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STOCHASTIC MODELING OF FATIGUE CRACK GROWTH IN 2024 T351 ALUMINUM ALLOY

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Abstract. *In this study, the stochastic simulation of fatigue crack propagation was performed for specimens made of a widely used structural aluminum alloy, specified as AA 2024 T351. The proposed modelling tool uses irregularity of crack advances observed in empirical studies as a key feature of the process. This approach is aimed at prediction of crack propagation curves scattering, usually observed in repeated tests under similar conditions. The model is based at a non-linear stochastic differential equation that includes two terms. The first one represents the predominant trend in the process, and the second one simulates the "noise" of a random nature. The fitting constants obtained using one experimental curve representing the average crack behavior were applied for generation of fifty predictions of crack growth from the same initial length and under same cyclic loading. The simulated scatter was compared with two sets of experimental data collected from technical literature. The model was able to reproduce well the real data scattering. However, random choice of basic experimental curve may produce less reliable simulations, including conservative predictions. Therefore, a discussion about an averaging process of the experimental data necessary for realistic scattering prediction is introduced in the paper.*

Keywords: *Fatigue Crack Growth, Modeling, Stochastic Differential Equation, Aluminum Alloy*

1. INTRODUCTION

Although the stochastic nature of metal fatigue is for long recognized and statistical tools are ordinarily applied to fatigue analysis by the stress-life approach, most of the conventional fracture mechanics applications for fatigue crack growth (FCG) are deterministic. In particular, the classical “Forman equation” is the most employed model in structural engineering (Forman *et al.*, 1967; Stephens, *et al.*, 2001). The sub-critical crack propagation in metallic sheets is a complex phenomenon and the experimental investigations necessary for the definition of the inspection methods and intervals in damage tolerant structures present significant methodological difficulties and high cost. In general, the studies are focused on the effect of one or more parameters on the resistance to crack nucleation and on the FCG rate in a mechanical component. However, when several identical specimens cut from the same sheet are tested in the same loading regime, a significant scattering is observed in the results, exposing the probabilistic nature of FCG.

Thus, it is important to analyze FCG data from a statistical viewpoint in order to take in account the scatter. Among the many efforts carried out in this aspect, the paper by Virkler *et al.* (1979) is perhaps one of the definitive studies illustrating the variability in crack growth rates. More recent contributions to this subject can also be cited. The work by Wu and Ni (2007) presented the results of tests performed with a considerable amount of specimens made of 2024 T351 aluminum alloy. According to these authors, it was confirmed that the inhomogeneity of material is the cause of the scatter observed in FCG data. Moreover, the obtained datasets were used for verification of various probabilistic crack growth models. The randomized Paris-Erdogan and polynomial stochastic crack growth models were considered more appropriate to describe the FCG data presented in that paper.

Recently, Jones *et al.* (2014) discussed the factors that lead to scatter in the fatigue lives of nominally identical monolithic metallic specimens. Their paper has concentrated on the contribution to scatter due to variations in the cyclic threshold Stress Intensity Factor, and the authors concluded that the study of the scatter shown by cracks in operational aircraft growth requires tests involving cracks that grow from small naturally occurring material discontinuities. The Lead crack fatigue lifing framework, developed by the Defence Science and Technology Organisation (DSTO) in Australia (Molent *et al.*, 2011), had hitherto been on this track by using data for real cracks instead of assuming initial crack sizes and deriving early crack growth behavior from back-extrapolation of data for long cracks. The Lead Crack framework states that, if a particular region of a structure has the propensity to crack, it is possible that a number of cracks will nucleate and grow. The Lead crack is the crack that grows fastest in this region and its growth may be represented by an exponential equation.

Previous works on FCG analysis from a statistical point of view were also published by our research group; some of which are mentioned below. First, a predictive model of FCG that considers the growth of a main crack as a sequence of failures of damage accumulating volume elements along the crack path was developed (Baptista and Pastoukhov, 2003). It was assumed that the scattering of the stress-life data could be reproduced in the volume elements ahead of the crack, thus allowing probabilistic predictions to be performed. Experimental results of crack growth tests performed with commercial purity titanium specimens showed good agreement with those obtained from computer simulations. The next effort (Siqueira *et al.* 2007) explored the possibilities of two promising methods, namely the Artificial Neural Networks and Stochastic Differential Equations (SDE), in determining the residual life and the crack growth history taking in account the observed scattering of the data. In a joint paper using DSTO data, Siqueira *et al.* (2013) presented a method of estimating the scatter factor (SF) for fatigue lives based on the statistical distribution of nucleated cracks in an aluminium alloy critical airframe location. Finally, a recent work (Siqueira *et al.*, 2014) proposed the use of a non-linear Stochastic Differential Equation (SDE) to simulate the propagation of cracks and to predict its scattering under constant amplitude loading. The SDE parameters estimation was performed with basis on the distribution of the instantaneous FCG rate taken as a finite difference. The process simulations have shown good agreement with the available data for commercially pure titanium.

The aim of the present investigation is to introduce an auxiliary method for assessing the scattering of the FCG data with basis on a reduced number of tests. The starting point is the variability in the crack growth process itself, which actually shows local advances and pauses, and does not present a continuous acceleration as suggested by the conventional kinetic equations. The main idea is to fit FCG data, in which the crack size is related to the number of loading cycles, by using a non-linear SDE. This approach allows for the probabilistic simulation of FCG curves, including the mean data and a computational confidence interval, with the use of only one FCG dataset, or a reduced number of test results. The FCG data used in this work, previously published by Wu and Ni (2007), refer to constant and variable amplitude loading tests in samples made of 2024 T351 aluminum alloy.

2. DEVELOPMENT

Some differential equations have a noise term and, by this reason, are termed Stochastic Differential Equations. These equations, also known as Langevin equations, are important tools for the analysis of phenomena showing a certain amount of scattering (Scherer, 2005) and have applications in many fields. There is a similarity between a classical differential equation and a SDE, although the latter allows for adding some randomness coefficients. Furthermore, some rules of the Stochastic Calculus, employed in the interpretation and solution of SDEs, are different from those of the usual calculus (Riemann integrals). In the previous paper (Siqueira *et al.*, 2014) it was verified that a non-linear SDE can be employed to simulate the rate of growth of a fatigue crack in metallic materials. In the present work, the non-linear SDE is adopted for the simulation of the crack propagation curves in terms of crack size *versus* number of cycles for either constant amplitude or random loading conditions.

2.1 Stochastic model

The so-called "Wiener process" or $W(t)$ is a continuous-time stochastic process that gives a mathematical description of the Brownian motion. According to Durrett (1996), this process can be defined as a continuous Gaussian process (i.e., a stochastic process defined by its mean and covariance functions) having independent increments and the conditions: $E(W(t)) = 0$ and $Var(W(t) - W(s)) = t - s$. Besides, the probability distribution of $W(t) - W(s)$ is a Normal distribution with zero mean and variance of $(t - s)$. The Langevin equations have a term that would be a derivative of a Wiener process. However, it is known that the Wiener process is nowhere differentiable; thus, it requires its own rules of calculus (Bichteler, 2002). There are two dominating versions of stochastic calculus, proposed respectively by K. Itô and R. L. Stratonovich. The Itô and Stratonovich integrals not only apply to different classes of integrands, but also give different results when applied to the same integrand. The choice between the two integrals depends on the purpose (Bichteler, 2002). For the present work, the Itô's non-linear SDE is chosen, as shown in Eq. (1).

$$dX_t = aX_t^b + cX_t dW_t \tag{1}$$

In the proposed model, X_t is the crack length (in millimeters), (a, b, c) are the fitting parameters, dW_t is a vector of Wiener process and t is the time, considered in terms of the number of load cycles for analysis of fatigue crack growth. Since loading and subsequent unloading is a physically continuous process and current number of cycles is a real and not a natural number, t plays a role of continuous variable in the modeling and simulation processes. One of the main challenges of the stochastic modeling is the estimation of the fitting parameters of the equation employing available experimental data (Erdoglu *et al.*, 2012; Kleppe *et al.*, 2014). Based on the method proposed by Kelly *et al.* (2004), reasonable estimators of (a, b, c) are obtained by using Eq. (2), Eq. (3) and Eq. (4). In these equations, (t_i, x_i) represent the number of cycles and crack length of the *i-th* experimental point. The *er* number, defined by Eq. (4), is used for estimation of parameter *b*.

$$a = \frac{\sum_{i=1}^{n-1} \frac{x_{i+1} - x_i}{t_{i+1} - t_i}}{\sum_{i=1}^{n} x_i^b} \tag{2}$$

$$c = \sqrt{\frac{\sum_{i=1}^{n-1} \frac{\left(x_{i+1}^{0.5} - x_i^{0.5}\right)^2}{t_{i+1} - t_i}}{\sum_{i=1}^{n-1} x_{i+1} - x_i}} \tag{3}$$

$$er = \left| \frac{\sum_{i=1}^{n} x_i^b}{\sum_{i=1}^{n} x_i^{b-0.5}} - \frac{\sum_{i=1}^{n-1} \frac{x_{i+1} - x_i}{t_{i+1} - t_i}}{2\sum_{i=1}^{n-1} \frac{\left(x_{i+1} - x_i\right)^{0.5}}{t_{i+1} - t_i} + \frac{c^2}{4} \sum_{i=1}^{n} x_i^{0.5}} \right| \tag{4}$$

The procedure used to estimate the fitting parameters *(a, b, c)* is defined by the following algorithm:
Step One: Calculate *c* using Eq. (3);
Step Two: Find *b* such that *er* given by Eq. (4) has order of magnitude of 10^{-3};
Step Three: Calculate *a* using Eq. (2).

2.2 Experimental data

The FCG experimental data employed in this work were originally published by Wu and Ni (2007) and were obtained using compact tension (CT) specimens cut from a 2024-T351 aluminum alloy plate. Two sets of experiments are used in the present work. The constant-amplitude (CA) loading consists of 30 specimens tested under sinusoidal loading with maximum stress 70.41 MPa and stress ratio R = 0.2. The variable-amplitude (VA) loading consists of 25 specimens tested under random loading generated in accordance to the spectral theory and oscillating, in terms of mean value, between the minimum stress of 60.74 MPa and the maximum 95.72 MPa. The experimental datasheets of these tests were kindly provided to us by the authors.

The available data are given with precision of 0.01 mm in terms of crack length. For some of the simulations performed in this work, the average of two experimental curves was considered. The average curves of any two experimental ones were calculated point to point in order to determine a mean trend of both, looking at maintenance of typical behavior of original data, and using three basic techniques. For point of two curves with exactly the same crack length, the average of number of cycles was adopted; for points with very similar crack length, the average crack length and average number of cycles. Some additional values of "number of cycles *versus* crack length" were created through linear interpolation, providing pairs at same crack length for points available in other curve. Thus, the average of the existing point and such result of interpolation was calculated in order to complete these sets until typical number of points in each experimental curve is achieved.

2.3 Simulation of crack propagation and confidence intervals

After determining the set of parameters *(a, b, c)* by using one experimental curve or the average of two curves as described in section 2.2, it is possible to obtain numerical simulations of Eq. (1). To do so, a discretization of this equation is necessary, resulting in Eq. (5).

$$X_{i+1} = X_i + aX_t^b \Delta t + cX_i \Delta W_i \tag{5}$$

Here ΔW_i is the increment of the Wiener process, which can be calculated through a normal distribution with zero mean and variance of Δt. Notice that in Eq. (5) all the necessary information to estimate X_{i+1} depend only on the information at moment t_i. The numerical simulations were performed using the freeware "Scilab 5.4.1", a high-level programming language that is available free of charge from: http://www.scilab.org/. In all of the numerical simulations the adopted time increment was Δt = 100. For each set of parameters *(a, b, c)*, 50 FCG simulations were done.

This number was adopted from the finding that increasing the number of simulations did not change the statistical properties of the obtained results. From each set of 50 numerical simulations, an average simulated curve and a computational confidence interval were obtained as follows: at each moment t_i the 50 crack length values were sorted in ascending order. Thus, the envelope was built considering the positions 3 and 48 of the crack length vector at each moment, resulting in a 92% confidence interval.

3. RESULTS AND DISCUSSION

The 30 experimental FCG curves from CA tests are plotted in Fig. 1 as green points, together with the one randomly chosen (blue points) for the estimation of the parameters *(a, b, c)* and the resulting average simulated curve (red points) and confidence interval (black lines). The calculated fitting parameters are: $a = 2.5 \times 10^{-12}$, $b = 5.9$ and $c = 2.29 \times 10^{-4}$.

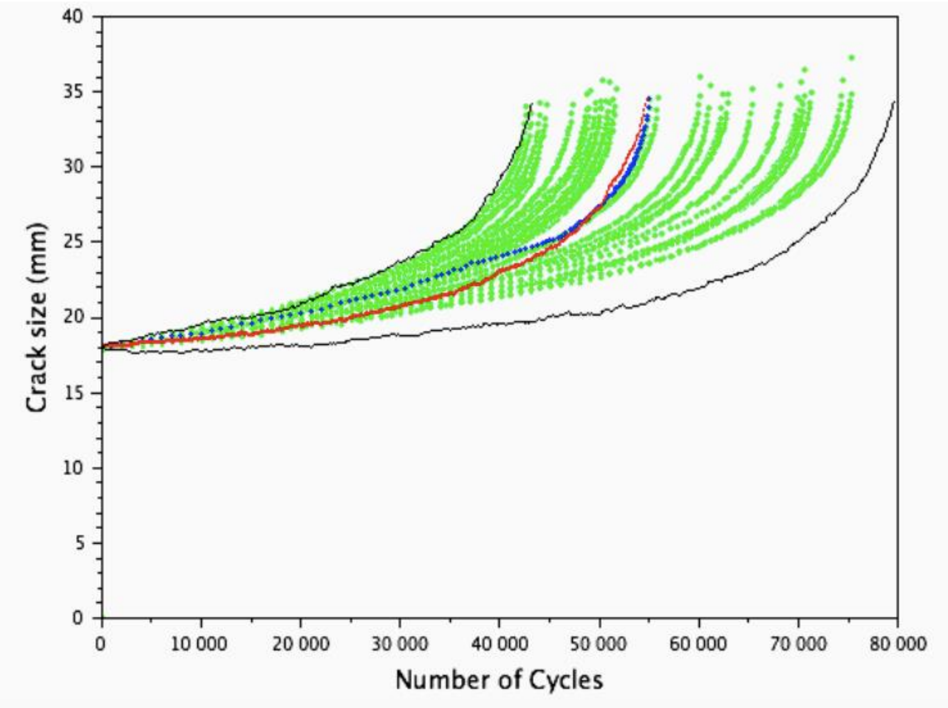

Figure 1. CA FCG curves (green points), the experimental curve adopted for parameter estimation (blue), average simulated curve (red) and computational 92% confidence interval of the simulations (black)

Figure 1 shows that, besides of being able to describe the mean FCG behavior observed in the experiments, the proposed stochastic model makes it possible to reproduce the scattering of the experimental data in the computational simulations by considering the irregularities present in one single experimental curve. This is an encouraging result, since one of the objectives of this work is to assess the scattering of FCG behavior with a reduced number of tests. However, it is evident in Fig. 1 that the "randomly" chosen curve (blue points) is, in fact, one representative of the mean behavior of the experimental points. In order to explore the potential of the proposed model it is necessary to observe what would happen to the results if the available data were, for instance, one of the extreme experimental curves.

Figure 2 and Figure 3 show the obtained results when the experimental curve chosen for parameter estimation is, respectively the fastest one (left extreme) of the slowest one (right extreme). In these figures, the color scheme is the same as used in Fig. 1. For each case, from the 50 performed simulations, the mean behavior correspond to that of the chosen curve, i.e., the model does not reproduce the average material behavior, but the corresponding extreme. It means that, if perhaps the only available data describe an extreme behavior, so will do the model. Another aspect of these plots refers to the confidence intervals. The computational confidence intervals observed in Fig. 2 and Fig. 3 are very un-symmetric, the right (slower growth) extremity being more far from the average simulated curve than the left (faster growth) extremity. Therefore, it is possible to infer that the final width of the confidence interval, in terms of the absolute number of cycles, becomes higher as slower crack growth curves are taken to estimate the fitting parameters of the model. Of course, if the interval width is taken relatively to the mean, it remains approximately the same for all situations. Anyway, if one takes the left extremes of the computational intervals as the lower bound FCG curves, it is verified that, for the fastest experimental curve (Fig. 2), the model is conservative, and for the slowest experimental curve (Fig. 3), the predicted lower bound scattering is very close to the experimental behavior. However, it is evident that, if by chance only one extreme test result is available to the determination of the model parameters, the mean material behavior and corresponding scattering will not be adequately described.

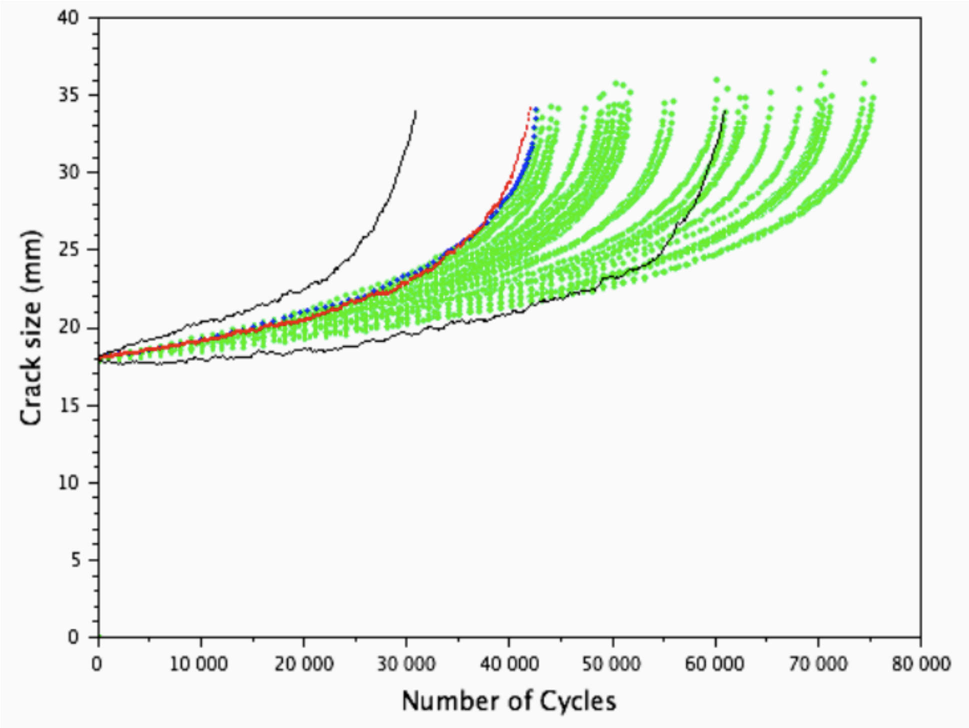

Figure 2. CA FCG data and simulation results when the left extreme curve is adopted for parameter estimation

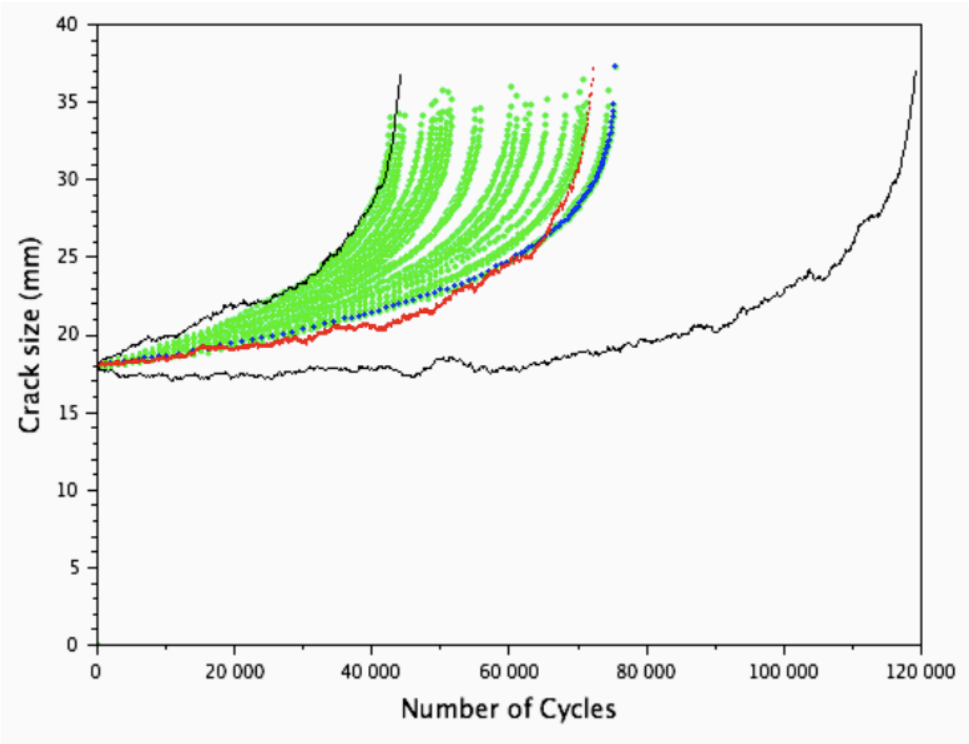

Figure 3. CA FCG data and simulation results when the right extreme curve is adopted for parameter estimation

One way to obtain less conservative predictions is to employ two or more tests for the parameter estimation. In this case, the two or more available results should be previously submitted to a numerical procedure, e.g. for determining an "average" of the experimental curves as described in section 2.2, which would be used for the parameter estimation. This procedure was done for some pairs of experimental curves with promising results. Besides some situations in which the experimental curves used in the procedure were randomly chosen, in three cases the average one was determined from the extreme results, i.e. the fastest and slowest experiment, the second fastest and the last but one curves, and the average from the third fastest and the last but two curves. In all of these analyses, the resulting simulations were compatible with the set of experimental results. As an example, Fig. 4 shows the obtained results when the average of the extreme (fastest and slowest) crack propagation curves was taken for the model parameter estimation. The color scheme is the same adopted for previous figures, except that in this case, the blue points represent the average of the two extreme experiments. It is also noteworthy that the fitting parameters obtained when the average of two curves is employed in the model are consistent to the set of parameters obtained for individual curves. Figure 5 demonstrates this by plotting parameter b against $\ln a$ for the individual experimental curves (blue points) and for average curves (red points), showing that the same trend is followed for both sets of parameters. It should also be noted that a relationship is expected to exist between a and b since the latter is employed in the estimation of the former, as seen in eq. (2).

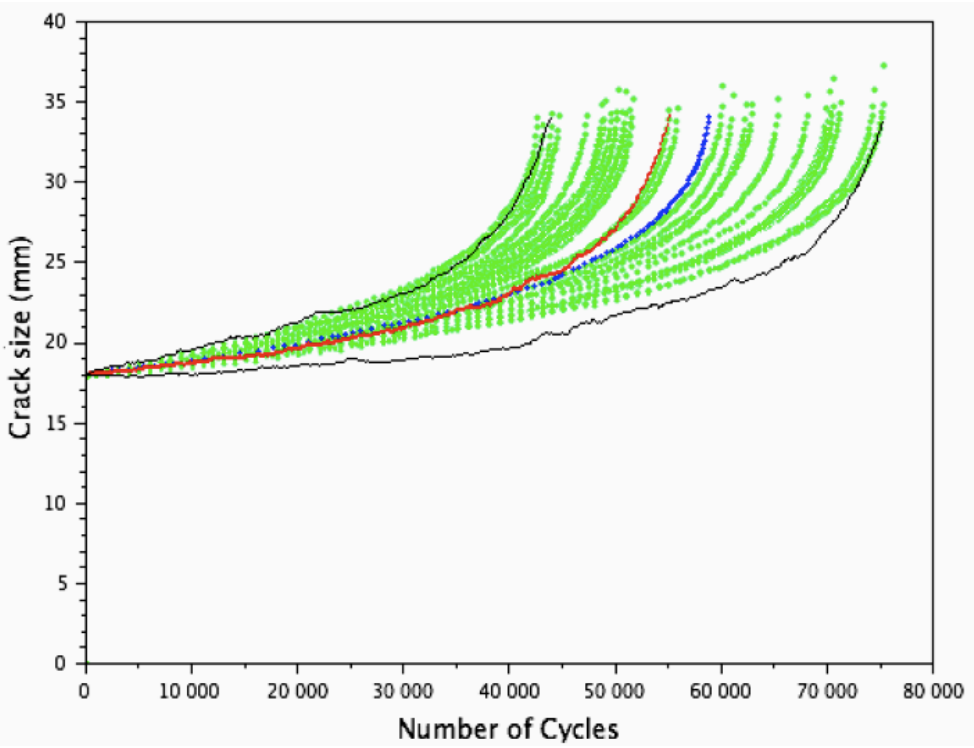

Figure 4. CA FCG data and simulation results when the average of extreme curves is adopted for parameter estimation

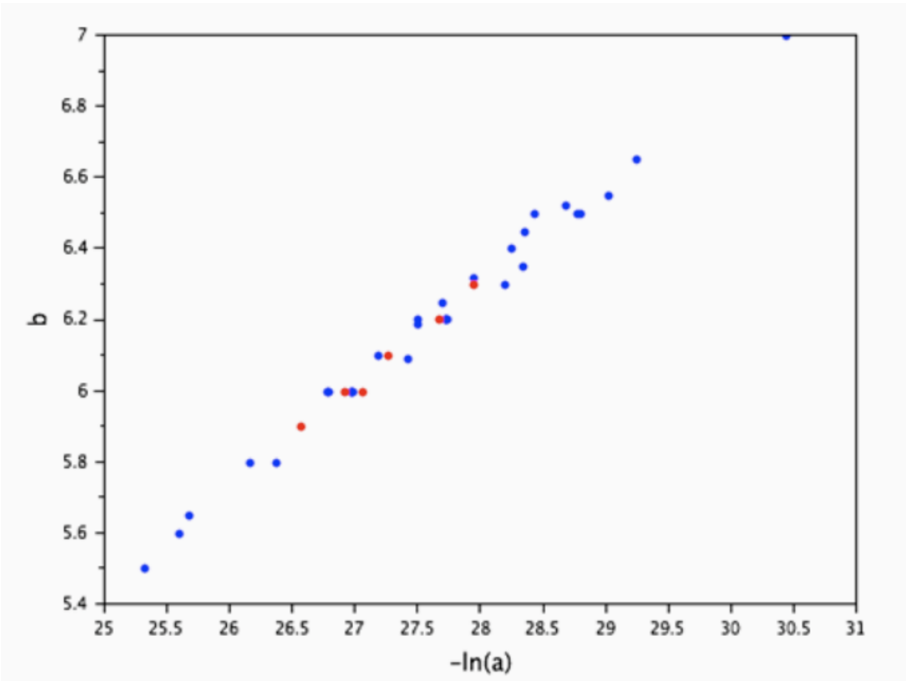

Figure 5. Relationship between a and b obtained from experimental (blue points) and average of two curves (red points)

A suchlike analysis was performed using the variable-amplitude (VA) loading crack growth data. Again, it was observed that when a single curve representing the mean material behavior is employed for the fitting parameters determination, the resulting computational simulations give a suitable description of the average and scattering observed in the experimental results. Moreover, when the average of two curves (determined according section 2.2) is employed for the estimation of fitting parameters *(a, b, c)*, the resulting computational simulations are also successful in describing the experimental results. Figure 6 shows the experimental FCG curves from VA tests (green points) plotted together the average of fastest and slowest experimental curves (blue points) and the resulting mean and computational 92% confidence interval (red points and black lines, respectively). Figure 7 is a similar plot for the case in which the average (blue points) is taken from the second fastest and the slowest but one experimental curves. Also for the VA FCG data, a linear relationship between *b* and *ln a*, similar to that shown in Fig. 5, was verified.

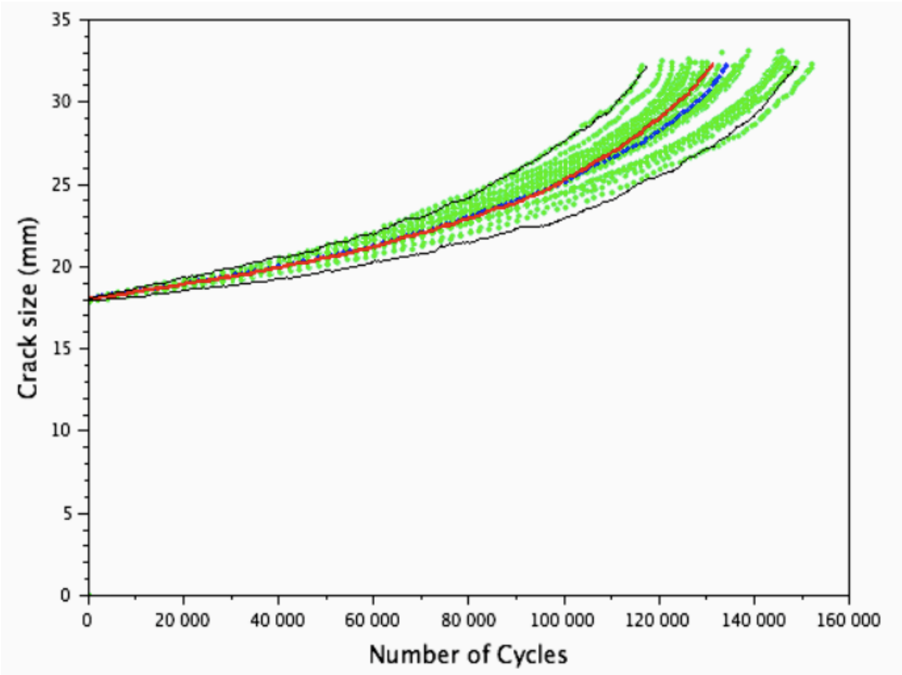

Figure 6. VA FCG data and simulation results when the average of extreme curves is adopted for parameter estimation

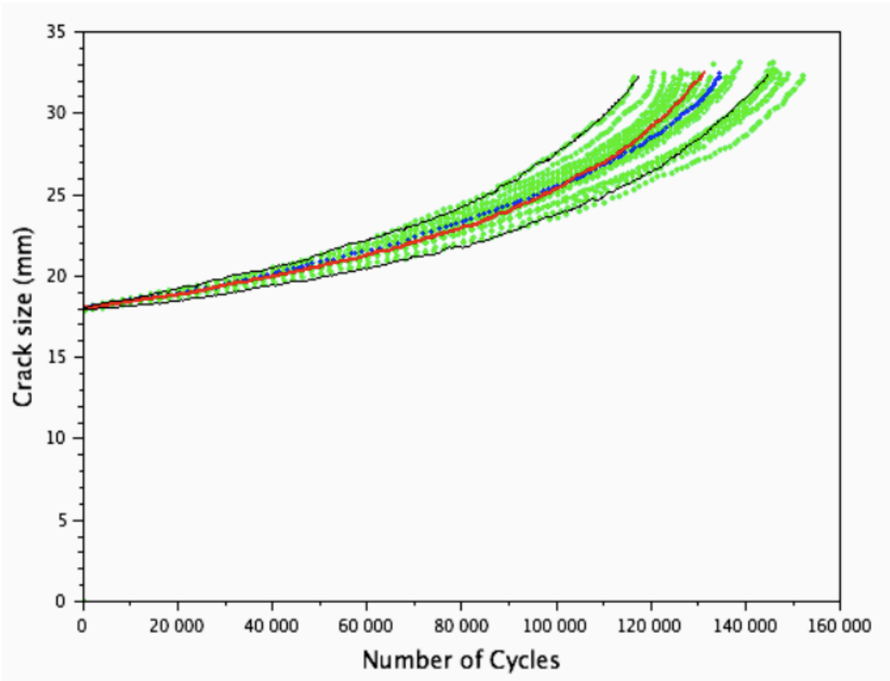

Figure 7. VA FCG data and simulation results when the average of second extreme curves is adopted for parameter estimation

4. CONCLUSION

In this paper, a non-linear SDE was employed in order to assess the scattering of the FCG data with basis on a reduced number of tests. Stochastic simulations of fatigue crack propagation were performed for a widely used structural aluminum alloy, specified as AA 2024 T351, whose FCG data referred to constant and variable amplitude loading were obtained from literature. The proposed modelling tool uses irregularity of crack advances observed in the empirical studies as a key feature of the process. The adopted SDE was able to reasonably reproduce the mean material behavior and the scattering of FCG curves by using information from only one experimental curve. The adopted averaging procedure of two experimental curves was effective in reproducing the scattering as mitigating the observed conservative effect when extreme experimental data were employed in modeling.

5. ACKNOWLEDGEMENTS

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