23rd International Congress of Mechanical Engineering
December 6-11, 2015, Rio de Janeiro, RJ, Brazil

A FAST TOOL FOR BUCKLING ANALYSIS OF STIFFENED PLATES

Francisco Alex Correia Monteiro
Eliseu Lucena Neto
Lucas Nobumichi Yshii
Instituto Tecnológico de Aeronaútica, 12228-900, São José dos Campos, SP, Brazil.
facm@ita.br, eliseu@ita.br, lucasyshii@gmail.com

Abstract. *A fast tool for buckling analysis that can be used during the design phases of rectangular isotropic plates is presented. The plate is subjected to usual in-plane loading and its boundary edges can be elastically restrained against rotation by stiffeners. The solution is obtained by a Ritz procedure using displacement approximations derived from Legendre polynomials. Most boundary conditions may be introduced a posteriori into the discrete problem, in the same manner as is done in finite element procedures. Excellent accuracy are attested by numerical results.*

Keywords: *buckling, stiffened plates, Ritz method*

1. INTRODUCTION

Stiffened plates have wide applications in many civil engineering, aerospace and marine structures. They are used in box girders, plate girders, ship hulls and wing structures. Interest in stiffened plate construction has been widespread in last decades due to its economic and structural benefits. The advantage of stiffening a plate lies in achieving an economical, light weight design. While the stiffening elements add small weight to the overall structure, their influence on strength and stability can be significant. Adding stiffeners to the plate complicates the analysis and several assumptions must be made in order to facilitate a solution to the problem.

There is one important point to be made at the outset, which is that there exist different distinct buckling modes for a stiffened panel. Depending on the spacing and the size of the frames (transversal stiffeners) and stringers (longitudinal stiffeners) relative to the plate thickness, buckling deflections could be developed within each plate panel or encompass a number of stiffened panels. In the first case, commonly referred to as skin buckling, the line of connection between skin and stiffeners remains virtually straight, with the stiffeners exhibiting only minor movement. The second case, known as overall buckling, involves a variable displacement of the stiffeners. These modes are neither mutually exclusive nor independent.

For a stiffened plate subjected to compression the overall (global) buckling and the skin (local) buckling modes are usually distinct from each other. As sudden global buckling is undesirable at design limit load levels, typical designs exhibit skin buckling first, followed by load re-distribution to the stiffeners. Therefore, an investigation into this mode can be carried out by examining the behavior of a panel between two successive frames and stringers. In such analysis the effect of stiffeners can only be implicitily accounted for in the boundary conditions along the straight edges of the panel. The prediction of the local buckling of single or multi-panel stiffened plates is simplified by design standards by assuming the plate and the stiffeners are either hinged or clamped along their lines of intersections, thus ignoring the interaction between plate and stiffener elements. The simply supported condition assumes the attached stiffeners have zero rotational stiffness and the clamped condition assumes this value is infinite. In reality, the stiffeners possess finite rotation capacity posing an intermediate case.

Bedair and Sherbourne (1993) and Sherbourne and Bedair (1993) deal with local buckling of stiffned plates under uniform compression. The influence of the rotational restraint, in-plane bending and translation restraints upon the local buckling and post-buckling behavior were investigated. The geometric properties of the plate/stiffeners proportions that determine the intensities of these restraints were defined and several boundaries were modeled for torsional rigidity of the stiffeners, from rotationally free to rotationally clamped. Bedair and Sherbourne (1995) present a semi-analytical approach to determine the local buckling load of stiffened plates under any combination of in-plane loading (compression, shear and in-plane bending). The edges were modeled as partially restrained against rotation and the plate was treated as infinitely long. The energy method was then used to derive a generalized K factor in terms of structure parameters that defines the idealized buckling mode. The buckling load was then computed using sequential quadratic programming to find the particular combination of these parameters in the idealized buckling mode that minimizes the coefficient K. Modifications factors were then suggested to compute the local buckling load for plates of finite length.

Since the stiffeners provide a continuous torsional rigidity along panel edges, they could be modeled as torsion bars. Such assumption seems to be correct if the stiffeners do not buckle before the skin buckling takes place, which is the case of a typical aeronautical design. Bearing this in mind, the stability behavior of the complete structure can be studied considering only the portion of a skin panel between adjacent frames and stringers, converting the complex original problem to the study of stability behavior of an individual skin panel elastically restrained along the reinforced edges. This key ideia is exploited by several authors. For instance, Paik and Thayamballi (2000) have employed the Lévy method to develop design formulae for buckling strength as a function of the torsional rigidity of support members that provide the rotational restraints along edges. In a simple manner, Bisagni and Vescovini (2009) present a Lévy and Ritz solutions for the linearized local skin buckling of longitudinally reinforced plates.

A significant amount of research has been directed towards experimental modelling of thin-walled plates, as well as towards the development of analytical and numerical approximated methods to improve their design against buckling. Using the finite element method it is possible to access the buckling strenght of stiffened plates without having to place emphases on simplifying assumptions. However, the use of finite element method is not recommended for early design phases, where a large amount of preliminary optimization iterations are performed, due to computational and simulation time costs. The development of a fast tool for buckling analysis of plates under any combination of classical boundary conditions, including or not attached stiffeners, and subjected to typical in-plane loading is the purpose of the present work.

2. BUCKLING PROBLEM

The homogeneous isotropic rectangular plate shown in Fig. 1, of length a and width b, is subjected to uniform in-plane loads p_x, p_y, p_{xy} and has the midsurface in the xy-plane of the Cartesian coordinate system xyz. According to the Kirchhoff theory, the plate buckling is described by

$$D_{11} w_{,xxxx} + 2\left(D_{12} + 2D_{66}\right) w_{,xxyy} + D_{22} w_{,yyyy} + p_x w_{,xx} + 2p_{xy} w_{,xy} + p_y w_{,yy} = 0 \tag{1}$$

where w is the displacement in the z direction, subscripts x and y preceded by commas denote differentiation with respect to x or y, and D_{ij} are the bending stiffnesses of the plate. The loading p_x and p_y have uniform (p_{x0}, p_{y0}) and linear (p_{x1}, p_{y1}) components: $p_x = p_{x0} + p_{x1}(2y/b - 1)$ and $p_y = p_{y0} + p_{y1}(2x/a - 1)$.

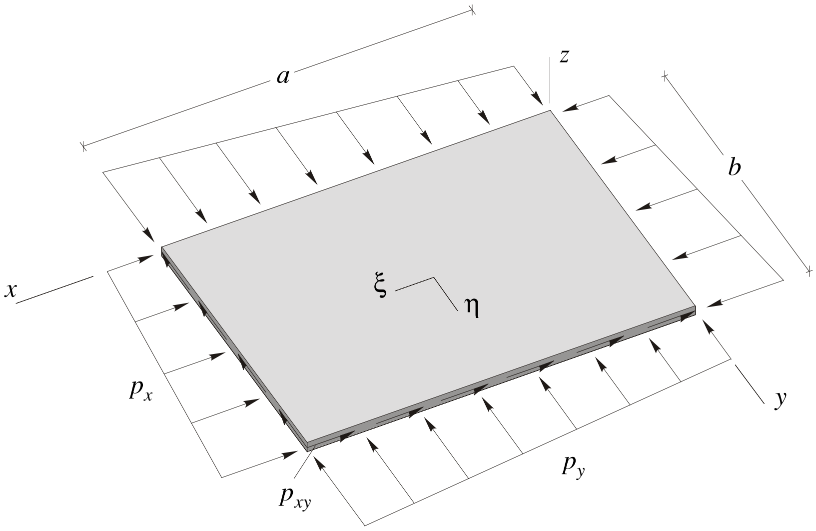

Figure 1. Rectangular isotropic plate subjected to in-plane loads p_x, p_y, p_{xy}.

Herein, the solution of Eq (1) should satisfy the prescribed values of

$$\begin{array}{llll} w \text{ or } V_x & w_{,x} \text{ or } M_x & \text{on} & x = 0, a \\ w \text{ or } V_y & w_{,y} \text{ or } M_y & \text{on} & y = 0, b \end{array} \tag{2}$$

where V_x, V_y are the effective shear forces and M_x, M_y are the bending moments. One element of each pair (w, V_x), (w, V_y), ($w_{,x}$, M_x) and ($w_{,y}$, M_y) (but not both elements of the same pair) may be specified. Moreover, the displacement or the twisting moment M_{xy} (but not both) should be specified at the plate corners. A boundary condition is called geometric when w, $w_{,x}$ or $w_{,y}$ are specified and is called mechanical when V_x, V_y, M_x, or M_y are specified. If a plate edge is restrained against rotation by a stiffener then $M_x = M_f$ or $M_y = M_s$, where M_f and M_s are the external moments exerted by the frames on the panel edges $x = 0, a$ and by the stringers on the panel edges $y = 0, b$:

$$\begin{array}{ll} M_f\left(0,y\right) = \theta_f^0 w_{,xyy}\left(0,y\right) - \varphi_f^0 w_{,xyyyy}\left(0,y\right) & M_f\left(a,y\right) = -\theta_f^a w_{,xyy}\left(a,y\right) + \varphi_f^a w_{,xyyyy}\left(a,y\right) \\ M_s\left(x,0\right) = \theta_s^0 w_{,xxy}\left(x,0\right) - \varphi_s^0 w_{,xxxxy}\left(x,0\right) & M_s\left(x,b\right) = -\theta_s^b w_{,xxy}\left(x,b\right) + \varphi_s^b w_{,xxxxy}\left(x,b\right). \end{array} \tag{3}$$

The stiffeners are supposed to resist torsion in two ways: the first is denoted by Saint-Venant torsion, the second by warping torsion (Kollbrunner and Basler, 1969). The quantities $\theta_f = G_f J_f$, $\theta_s = G_s J_s$ and $\varphi_f = E_f \Gamma_f$, $\varphi_s = E_s \Gamma_s$ are related with Saint-Venant and warping rigidities. The stiffener materials are supposed isotropic with Young's and shear moduli (E_f, G_f) for frames and (E_s, G_s) for stringers. The pairs (J_f, Γ_f) and (J_s, Γ_s) define the torsion and warping constants.

The plate buckling described by Eq (1) subject to Eq (2) can be stated as

$$\delta W_i + \delta W_e = 0 \tag{4}$$

where δW_i and δW_e are the internal and external virtual work:

$$\begin{aligned}
\delta W_i &= -\int_0^b \int_0^a \left[\left(D_{11} w_{,xx} + D_{12} w_{,yy}\right) \delta w_{,xx} + 4 D_{66} w_{,xy} \delta w_{,xy} + \left(D_{12} w_{,xx} + D_{22} w_{,yy}\right) \delta w_{,yy}\right] dxdy \\
\delta W_e &= \int_0^b \int_0^a \left[p_x w_{,x} \delta w_{,x} + p_{xy}\left(w_{,y} \delta w_{,x} + w_{,x} \delta w_{,y}\right) + p_y w_{,y} \delta w_{,y}\right] dxdy \\
&\quad + \int_0^a \left[M_s(x,0)\, \delta w_{,y}(x,0) - M_s(x,b)\, \delta w_{,y}(x,b)\right] dx \\
&\quad + \int_0^b \left[M_f(0,y)\, \delta w_{,x}(0,y) - M_f(a,y)\, \delta w_{,x}(a,y)\right] dy.
\end{aligned} \tag{5}$$

3. RITZ EQUATIONS

It is convenient to nondimensionalize Eq. (4) by adopting the coordinates $\xi = (2x - a)/a$ and $\eta = (2y - b)/b$ to obtain

$$\xi_{,x} = \frac{2}{a} \qquad dx = \frac{a}{2} d\xi \qquad \eta_{,y} = \frac{2}{b} \qquad dy = \frac{b}{2} d\eta \tag{6}$$

and

$$\begin{aligned}
w_{,x} &= \frac{2}{a} w_{,\xi} \qquad w_{,y} = \frac{2}{b} w_{,\eta} \qquad w_{,xx} = \frac{4}{a^2} w_{,\xi\xi} \qquad w_{,xy} = \frac{4}{ab} w_{,\xi\eta} \\
w_{,yy} &= \frac{4}{b^2} w_{,\eta\eta} \quad w_{,xxy} = \frac{8}{a^2 b} w_{,\xi\xi\eta} \quad w_{,xyy} = \frac{8}{ab^2} w_{,\xi\eta\eta} \quad w_{,xxxxy} = \frac{32}{a^4 b} w_{,\xi\xi\xi\xi\eta} \quad w_{,xyyyy} = \frac{32}{ab^4} w_{,\xi\eta\eta\eta\eta}
\end{aligned} \tag{7}$$

from which

$$\begin{aligned}
\delta W_i &= -\int_{-1}^1 \int_{-1}^1 \frac{4}{ab} \left[\left(\frac{b^2}{a^2} D_{11} w_{,\xi\xi} + D_{12} w_{,\eta\eta}\right) \delta w_{,\xi\xi} + 4 D_{66} w_{,\xi\eta} \delta w_{,\xi\eta} + \left(D_{12} w_{,\xi\xi} + \frac{a^2}{b^2} D_{22} w_{,\eta\eta}\right) \delta w_{,\eta\eta}\right] d\xi d\eta \\
\delta W_e &= \int_{-1}^1 \int_{-1}^1 \left[\left(p_{x0} + p_{x1}\eta\right) \frac{b}{a} w_{,\xi} \delta w_{,\xi} + p_{xy}\left(w_{,\eta} \delta w_{,\xi} + w_{,\xi} \delta w_{,\eta}\right) + \left(p_{y0} + p_{y1}\xi\right) \frac{a}{b} w_{,\eta} \delta w_{,\eta}\right] d\xi\, d\eta \\
&\quad + \int_{-1}^1 \frac{8}{ab^2} \left[\theta_s^0 \left(w_{,\xi\xi\eta} \delta w_{,\eta}\right)\big|_{\eta=-1} + \theta_s^b \left(w_{,\xi\xi\eta} \delta w_{,\eta}\right)\big|_{\eta=1}\right] d\xi \\
&\quad - \int_{-1}^1 \frac{32}{a^3 b^2} \left[\varphi_s^0 \left(w_{,\xi\xi\xi\xi\eta} \delta w_{,\eta}\right)\big|_{\eta=-1} + \varphi_s^b \left(w_{,\xi\xi\xi\xi\eta} \delta w_{,\eta}\right)\big|_{\eta=1}\right] d\xi \\
&\quad + \int_{-1}^1 \frac{8}{a^2 b} \left[\theta_f^0 \left(w_{,\xi\eta\eta} \delta w_{,\xi}\right)\big|_{\xi=-1} + \theta_f^a \left(w_{,\xi\eta\eta} \delta w_{,\xi}\right)\big|_{\xi=1}\right] d\eta \\
&\quad - \int_{-1}^1 \frac{32}{a^2 b^3} \left[\varphi_f^0 \left(w_{,\xi\eta\eta\eta\eta} \delta w_{,\xi}\right)\big|_{\xi=-1} + \varphi_f^a \left(w_{,\xi\eta\eta\eta\eta} \delta w_{,\xi}\right)\big|_{\xi=1}\right] d\eta.
\end{aligned} \tag{8}$$

The solution of the buckling problem is approximately sought by the Ritz method in the form

$$w(\xi,\eta) \approx \sum_{i=1}^m \sum_{j=1}^n w_{ij} X_i(\xi) Y_j(\eta) = \sum_{i=1}^m \lfloor\ w_{i1} \ \ \dots \ \ w_{in}\ \rfloor X_i \lfloor\ Y_1 \ \ \dots \ \ Y_n\ \rfloor^T = \sum_{i=1}^m \{w_i\}^T X_i \{Y\}. \tag{9}$$

Substitution of Eq. (9) into Eq. (4) leads to

$$\begin{aligned}
\delta W_i &= -\sum_{i=1}^{m}\sum_{k=1}^{m}\{\delta w_k\}^T[R_w]_{ki}\{w_i\} \\
\delta W_e &= \sum_{i=1}^{m}\sum_{k=1}^{m}\{\delta w_k\}^T\left([R_p]_{ki}+[R_\theta]_{ki}-[R_\varphi]_{ki}\right)\{w_i\}
\end{aligned} \tag{10}$$

with

$$\begin{aligned}
[R_w]_{ki} &= \frac{4}{ab}\left(\frac{b^2}{a^2}D_{11}\left[\Delta^{2200}\right]_{ki}+\frac{a^2}{b^2}D_{22}\left[\Delta^{0022}\right]_{ki}+4D_{66}\left[\Delta^{1111}\right]_{ki}+D_{12}\left(\left[\Delta^{2002}\right]_{ki}+\left[\Delta^{0220}\right]_{ki}\right)\right) \\
[R_p]_{ki} &= p_{x0}\frac{b}{a}\left[\Delta^{1100}\right]_{ki}+p_{x1}\frac{b}{a}\left[\bar{\Delta}^{1100}\right]_{ki}+p_{xy}\left(\left[\Delta^{1001}\right]_{ki}+\left[\Delta^{0110}\right]_{ki}\right)+p_{y0}\frac{a}{b}\left[\Delta^{0011}\right]_{ki}+p_{y1}\frac{a}{b}\left[\bar{\Delta}^{0011}\right]_{ki} \\
[R_\theta]_{ki} &= \frac{8}{ab^2}\left(\theta_s^0\left[\Theta_s(-1)\right]_{ki}+\theta_s^b\left[\Theta_s(1)\right]_{ki}\right)+\frac{8}{a^2b}\left(\theta_f^0\left[\Theta_f(-1)\right]_{ki}+\theta_f^a\left[\Theta_f(1)\right]_{ki}\right) \\
[R_\varphi]_{ki} &= \frac{32}{a^3b^2}\left(\varphi_s^0\left[\Phi_s(-1)\right]_{ki}+\varphi_s^b\left[\Phi_s(1)\right]_{ki}\right)+\frac{32}{a^2b^3}\left(\varphi_f^0\left[\Phi_f(-1)\right]_{ki}+\varphi_f^a\left[\Phi_f(1)\right]_{ki}\right)
\end{aligned} \tag{11}$$

where

$$\begin{aligned}
\left[\Delta^{pqrs}\right]_{ki} &= \int_{-1}^{1}\frac{d^pX_k}{d\xi^p}\frac{d^qX_i}{d\xi^q}\,d\xi\int_{-1}^{1}\frac{d^r\{Y\}}{d\eta^r}\frac{d^s\{Y\}^T}{d\eta^s}\,d\eta \\
\left[\bar{\Delta}^{1100}\right]_{ki} &= \int_{-1}^{1}X_k'X_i'\,d\xi\int_{-1}^{1}\{Y\}\{Y\}^T\eta d\eta \\
\left[\bar{\Delta}^{0011}\right]_{ki} &= \int_{-1}^{1}X_kX_i\xi\,d\xi\int_{-1}^{1}\{Y'\}\{Y'\}^T d\eta
\end{aligned} \tag{12}$$

and

$$\begin{aligned}
\left[\Theta_s(\bar{\eta})\right]_{ki} &= \int_{-1}^{1}X_kX_i''\,d\xi\,\{Y'(\bar{\eta})\}\{Y'(\bar{\eta})\}^T \qquad \left[\Theta_f(\bar{\xi})\right]_{ki} = X_k'(\bar{\xi})\,X_i'(\bar{\xi})\int_{-1}^{1}\{Y\}\{Y''\}^T d\eta \\
\left[\Phi_s(\bar{\eta})\right]_{ki} &= \int_{-1}^{1}X_kX_i''''\,d\xi\,\{Y'(\bar{\eta})\}\{Y'(\bar{\eta})\}^T \qquad \left[\Phi_f(\bar{\xi})\right]_{ki} = X_k'(\bar{\xi})\,X_i'(\bar{\xi})\int_{-1}^{1}\{Y\}\{Y''''\}^T d\eta.
\end{aligned} \tag{13}$$

Since the components of $\{\delta w_k\}$ are arbitrary and independent, the discretized version of the buckling problem is given by

$$\left([R_w]-[R_\theta]+[R_\varphi]-[R_p]\right)\{w\}=\{0\} \tag{14}$$

with

$$\begin{aligned}
[R_w] &= \frac{4}{ab}\left(\frac{b^2}{a^2}D_{11}\left[\Delta^{2200}\right]+\frac{a^2}{b^2}D_{22}\left[\Delta^{0022}\right]+4D_{66}\left[\Delta^{1111}\right]+D_{12}\left(\left[\Delta^{2002}\right]+\left[\Delta^{0220}\right]\right)\right. \\
&\qquad \left.+\frac{2a}{b}D_{26}\left(\left[\Delta^{1012}\right]+\left[\Delta^{0121}\right]\right)+\frac{2b}{a}D_{16}\left(\left[\Delta^{2101}\right]+\left[\Delta^{1210}\right]\right)\right) \\
[R_\theta] &= \frac{8}{ab^2}\left(\theta_s^0\left[\Theta_s(-1)\right]+\theta_s^b\left[\Theta_s(1)\right]\right)+\frac{8}{a^2b}\left(\theta_f^0\left[\Theta_f(-1)\right]+\theta_f^a\left[\Theta_f(1)\right]\right) \\
[R_\varphi] &= \frac{32}{a^3b^2}\left(\varphi_s^0\left[\Phi_s(-1)\right]+\varphi_s^b\left[\Phi_s(1)\right]\right)+\frac{32}{a^2b^3}\left(\varphi_f^0\left[\Phi_f(-1)\right]+\varphi_f^a\left[\Phi_f(1)\right]\right) \\
[R_p] &= p_{x0}\frac{b}{a}\left[\Delta^{1100}\right]+p_{x1}\frac{b}{a}\left[\bar{\Delta}^{1100}\right]+p_{xy}\left(\left[\Delta^{1001}\right]+\left[\Delta^{0110}\right]\right)+p_{y0}\frac{a}{b}\left[\Delta^{0011}\right]+p_{y1}\frac{a}{b}\left[\bar{\Delta}^{0011}\right] \\
\{w\} &= \lfloor\ \lfloor w_1\rfloor\quad\lfloor w_2\rfloor\quad\ldots\quad\lfloor w_m\rfloor\ \rfloor^T
\end{aligned} \tag{15}$$

where

$$
\begin{aligned}
\left[\Delta^{pqrs}\right] &= \begin{bmatrix} [\Delta^{pqrs}]_{11} & \cdots & [\Delta^{pqrs}]_{1m} \\ \vdots & & \vdots \\ [\Delta^{pqrs}]_{m1} & \cdots & [\Delta^{pqrs}]_{mm} \end{bmatrix} \\
[\Theta_s(\bar{\eta})] &= \begin{bmatrix} [\Theta_s(\bar{\eta})]_{11} & \cdots & [\Theta_s(\bar{\eta})]_{1m} \\ \vdots & & \vdots \\ [\Theta_s(\bar{\eta})]_{m1} & \cdots & [\Theta_s(\bar{\eta})]_{mm} \end{bmatrix} \qquad [\Theta_f(\bar{\eta})] = \begin{bmatrix} [\Theta_f(\bar{\eta})]_{11} & \cdots & [\Theta_s(\bar{\eta})]_{1m} \\ \vdots & & \vdots \\ [\Theta_f(\bar{\eta})]_{m1} & \cdots & [\Theta_s(\bar{\eta})]_{mm} \end{bmatrix} \\
[\Phi_s(\bar{\eta})] &= \begin{bmatrix} [\Phi_s(\bar{\eta})]_{11} & \cdots & [\Phi_s(\bar{\eta})]_{1m} \\ \vdots & & \vdots \\ [\Phi_s(\bar{\eta})]_{m1} & \cdots & [\Phi_s(\bar{\eta})]_{mm} \end{bmatrix} \qquad \left[\Phi_f(\bar{\xi})\right] = \begin{bmatrix} \left[\Phi_f(\bar{\xi})\right]_{11} & \cdots & \left[\Phi_f(\bar{\xi})\right]_{1m} \\ \vdots & & \vdots \\ \left[\Phi_f(\bar{\xi})\right]_{m1} & \cdots & \left[\Phi_f(\bar{\xi})\right]_{mm} \end{bmatrix} \\
\left[\bar{\Delta}^{1100}\right] &= \begin{bmatrix} \left[\bar{\Delta}^{1100}\right]_{11} & \cdots & \left[\bar{\Delta}^{1100}\right]_{1m} \\ \vdots & & \vdots \\ \left[\bar{\Delta}^{1100}\right]_{m1} & \cdots & \left[\bar{\Delta}^{1100}\right]_{mm} \end{bmatrix} \qquad \left[\bar{\Delta}^{0011}\right] = \begin{bmatrix} \left[\bar{\Delta}^{0011}\right]_{11} & \cdots & \left[\bar{\Delta}^{0011}\right]_{1m} \\ \vdots & & \vdots \\ \left[\bar{\Delta}^{0011}\right]_{m1} & \cdots & \left[\bar{\Delta}^{0011}\right]_{mm} \end{bmatrix}.
\end{aligned}
\tag{16}
$$

Equation (14) can be written as the linear eigenvalue problem

$$([R_c] - \lambda [R_p]) \{w\} = \{0\}, \tag{17}$$

with the constitutive stiffness matrix

$$[R_c] = [R_w] - [R_\theta] + [R_\varphi] \tag{18}$$

and the geometric stiffness matrix $[R_p]$ varying with the loading type.

4. NUMERICAL RESULTS

The mode functions $X_i(\xi) = F_i(\xi)$ and $Y_j(\eta) = F_j(\eta)$ in Eq. (9) are taken as the hierarchical polynomial set

$$
\begin{aligned}
F_1(\xi) &= \frac{1}{2} - \frac{3}{4}\xi + \frac{1}{4}\xi^3 \qquad F_2(\xi) = \frac{1}{8} - \frac{1}{8}\xi - \frac{1}{8}\xi^2 + \frac{1}{8}\xi^3 \\
F_3(\xi) &= \frac{1}{2} + \frac{3}{4}\xi - \frac{1}{4}\xi^3 \qquad F_4(\xi) = -\frac{1}{8} - \frac{1}{8}\xi + \frac{1}{8}\xi^2 + \frac{1}{8}\xi^3
\end{aligned}
\tag{19}
$$

and

$$F_i(\xi) = \sum_{n=0}^{(i-1)/2} \frac{(-1)^n}{2^n n!} \frac{(2i-2n-7)!!}{(i-2n-1)!} \xi^{i-2n-1} \qquad i > 4 \tag{20}$$

where $i!! = i(i-2)\ldots(2 \text{ or } 1)$, $0!! = (-1)!! = 1$, $(i-1)/2$ denotes its own integer part (Bardell, 1991). The first four modes are identical to the Hermite cubics and control the boundary conditions on displacement and rotation at $\xi = \pm 1$, while the higher order modes ($i > 4$) have been generated from Legendre polynomials and possess both zero displacement and zero slope at each end. This features are significant since the higher modes contribute only to the internal displacement field, and all the classical geometric boundary conditions could be matched just by removing some of the first four basis functions. In the same manner as is done on finite element procedures, the boundary conditions may be introduced a posteriori into Eq. (17).

We have used symbolic mathematics to evaluate the integrals in Eqs. (12) and (13), and written a numerical code in MATLAB language to solve the linear eigenvalue problem. The following plate data are required: geometry and material properties, boundary conditions (free, simply supported, clamped), and loading pattern. If a stiffener is assigned to a particular edge, then the edge condition differs from the simply supported one in the sense that the torsion resisted by the stiffener is also taken into account. Stiffener data will then be required.

A number of studies have been conducted using several combinations of boundary conditions and loading in order to verify the behavior of the proposed methodology. The following data were used in all of the computations:

$$
\begin{aligned}
a &= 1000\,\text{mm} \qquad D_{11} = D_{22} = D_{12} + 2D_{66} = 1.92 \times 10^7\,\text{N}\,\text{mm} \\
\theta_f^0 &= \theta_f^a = \theta_s^0 = \theta_s^b = 4.26 \times 10^6\,\text{N}\,\text{mm}^2 \\
\varphi_f^0 &= \varphi_f^a = \varphi_s^0 = \varphi_s^b = 2.69 \times 10^9\,\text{N}\,\text{mm}^4.
\end{aligned}
\tag{21}
$$

For purpose of description, the boundary conditions at the plate edges are denoted by: SSSS (all the edges are simply supported), SSCF (edges are simply supported at $x = 0, a$, clamped at $y = 0$ and free at $y = b$), SSCC (edges are simply supported at $x = 0, a$ and clamped at $y = 0, b$), CCCC (all the edges are clampled). All the analyzed plates are subject to one of the following loading: axial compression p_{x0}, biaxial compression with $p_{x0} = p_{y0}$, shear p_{xy}, in-plane bending p_{x1}, in-plane bending p_{y1}. Table 1 shows the effect of plate aspect ratio and the loading pattern on the critical buckling loads $k = pa^2/\pi^2 D_{11}$ considering $m = n = 20$ number of terms, the quantity p stands for the associated load type. For comparison purpose, the accuracy of the proposed formulation is compared with a more rigorous numerical procedure provided by the finite element commercial code NASTRAN, differences from finite element results are tabled in parentheses. Table 2 shows the effect of plate aspect ratio and the loading pattern on the critical buckling loads now considering a SSSS plate with all the edges reinforced by stiffeners.

Table 1. Effect of plate aspect ratio and the loading pattern on the critical buckling loads k.

	a/b	axial	biaxial	shear	x-bending	y-bending
SSSS	1.0	4.000 (0.01%)	2.000 (0.01%)	9.325 (-0.06%)	25.528 (0.01%)	25.528 (0.01%)
	1.5	9.766 (0.03%)	3.250 (0.01%)	15.907 (-0.07%)	54.252 (0.02%)	23.882 (-0.02%)
	2.0	16.000 (0.06%)	5.000 (0.01%)	26.184 (-0.08%)	95.527 (0.08%)	25.528 (-0.01%)
	2.5	25.840 (0.08%)	7.250 (0.01%)	37.706 (-0.07%)	149.871 (0.14%)	29.099 (0.00%)
	3.0	36.000 (0.12%)	10.000 (0.00%)	52.561 (-0.07%)	217.006 (0.19%)	33.817 (0.00%)

	a/b	axial	biaxial	shear	x-bending	y-bending
SSCF	1.0	1.653 (0.02%)	1.144 (0.03%)	4.966 (0.00%)	2.791 (0.03%)	12.837 (0.06%)
	1.5	2.905 (0.05%)	1.420 (0.05%)	6.458 (0.02%)	4.868 (0.05%)	13.090 (0.06%)
	2.0	5.344 (0.07%)	1.835 (0.06%)	9.148 (0.03%)	8.865 (0.07%)	13.454 (0.06%)
	2.5	8.658 (0.06%)	2.382 (0.07%)	13.031 (0.02%)	14.587 (0.07%)	14.122 (0.06%)
	3.0	11.621 (0.09%)	3.060 (0.07%)	17.979 (0.01%)	19.473 (0.11%)	15.154 (0.06%)

	a/b	axial	biaxial	shear	x-bending	y-bending
SSCC	1.0	7.691 (0.05%)	3.830 (0.00%)	12.565 (-0.07%)	39.672 (0.08%)	31.997 (-0.02%)
	1.5	16.011 (0.09%)	8.477 (-0.02%)	24.259 (-0.09%)	89.262 (0.14%)	38.899 (-0.03%)
	2.0	27.886 (0.13%)	15.299 (-0.04%)	40.027 (-0.09%)	158.687 (0.20%)	50.148 (-0.03%)
	2.5	43.743 (0.16%)	23.492 (0.02%)	60.262 (-0.10%)	247.949 (0.27%)	64.483 (-0.03%)
	3.0	63.497 (0.18%)	33.908 (0.01%)	85.334 (-0.11%)	357.047 (0.34%)	81.427 (-0.03%)

	a/b	axial	biaxial	shear	x-bending	y-bending
CCCC	1.0	10.074 (0.01%)	5.304 (0.00%)	14.642 (-0.07%)	47.754 (0.06%)	47.754 (0.06%)
	1.5	18.789 (0.03%)	9.273 (-0.01%)	25.781 (-0.10%)	97.341 (0.10%)	56.663 (0.03%)
	2.0	31.468 (0.06%)	15.694 (-0.03%)	40.992 (-0.10%)	166.582 (0.16%)	65.872 (0.02%)
	2.5	47.332 (0.12%)	24.332 (-0.03%)	61.621 (-0.11%)	255.554 (0.22%)	79.417 (0.03%)
	3.0	66.234 (0.16%)	34.745 (0.03%)	85.809 (-0.11%)	364.310 (0.28%)	96.231 (0.03%)

Table 2. SSSS plate with all the edges reinforced by stiffeners.

a/b	axial	biaxial	shear	x-bending	y-bending
1.0	4,001 (-0,01%)	2,000 (-0,01%)	9,326 (-0,08%)	25,534 (-0,01%)	25,534 (-0,01%)
1.5	9,768 (0,00%)	3,251 (-0,01%)	15,910 (-0,08%)	54,267 (0,00%)	23,888 (-0,05%)
2.0	16,005 (0,02%)	5,001 (-0,01%)	26,188 (-0,09%)	95,563 (0,05%)	25,537 (-0,04%)
2.5	25,850 (0,04%)	7,251 (-0,01%)	37,714 (-0,09%)	149,940 (0,10%)	29,109 (-0,04%)
3.0	36,017 (0,08%)	10,002 (-0,01%)	52,573 (-0,09%)	217,117 (0,15%)	33,830 (-0,03%)

5. CONCLUSIONS

The following conclusions are drawn from the results:

(a) the implemented Ritz procedure using a specialized Legendre hierachical basis functions and finite element solutions compare well;

(b) the influence of the rotational restraint upon the buckling load has insignificant effect for the adopted geometry properties;

(c) the proposed methodology provides an efficient and fast tool to predict linear buckling loads of rectangular isotropic plates under any combination of classical boundary conditions, including or not attached stiffeners, and subjected to typical in-plane loading.

6. REFERENCES

Bardell, N.S., 1991. "Free vibration analysis of a flat plate using the hierarquical finite element method". *Journal of Sound and Vibration*, Vol. 151, pp. 263–289.

Bedair, O.K. and Sherbourne, A.N., 1993. "Plate/stiffener assemblies in uniform compression: Part I – buckling". *Journal of Engineering Mechanics (ASCE)*, Vol. 119, pp. 1937–1955.

Bedair, O.K. and Sherbourne, A.N., 1995. "Unified approach to the local stability of plate/stiffener assemblies". *Journal of Engineering Mechanics (ASCE)*, Vol. 121, pp. 214–229.

Bisagni, C. and Vescovini, R., 2009. "Analytical formulation for local buckling and post-buckling analysis of stiffened laminated panels". *Thin-Walled Structures*, Vol. 47, pp. 318–334.

Kollbrunner, C.F. and Basler, K., 1969. *Torsion in structures*. Springer, Berlin.

Paik, J.K. and Thayamballi, A.K., 2000. "Buckling strength of steel plating with elastically restrained edges". *Thin-Walled Structures*, Vol. 37, pp. 27–55.

Sherbourne, A.N. and Bedair, O.K., 1993. "Plate/stiffener assemblies in uniform compression: Part II – post-buckling". *Journal of Engineering Mechanics (ASCE)*, Vol. 119, pp. 1956–1972.

7. RESPONSIBILITY NOTICE

The authors are the only responsible for the printed material included in this paper.