured experimentally by applying force and recording the resulting displacement. The parameters of the mechanical system are (in SI units): (X_A, Y_A)=$(0.2237, 0)$, (X_C, Y_C)=$(0.8966, 0.0641)$, R_1=0.0375, R_2=0.1016, m_1=2.1543, m_2=7.8269, J_1=0.001702, J_2=0.032651, L_0=1.7923, k_{t1}=2425300, EA=120000 and EI=0.02. The linear density belt is ρ=0.11915 kg/m.

Through experimental modal analysis of the system in static condition, frequency spectrum were determined as function of the belt tension (Fig. 3b), to be confronted with the natural frequencies of the proposed mathematical model.

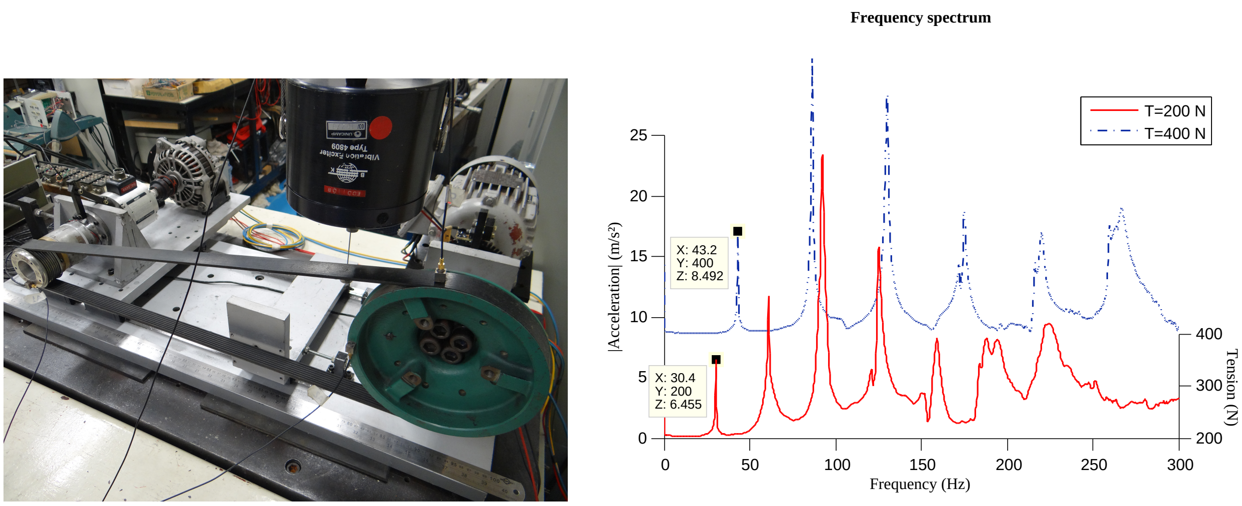

Figure 3. a) test bench built; b) experimental frequency spectrum.

4. NUMERICAL SIMULATION AND EXPERIMENTAL VERIFICATION

To verify the model ability to predict modal parameters of the system according tension level applied to belt, two numeric tests was performed. In the first, the belt is tensioned at 200 N, while in the second, a force of 400 N is applied. In the simulations the flexible elements were discretized in a mesh with 10 ones.

The natural frequencies of the system are presented in Tab. 1 (rigid body mode are dismissed), and the modal shapes are shown in Fig. 4.

Table 1. System's natural frequencies as function of belt tension level (Hz).

	T=200 N			T=400 N		
Mode	Theoretical	Experimental	Modal shape	Theoretical	Experimental	Modal shape
1	30.5	30.3	1st transverse	43.2	43.1	1st transverse
2	61.3	61.1	2nd transverse	86.6	86.5	2nd transverse
3	92.4	92.4	3rd transverse	101.0	-	1st rotational
4	100.8	-	1st rotational	130.3	130.1	3rd transverse
5	124.2	125.2	4th transverse	174.4	175.0	4th transverse
6	156.7	159.2	5th transverse	178.6	-	1st translational
7	178.5	-	1st translational	219.1	220.0	5th transverse
8	190.3	194.9	6th transverse	264.5	266.8	6th transverse

It is observed that the natural frequencies predicted by the model are well correlated with experimental and have good accuracy. In general modes have linear coupling, generating predominantly transverse, axial, rotational or translational vibration. However, depending on the mode and system parameters, the degree of coupling can be larger or smaller. The translational mode is associated with the bearing vibration, and couples the upper span transverse movement, while the rotational couples the belt's axial movement. The bottom span also has transverse modes with the same frequency of the upper span. It was observed in simulations that transverse modes couples the pulley's rotational motion - even in very low amplitude, showing the coupling of transverse and rotational degrees of freedom. The span's longitudinal modes can be also observed in simulations, but in higher frequency.

When the force applied on the belt increases, it can be seen that all the frequencies associated with the transverse modes are changed. However, the rotational, translational and longitudinal span modes are not affected by the belt tension level, because these modes depend predominantly on the parameters EA and k_{t1}.

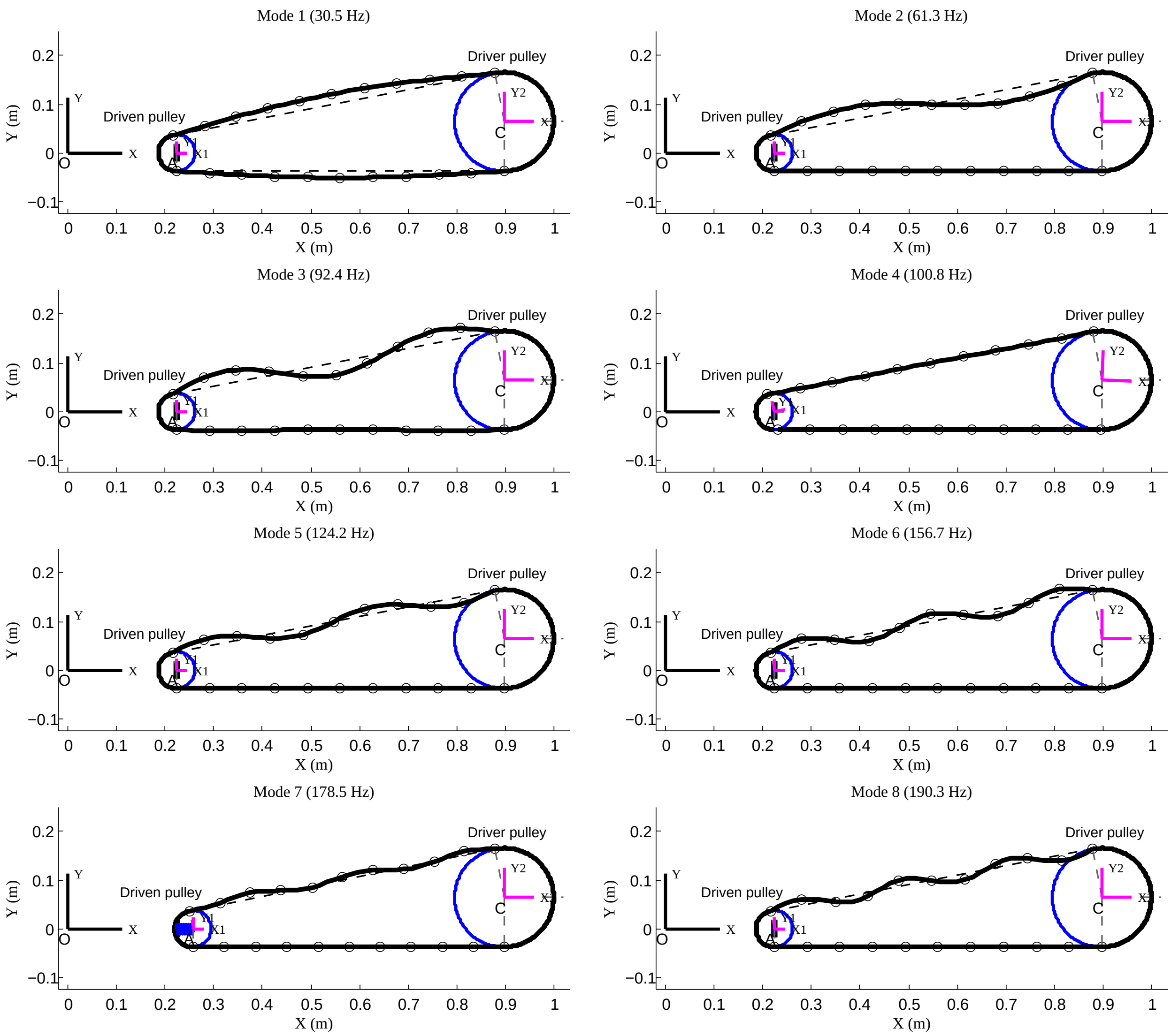

Figure 4. Modal shapes (T=200 N).

To verify the influence of the transverse rigidity modulus on modal parameters, a simulation was made considering EI=100 $\mathrm{N.m}^2$. It can be seen in Tab. 2 that rotational, translational and longitudinal modes of the belt remain unchanged, while the transverse ones are sensitive to the variation of this parameter.

Nonlinear analysis can also be performed to verify the influence of nonlinearities geometric upon rotational excitations of the driver pulley and the effect of span's transverse modes on the rotational ones of the driven pulley. When large strain occurs, it was observed the presence of harmonics in the time response obtained from the numerical integration of the equations for the nonlinear model, indicating that possible nonlinearities are present in the system that need to be taken into consideration when the assumptions of small displacements and deformations are not longer valid.

5. CONCLUSIONS

In this paper it was proposed a flexible multibody model to analyze the dynamic behavior of a belt drive system. The model was able to capture the existing linear coupling of the system between the spans and pulleys. It was shown that the model is sensitive to the belt tension and other system parameters, presenting itself as a robust and promising tool to

Table 2. Natural frequencies simulated for EI=100 $\mathrm{N.m^2}$.

Mode	Frequency (Hz)	Modal shape
1	100.8	1st rotational
2	105.4	1st transverse
3	178.5	1st translational
4	408.3	2nd transverse
5	761.7	1st longitudinal
6	769.6	2nd longitudinal
7	913.3	3rd transverse
8	1554.6	3rd longitudinal

assist in analysis and design stage.

6. ACKNOWLEDGEMENTS

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