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LIMIT ANALYSIS WITH PRESCRIBED VELOCITIES AND UNILATERAL CONDITIONS WITH FRICTION APPLIED IN SCRATCH TEST

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Abstract. *The aim of this paper is to propose a mixed limit analysis formulation with prescribed velocities and frictional unilateral conditions in order to analyze problems under plastic collapse and dissipation by friction at a contact interface. Then, the external power supplied by the rigid body is not converted only in plastic dissipation; some part is dissipated by friction. A bipotential for Coulomb friction law is proposed by De Saxcé and it is applied in order to quantify the friction dissipation power. After deriving the limit analysis continuum formulation, a discretization by triangular elements is proposed and the velocity and stress fields are interpolated. Then, the optimum conditions are derived and the limit analysis problem is solved as an optimization problem, solved by an algorithm based on relaxation and contraction techniques and Quasi-Newton method. The contact problem is solved separately by condensation of degrees of freedom technique and a complementarity problem is solved. As an application, the scratch test problem is modeled considering a rigid indenter in contact with a deformable body. The influence of friction on scratch hardness calculation is investigated as well as the occurrence of sliding or adhesion at the contact interface.*

Keywords: *Limit Analysis, Frictional Unilateral Contact, Scratch Test*

1. INTRODUCTION

Scratch test is a method to determine mechanical properties, such as adhesion of coatings or strength of materials. In order to execute the experiment, a rigid indenter at a constant depth is dragged on the material test surface and the horizontal e vertical forces are monitored. From these results, the scratch hardness H_t is determined by relating the horizontal force and the vertical projected area ($H_t = F_h/w.d$), defined by height d and thickness w according to Figure 1:

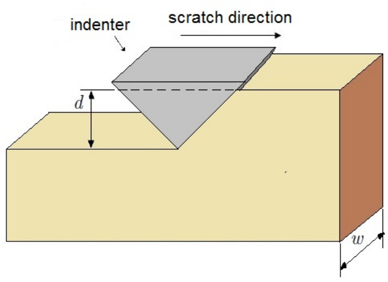

Figure 1. Scratch test.

Then, modeling this process is an important issue since experimental and numerical results can be related and it allows the characterization of material properties and the friction coefficient at tool-specimen contact interface. This problem can be modeled under plasticity theory, since plastic flow occurs during the dragging process. Nevertheless, if there is friction at tool-specimen interface, some part of the external power supplied by the tool is dissipated by friction.

This process in modeled by limit analysis method and a prescribed velocities field is applied in order to simulate the move action of the rigid tool. This boundary condition is convenient since the move action is known and when the contact surface is not planar. In this proposed formulation, the prescription of velocities and verification of unilateral conditions occurs at the same boundary. It allows the determination of relative velocities and the occurrence of sliding or adhesion at contact region and the occurrence of friction dissipation. Most models in literature consider distinct boundaries and the unilateral contact is treat as a fixed support.

In the study of unilateral contact with friction, one verifies the violation of the basic assumptions of normality and convexity of limit analysis principles since the sliding direction is not normal to the Coulomb cone. Then, a bipotential approach for Coulomb friction law is proposed by (Saxcé and Feng, 1998; Saxcé and Bousshine, 1998), in which a subdifferential relation between reaction forces and relative velocities is proposed and the concept of normality and convexity of Standard Materials is extended to non-Standard Materials. It is also shown in (Michalowski and Mroz, 1978) that sliding occurrence depends only on tangential reaction forces and normality is imposed implicity if normal reaction force is fixed. Therefore, under these assumptions, a limit analysis formulation with friction dissipation is derived. The continuum form is discretized into 2-D finite elements, under plane strain hypothesis. The velocity field is quadratic interpolated and the stress field linearly. The contact problem is discretized by collocation method. The contact problem is solved by condensation techniques, as proposed in (Stavroulakis and Antes, 2000) and applied by (Naccarato and Borges, 2007) to model and solve orthogonal cutting problems. The optimum conditions from discretized limit analysis are derived and solved the limit analysis problem is solved by relaxation and contraction techniques and *Quasi*-Newton method.

As application, the scratch test problem is modeled and solved considering a von Mises material. The influence of friction coefficient on scratch hardness is evaluated. The proposed methodology also allows the verification of sliding and adhesion at contact region. The results are compared to the analytical solution based on lower bound method proposed by (Bard and Ulm, 2011; Bard, 2010).

2. LIMIT ANALYSIS WITH PRESCRIBED VELOCITIES AND FRICTIONAL UNILATERAL CONDITIONS

Limit analysis is a direct method applied in order to solve problems in plasticity and it aims the determination of a collapse load or a external power that causes the phenomenon of incipient plastic collapse. It is a direct method and there is no need to calculate the stress state at each load step as in an incremental method. The limit analysis principles are derived from kinematics, equilibrium and constitutive relations, deduced from plastic dissipation sub-differential concept and convexity of plastic domain. Figure 2 shows a deformable body $\mathcal{B}$ and their boundaries:

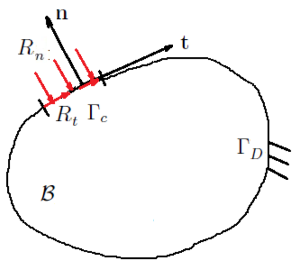

Figure 2. Deformable body $\mathcal{B}$ and their boundaries.

where Γ_D is the boundary where homogeneous velocities are prescribed and Γ_c is the boundary where there is frictional unilateral contact and non-homogeneous velocities are prescribed.

2.1 Kinematics, Equilibrium and Constitutive Relations for Limit Analysis Formulation with Prescribed Velocities

From kinematics,V is defined as the set of all kinematically admissible velocities and their subsets V^0 and $\bar{V}$ so that $V = V^0 + \overline{V}$:

$$V^0 = \{v \in V | v = 0 \; in \; \Gamma_D\} \tag{1}$$
$$\overline{V} = \{v \in V | v = \bar{v} \; in \; \Gamma_c\} \tag{2}$$

The strain rates $\mathbf{D}^p \in W$ are related to the velocity field $\mathbf{v}$ by the tangent linear operator $\mathcal{D}$, according to Equation (3):

$$\mathbf{D}^p = \mathcal{D}\mathbf{v} \qquad \mathbf{v} \in V \tag{3}$$

From equilibrium requirements, the stress field $\mathbf{T}$ is said self-balanced according to the principle of virtual power established in Equation (4). The stress field $\mathbf{T}$ must fulfill the condition of plastic admissibility, defined by the convex set *P* in Equation (5):

$$S_0 = \{\mathbf{T} \in W' | \langle \mathbf{T}, \mathcal{D}\hat{\mathbf{v}} \rangle = 0, \forall \hat{\mathbf{v}} \in V^0\} \tag{4}$$

$$P = \{\mathbf{T} \in W^{'} | f(\mathbf{T}) \leq 0\} \tag{5}$$

If $\mathbf{T}$ is a collapse stress field, considering that $\hat{\mathbf{v}} = \mathbf{v} - \bar{\mathbf{v}}$ and from (4), the external collapse power is deduced in Eq. (6). If $\mathbf{n}$ is the external unitary normal vector at contact boundary:

$$\Pi_e = \langle \mathbf{Tn}, \overline{\mathbf{v}} \rangle_{\Gamma_c} = \int_{\Gamma_c} \mathbf{Tn} \cdot \overline{\mathbf{v}} d\Gamma_c \tag{6}$$

The constitutive relations are derived from maximum dissipation principle and based on subdifferential of plastic dissipation function $\chi(D^p)$. The consequence of this statement leads to the normality law. In a compact way, the constitutive relations are expressed in Equation (8), according to (Borges *et al.*, 1996):

$$\mathbf{T} \in S_0 \cap P \tag{7}$$

$$\mathbf{T} \in \partial\chi(\mathbf{D}^p) \longleftrightarrow \mathbf{D}^p \in C_p(\mathbf{T}) \tag{8}$$

where C_p is the cone of normals and it leads to normality law if the yielding surface is smooth.

2.2 Unilateral Conditions and Dissipation Power by Friction

The boundary Γ_c is defined as a contact region between the rigid tool and the deformable body. At this boundary the action of the rigid body is represented by the prescribed velocity field $\bar{\mathbf{v}}$. Moreover, the unilateral contact restrictions and the constitutive relations for tangential sliding are evaluated at the same boundary.

The occurrence of normal contact and sliding are governed by normal and tangential relative velocities. At normal direction, it is allowed contact or separation between bodies and it implies $v_n \leq 0$. The occurrence of normal reaction forces R_n is related to normal relative velocity v_n, under a complementary or Signorini condition, according to (Raous, 1999) and Curnier (1999) in Equation 9. It represents the separation and contact conditions: if there is contact, $v_n = 0$ and $R_n < 0$. Otherwise, separation and $v_n < 0$ and $R_n = 0$.

$$v_n \leq 0, \; R_n \leq 0, \; R_n v_n = 0 \tag{9}$$

At tangential direction, the sliding or adherence occurrence is related to the Coulomb friction cone, as in Figure 4. It implies that if $|\mathbf{R}_t| < -\mu Rn$, there is adhesion and the reaction forces belong to K_μ. If $|\mathbf{R}_t| = -\mu Rn$, there is sliding and the reaction forces belong to the cone boundary ∂K_μ. At sliding condition and in order to satisfy inequality in 9, the tangential relative velocity is not normal to Coulomb cone and it is parallel to $-\mathbf{R}_t$, according to Equation 10.

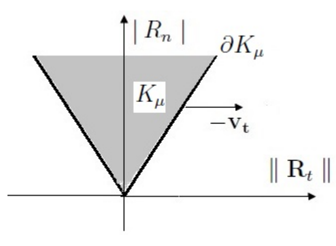

Figure 3. Coulomb cone and relative tangential velocity.

$$\mathbf{v}_t = -\lambda \nabla_{\mathbf{R}_t} f(\mathbf{R}_t, R_n) \tag{10}$$

Analyzing the Equation 10 and Figure 4, one notes that the non-normality of relative tangential velocity vector violates the normality and convexity basis of limit analysis principles. Then, based on bi-potential for Coulomb friction law proposed by (Saxcé and Feng, 1998) and (Saxcé and Bousshine, 1998), there is an attempt to recover the properties of normality and convexity established for Standard Materials in an implicity form. Equation 11 represents also the dissipation power by friction:

$$b_c(v,R) = \langle -\mu R_N, |v_t| \rangle \geq \langle -R_t, v_t \rangle + \langle -R_n, v_n \rangle \geq 0 \tag{11}$$

From the hypothesis that there is permanent contact between the bodies and the contact region is known, for a fixed normal reaction force R_n, the occurrence of sliding depends on tangential force. For a fixed R_n, Figure shows the implicity normality at a plane $R_n = constant$ from Figure 4, as seen in (Michalowski and Mroz, 1978):

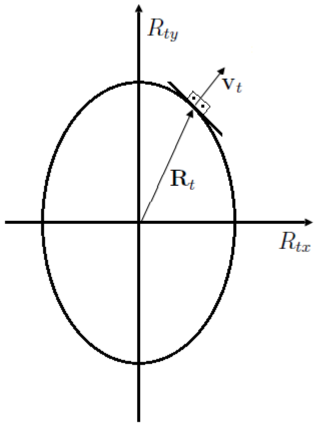

Figure 4. Tangential sliding at a fixed normal reaction force R_n.

Therefore, a mixed limit analysis principle for prescribed velocities and dissipation by friction is deduced as in Equation (12). Considering $\mathbf{R} = [R_n, R_t]$ and $\mathbf{v}_c = [v_n, v_t]$:

$$\Pi_e = \inf_{v^*} [\sup_{T^{**}, R^{**}} \langle \mathbf{T}^{**}, \mathcal{D}\mathbf{v}^* \rangle + \langle \mathbf{R}^{**}, \mathbf{v}_c^* \rangle_{\Gamma_c}], \mathbf{v} = \hat{\mathbf{v}} + \bar{\mathbf{v}}, \tilde{\mathbf{v}} \in V^0 \tag{12}$$

and the unilateral conditions with friction:

$$v_n \leq 0, \ R_n \leq 0 \longleftrightarrow R_n v_n = 0 \tag{13}$$

$$-|\mathbf{v}_t| \leq 0, \ |R_t| - \mu R_n \leq 0 \longleftrightarrow (|R_t| - \mu R_n)|\mathbf{v}_t| = 0 \tag{14}$$

2.3 Discretization

The stress and velocity fields are defined over a space of infinite dimension and in order to find a numerical solution they are approximated by finite elements. The proposed element is a triangular one, where velocities are quadratic interpolated and stresses are linearly interpolated, according to Zouain *et al.* (2014). On discrete form, the strain discrete operator is denoted by *B*. For the contact parcel, the collocation method is applied.

From discretized model, the optimum conditions are calculated in relation to the set of variables $\{T, v, R, \lambda\}$. Based on degrees of freedom condensation technique proposed by Stavroulakis and Antes (2000) applied in Naccarato and Borges (2007) to solve the contact problem for the orthogonal cutting problem, this algorithm is adapted to the prescribed velocity case. Once solved the contact problem, the optimum equations from limit analysis formulation are solved by the algorithm proposed in Borges *et al.* (1996), based on relaxation and contraction techniques. Therefore, the discrete form of 12 is established:

$$\Pi_e = \min_{\mathbf{v}} \max_{\mathbf{T}} \mathbf{T} \cdot B\mathbf{v} - \mathbf{R} \cdot \mathbf{v}_c - \dot{\lambda}_i f_i(\mathbf{T}) \tag{15}$$

and the complementary conditions:

$$R_n v_n = 0 \tag{16}$$

$$\gamma \geq 0, \ \rho \geq 0 \ e \ \gamma\rho = 0 \tag{17}$$

where: γ and ρ are discrete forms of Coulomb law and tangential velocities respectively.

3. EVALUATION OF SCRATCH HARDNESS

In order to model the scratch test problem, the hypothesis of plane strain is considered. The indenter is treated as an undeformable material and the specimen as a deformable one under von Mises criterion. The specimen is discretized into triangular elements and an adaptive mesh is applied, as shown in Figure 5.

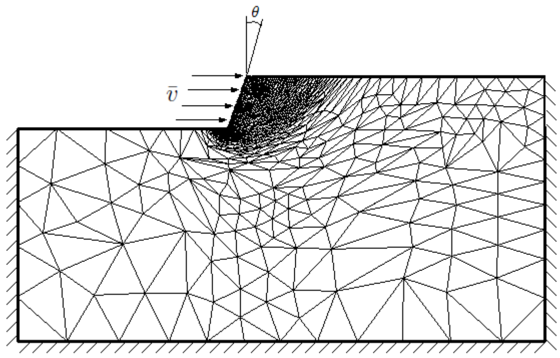

Figure 5. Adaptive mesh, prescribed velocity boundary and back-angle θ.

Under these assumptions and varying the friction coefficient between the rigid tool and the specimen, the scratch hardness is evaluated from the proposed mixed limit analysis method with prescribed velocities and frictional unilateral conditions. Also, the occurrence of sliding or adhesion is verified at contact boundary.

According to (Chen and Liu, 1975), the external collapse power is lower limited by frictionless case and upper limited by adhesion of entire contact region. As long as friction coefficient increases, adhesion occurs at some contact regions and it coexists with sliding regions. At a critical friction coefficient, the entire contact boundary is under adhesion and one verifies the invariance of scratch hardness represented by the plateau from $\mu = 0,175$, according to Figure 6.

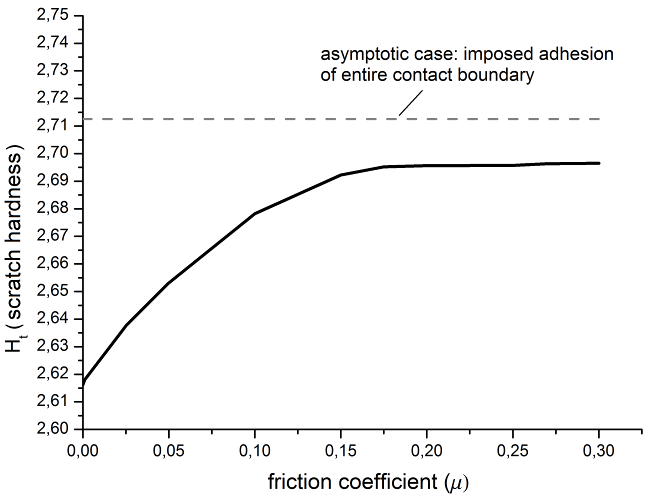

Figure 6. Scratch hardness and its variation with friction coefficient.

Also, an asymptotic case with imposed adhesion of all contact region was simulated in order to evaluate the upper limit of scratch hardness. This condition was reached by applying tangential restrictions at entire contact region. The scratch hardness solved by the contact problem with friction are below the asymptotic case. Between frictionless case ($\mu = 0$) and entire adhesion ($\mu = 0,175$) there is sliding and adhesion at contact boundary. Adhesion regions increases with friction coefficient, according to tangential relative velocities distribution along contact boundary in Figure 7. Figure 8 shows the tangential relative velocities at contact region and the friction dissipation power region for friction coefficient $\mu = 0,05$.

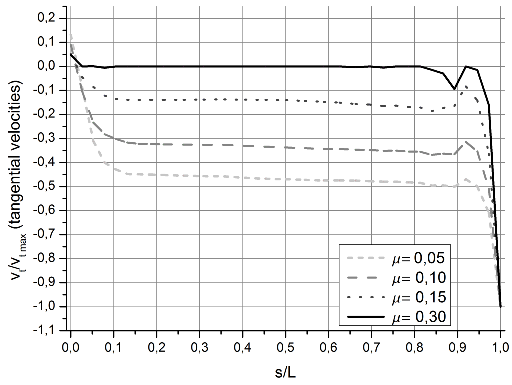

Figure 7. Tangential relative velocities distribution.

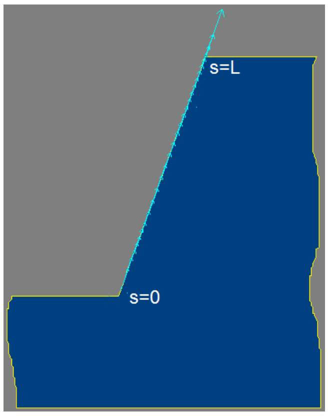

Figure 8. Tangential relative velocity for $\mu = 0,05$: sliding at entire contact region.

Figure 9 shows the tangential relative velocity at contact region for friction coefficient $\mu = 0,20$. One observe adhesion of almost entire contact region and some sliding around free surface at $s = L$. Figure 10 shows the plastic multiplier distribution in the specimen. The plastic multiplier is associated with plastic dissipation and plastic deformation norm.

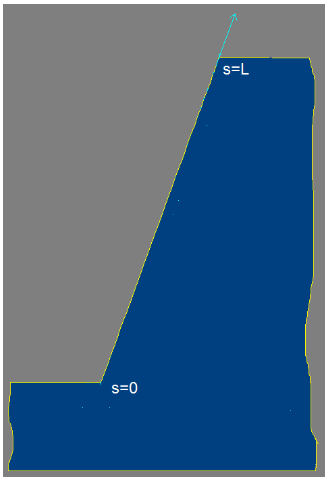

Figure 9. Tangential relative velocities distribution for $\mu = 0,20$.

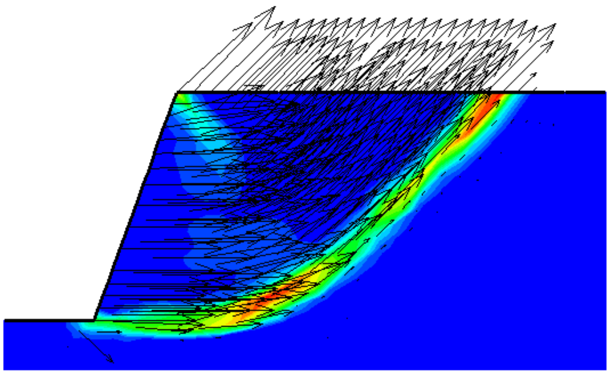

Figure 10. Plastic dissipation and plastic deformation.

From (Bard, 2010) and (Bard and Ulm, 2011), a analytical solution based on lower bound method is developed and the scratch hardness is determined according to friction coefficient. Also, the cited authors implemented an incremental solution in a commercial finite element software. Figure shows the comparison between the scratch hardness evaluated by the proposed method in this paper and the analytical lower-bound solution and the incremental one. A normal velocity field was prescribed in order to reproduce the same boundary conditions applied in (Bard, 2010).

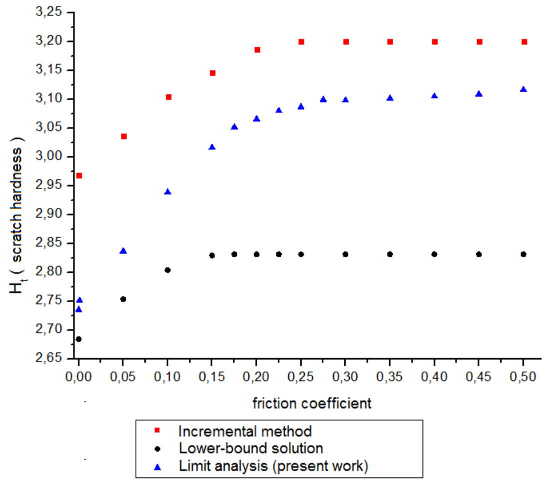

Figure 11. Scratch hardness and its variation with friction coefficient.

Comparing the solutions, from the analytical solution only sliding or adhesion of entire contact region is possible to evaluate. Then, the critical friction coefficient occurs at $\mu_{crit} = 0,15$. One may observe that scratch hardness evaluated from the mixed limit analysis solution is above lower-bound solution, as expected. However, the critical friction coefficient is greater than μ_{crit} from lower bound solution. This behavior is due finite element solution and a gradual change of sliding/adherence regime is observed with friction coefficient, until entire adhesion of all regions at μ_{crit}. The complete adhesion regime is equivalent to solve a problem with tangential restriction direction. Then, the hardness scratch reaches a plateau above μ_{crit}.

The kinematical incremental solution provides an upper bound solution and it is above the one proposed by mixed limit analysis solution. Since finite element method is also applied in the incremental solution, it is observed that the critical friction coefficient occurs at $\mu = 0,25$, around the critical value observed in the present work.

4. CONCLUSION

In this paper is proposed a limit analysis methodology in order to evaluate an external collapse power in problems with frictional unilateral conditions. A velocity field is prescribed in order to describe the action of the rigid body onto deformable one. The verification of unilateral conditions occurs at the velocity prescription boundary simultaneously. It implies an important contribution since most authors treat the prescription boundary and verification of unilateral conditions at different boundaries. As hypothesis, it is assumed permanent contact between bodies and only velocity prescription is allowed. Prescription of surface traction is not permitted.

At tangential direction, the the constitutive relation for sliding is governed by a non-normality law and the tangential relative velocity vector is not normal to Coulomb cone. Assuming a steady-state regime, the contact length is known *a priori*. Then, the normal reaction force is fixed since permanent contact is admitted. Also, from the bi-potential concept for Coulomb friction law is applied and the properties of normality and convexity of Standard Materials are extended implicity to non-Standard-Materials. Then, a mixed limit analysis with prescribed velocities and dissipation by friction is derived.

The proposed methodology is applied to solve the scratch test problem, under plane strain condition and von Mises material. The scratch hardness behavior is evaluated with friction coefficient at toll-specimen interface. The scratch hardness is lower limited by the frictionless case and upper limited by total adhesion. It is observed that as long as friction coefficient increases, sliding region decreases and adhesion occurs, according to the tangential relative velocity distribution along contact region. At a critical friction coefficient μ_{crit}, the entire contact region is under adhesion and the scratch hardness reaches a plateau. For any $\mu > \mu_{crit}$, the scratch hardness is invariant with friction coefficient. The growth of adhesion region is accompanied by the increasing of deformable body stiffness, and when total adhesion occurs, it is like imposing supports with tangential restrictions.

The results are also compared to a lower bound solution and an incremental method implemented in a finite element software. From limit analysis theorems, the static and kinematic methods provide respectively, a lower and an upper bound solution for the collapse power. Coherently, the results from limit analysis mixed formulation are within these limits. In the analytical solution, only sliding or adhesion of contact region is considered in the methodology. Then, the gradual regime change is not described by this model and the critical friction coefficient is lower than those one evaluated by finite element method.

5. ACKNOWLEDGEMENTS

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