

USE OF SIMPLIFIED THERMAL MODEL AND EARLY TIMES TO SOLVE INVERSE HEAT CONDUCTION PROBLEM

Luís Henrique da Silva Ignácio

Ana Paula Fernandes(*)

Gilmar Guimarães

Federal University of Uberlândia – UFU, School of Mechanical Engineering, Campus Santa Mônica, Av. João Naves de Ávila, 2121 - CEP 38400-902 Tel.:(34)3239-4148, Fax:(34) 3239-4206, Uberlândia, Minas Gerais, Brazil.

luishenrique.meta@yahoo.com.br; anapaula@mecanica.ufu.br; gguima@mecanica.ufu.br

(*)CAPES-PNPD-UFU Researcher

Abstract. Usually, thermal analysis in practical cases involves transient and three-dimensional models. In this case, numerical analysis or analytical solutions are needed to solve inverse heat conduction problem. The analytical solutions obtained in this case may have certain complexity, requiring a specific skill to implement and use. In some cases tridimensional problems can be reduced to bi or one dimensional without loss of precision. In others, one-dimensional problems with finite geometry can be reduced to simple geometry problems as the semi-infinite. The possibility of physical examination of these conditions may allow to obtain simplified solutions with the same confidence and precision in the most general and complex formulation. The study of the temperature for the early times can greatly simplify the thermal analysis and be sufficient to solve inverse heat conduction problema whithout loss of precision.

Keywords: temperature, analytical solution, early times, inverse heat conduction.

1. INTRODUCTION

The use of analytical solutions have an extremely important role in the thermal analysis of several practical engineering applications. For example, the modeling of thermal problems is always necessary in various manufacturing processes where the heat generation is present. In this example or in most real cases the problems are multidimensional and the boundary conditions are typically not homogeneous. Usually, the analytical solutions present in these cases may have certain complexity, requiring a specific skill to implement and use it (Fernandes *et al.*, 2015).

However, thermal problems with multidimensional or complex geometries can be simplified, depending on the physical phenomena present in the process. For example, three-dimensional problems sometimes can be reduced to one-dimensional or two-dimensional problems without loss of accuracy. In other cases, problems with finite geometry can be reduced to simple geometry problems as the semi-infinite. The analyses of these physical possibilities may allow to obtain simplified solutions with the same confidence and accuracy of the complex formulation. The potential use of these simplifications is large. One application, for example, can be the obtaining of an unknown heat flux, solving an inverse heat conduction problem, which is applied at inaccessible locals to instrumentation.

This aplication can use that simplified model only in special conditions. These conditions involves the concepts of early times to obtain the temperature data. After, with the estimation of heat flux, the direct problem can be solved considering the complete and original model for any time.

This paper proposes the solution of an inverse problem in a model of a flat plate with thickness L and an unknown heat flux imposed at frontal surface, while the opposite face is exposed to the environment with a convective heat transfer coefficient h and temperature $T_{\infty} = 25^{\circ}\text{C}$. The inverse problem solution is obtained using the inverse technique Transfer Function Based on Green's Function (TFBGF) proposed by Fernandes (2013). The novelty of the proposed procedure is the use of the concepts of early times to assure the utilization of simplified models to obtain the unknown heat flux.

2. FUNDAMENTALS

2.1 Direct problem

The direct problem is shown Fig. 1a. The flat plate is initially at equilibrium temperature T_0 , and suddenly at a time $t > 0$ an unknown heat flux (q'') is imposed at her frontal surface, while the opposite surface is exposed to environment with a convective heat transfer coefficient $h = 5 [W/m^2 K]$ and temperature $T_{\infty} = 25^{\circ}\text{C}$.

The governing equation of this problem, given by the diffusion equation, can be written by:

$$\frac{\partial^2 \theta}{\partial x^2} = \frac{1}{\alpha} \frac{\partial \theta}{\partial t} \quad (x, t) \in (0, L) \times (0, \infty) \quad (1a)$$

subjected to the boundary conditions:

$$-k \left. \frac{\partial \theta}{\partial x} \right|_{x=0} = q(t) \quad -k \left. \frac{\partial \theta}{\partial x} \right|_{x=L} = h\theta \quad (1b)$$

and the initial condition:

$$\theta(x, 0) = 0 \quad (1c)$$

In Equations (1a), (1b) and (1c) the variable θ is defined as $\theta(x, t) = T(x, t) - T_\infty$. The parameters k and α are, respectively, the thermal conductivity and diffusivity of the sample.

The direct thermal problem given by Eq. (1a), (1b) and (1c) can be solved using Green's functions method (Cole *et al.*, 2010). In this case, the solution can be given by:

$$\theta(x, t) = \alpha \int_0^t G(x, t|0, \tau) \frac{q(\tau)}{k} d\tau \quad (2)$$

where $G(x, t|x', \tau)$ is the Green function denoted by G_{X23} and give by:

$$G_{X23}(x, t|x', \tau) = \frac{2}{L} \sum_{m=1}^{\infty} e^{-\beta_m^2 \alpha (t-\tau)/L^2} \frac{\beta_m^2 + B_2^2}{\beta_m^2 + B_2^2 + B_2} \cos\left(\beta_m \frac{x}{L}\right) \cos\left(\beta_m \frac{x'}{L}\right) \quad (3)$$

β_m are the eigenvalues of the problem and are obtained though the transcendental solution of:

$$\beta_m \tan \beta_m = B_2 \quad (4a)$$

where

$$B_2 = \frac{hL}{k} \quad (4b)$$

Using the Green function $G_{X23}(x, t|x', \tau)$ in Eq. (2) obtain:

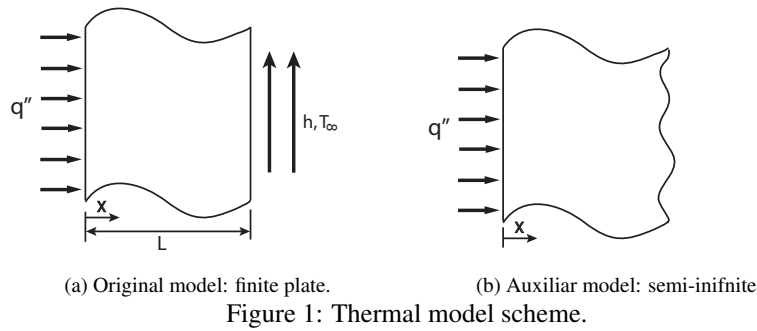
$$\theta_{X23}(x, t) = \frac{2\alpha}{kL} \sum_{m=1}^{\infty} e^{-\beta_m^2 \alpha t/L^2} \frac{\beta_m^2 + B_2^2}{\beta_m^2 + B_2^2 + B_2} \cos\left(\beta_m \frac{x}{L}\right) \int_0^t q(\tau) e^{\beta_m^2 \alpha \tau/L^2} d\tau \quad (5)$$

If $q(t)$ is equal a constant, that is $q(t) = q_0$, so without lose of generality it can be obtained:

$$\theta_{X23}(x, t) = \frac{2q_0L}{k} \sum_{m=1}^{\infty} \frac{\beta_m^2 + B_2^2}{\beta_m^2 + B_2^2 + B_2} \frac{\cos\left(\beta_m \frac{x}{L}\right)}{\beta_m^2} - \frac{2q_0L}{k} \sum_{m=1}^{\infty} e^{-\beta_m^2 \alpha t/L^2} \frac{\beta_m^2 + B_2^2}{\beta_m^2 + B_2^2 + B_2} \frac{\cos\left(\beta_m \frac{x}{L}\right)}{\beta_m^2} \quad (6)$$

It can be observed that Eq. (6) involves a sum in terms of eigenvalues β_m . These eigenvalues should be calculated solving the transcendental equation given by Eq. (4a). If the real problem is a 3D case, the temperature solution will have three sums and three transcendental equations to be calculated (Fernandes *et al.*, 2015).

The propose here is an alternative way that is to replace the finite plate by a semi-infinite, it means, to consider a plate of large thickness. Figure 1b shows a schematical representation of this model.



Obviously, just only in specific geometry, positions and physical conditions, the temperature evolution will be the same for both model. The idea here is, that in such conditions, if the heat flux imposed is the same, the semi-infinite model (auxiliar problem) can be used to estimate de heat flux component that is applied int the original problem (finite plate).

The Green function to the semi-infinite auxiliar problem is given by Cole *et al.* (2010):

$$G_{X20}(x, t|x', \tau) = \frac{1}{(4\pi\alpha(t - \tau))^{\frac{1}{2}}} \left[e^{-\frac{(x-x')^2}{4\alpha(t-\tau)}} + e^{-\frac{(x+x')^2}{4\alpha(t-\tau)}} \right] \quad (7)$$

Therefore, if Eq. (7) is substituted in Eq. (2) the temperature solution of the auxiliar problem can be given by:

$$\theta_{X20}(x, t) = \frac{2\alpha}{\sqrt{\pi}k} \int_0^t \frac{q(\tau)}{(4\alpha(t - \tau))^{\frac{1}{2}}} e^{-\frac{x^2}{4\alpha(t-\tau)}} d\tau \quad (8)$$

Similarly, considering the flux constant, the temperature solution can be written by:

$$\theta_{X20}(x, t) = \frac{q_0}{k} \sqrt{4\alpha t} \operatorname{ierfc} \left(\frac{x}{\sqrt{4\alpha t}} \right) \quad (9)$$

2.2 Heat penetration and deviation time

It can be observed that the analytical solutions of both thermal problem: finite flat plate (original problem) and semi-infinite (auxiliar problem) represent exact solutions of their respective models. However, the behavior of these solutions is identical, but just in certain physical and/or geometric conditions. In fact, once these regions has been determined, both solutions can be used interchangeably and furthermore, each solution verifies the other. This type of check is named of intrinsic verification (Cole *et al.*, 2014). This is necessary due to the fact that the solution has been numerically implemented, although it is an exact analytical equation.

Two parameters plays an important rule in the intrinsic verification: i) heat penetration time and ii) deviation time, which correspond to the early times. These parameters indicate, respectively, the influence of the active boundary conditions (heat flux) and the inative boundary condition (homogenous boundary conditions) inside the flat plate.

That is, in this work, the heat penetration time determines how long it takes for a given position x be influenced by the heat flux $q(t) = q_0$. It means, the 1D penetration time is defined to be the time for a given location at which the temperature just begins to change the fraction of 10^{-n} of the maximum heated surface temperature rise up to and including that time (de Monte *et al.*, 2008; Cole *et al.*, 2014).

Deviation time is closely related to the penetration time, except that it defines the time for a non-heated boundary to have an impact on a heated surface or interior temperature. The 1D deviation time for a plated heated at $x = 0$ is the time it takes for a point at the surface or interior to be affected by the presence of the homogenous boundary condition at $x = L$ (in this case the heat convection boundary condition). The key is to find the combined distance from the heated surface to the $x = L$ surface and then back to the point of interest. The distance is $2L - x$ (Cole *et al.*, 2014; de Monte *et al.*, 2008).

de Monte *et al.* (2008) defines the time penetration as:

$$\sigma_T(x, t) = \frac{T_{X20}(x, t)}{T_{X20}(0, t)} \quad (10)$$

It can be observed that heat time penetration and deviation time is both calculated at $x = 0$ using the semi-infinite model. It is also observed that $x = 0$ is the locate where the heat flux must be estimated.

Equation (10) should be rewritten to be equal to a fraction of acceptable change in temperature at the position x analyzed and so can be used in the calculations. Equation (11) shows how is applied this acceptable range, and the exponent n defines its size.

$$\frac{\theta_{X20}(x, t)}{\theta_{X20}(0, t)} = 10^{-n} \quad (11)$$

Table 1 presents various heat penetration times varying with the position on the plate and the exponent n .

Table 1: Heat penetration time [s].

x (mm) \ n	PVC			Cemented Carbide		
	2	4	10	2	4	10
2	3.3134	1.3217	0.4426	0.0089	0.0036	0.0012
4	13.2534	5.2866	1.7701	0.0356	0.0142	0.0048
6	29.8200	11.8949	3.9828	0.0801	0.0320	0.0107
8	53.0133	21.1465	7.0804	0.1423	0.0568	0.0191
10	82.8333	33.0414	11.0632	0.2224	0.0887	0.0297

From the knowledge of the heat penetration time (Tab. 1), the dimensionless heat penetration time defined by $t_{pen}^+ = \frac{\alpha t_{pen}}{x^2}$ can be calculated. These values are presented in the Tab. 2.

Table 2: Dimensionless heat penetration time.

x (mm) \ n	PVC			Cemented Carbide		
	2	4	10	2	4	10
2	0.0970	0.0387	0.0130	0.0971	0.0393	0.0131
4	0.0970	0.0387	0.0130	0.0971	0.0387	0.0131
6	0.0970	0.0387	0.0130	0.0971	0.0388	0.0130
8	0.0970	0.0387	0.0130	0.0970	0.0387	0.0130
10	0.0970	0.0387	0.0130	0.0970	0.0388	0.0130

It can be observed that the dimensionless heat penetration time varies only with the exponent n . In this case, the heat penetration time can be obtained directly. In the other side de Monte *et al.* (2008) suggest for the deviation time the expression:

$$\varepsilon_T(x, t) = \frac{T_{X23}(x, t) - T_{X20}(x, t)}{T_{X20}(0, t)} \quad (12)$$

Since the numerator of the equation is the temperature difference between the finite and semi-infinite model, the deviation time determines the instant that starts the discrepancy between the two models.

It is observed that the boundary condition on $x = L$ (being in this work $L = 10$ [mm]) should be homogeneous for the application of Eq. (12). This warranty is made with variable change defined by $\theta(x, t) = T(x, t) - T_\infty$, that is:

$$\varepsilon_T(x, t) = \frac{\theta_{X23}(x, t) - \theta_{X20}(x, t)}{\theta_{X20}(0, t)} \quad (13)$$

Similarly to the heat penetration time, Equation (13) can be rewritten so that the temperature difference in the position x calculated by both models is below a acceptable limit of decimal order, so:

$$\frac{\theta_{X23}(x, t) - \theta_{X20}(x, t)}{\theta_{X20}(0, t)} = 10^{-n} \quad (14)$$

The maximum time required for this limit is defined as the deviation time. Thus, from the same geometry and thermal properties, the values of deviation time and dimensionless heat deviation time are calculated and shown in Tables 3 and 4. Dimensionless heat deviation times are calculated by $t_{desv}^+ = \frac{\alpha t_{desv}}{(2L-x)^2}$.

Table 3: Deviation time [s].

x (mm)	n	PVC			Cemented Carbide		
		2	4	10	2	4	10
2		266.5070	107.0080	35.9754	0.6836	0.2867	0.0958
4		220.6730	84.8890	28.4250	0.5657	0.2273	0.0763
6		169.2000	65.0100	21.7629	0.4357	0.1739	0.0584
8		123.5760	47.7740	15.9891	0.3202	0.1277	0.0429
10		85.2940	33.1810	11.1035	0.2224	0.0888	0.0298

Table 4: Dimensionless heat deviation time.

x (mm)	n	PVC			Cemented Carbide		
		2	4	10	2	4	10
2		0.0963	0.0386	0.0130	0.0920	0.0386	0.0129
4		0.1009	0.0387	0.0130	0.0964	0.0387	0.0130
6		0.1011	0.0387	0.0130	0.0970	0.0387	0.0130
8		0.1005	0.0387	0.0130	0.0970	0.0387	0.0130
10		0.0999	0.0387	0.0130	0.0970	0.0387	0.0130

2.3 TFBGF inverse method application

For any dynamic system Fig. 2, the relation between input and output in the complex variable s domain is given by the multiplication expressed in Eq. (15) or in the time domain by the convolution. Thus, in terms of the heat flux/temperature pair we have:

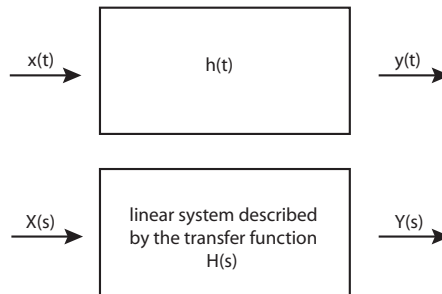


Figure 2: Dynamic system.

$$\mathcal{L}[T(x, t)] = \mathcal{L}[h(x, t) * q(t)] \Rightarrow T(x, s) = H(x, s) \cdot q(s) \quad (15)$$

Or, we can write:

$$q(s) = \frac{1}{H(x, s)} \cdot T(x, s) \quad (16)$$

Equation (16) in the time domain is equivalent to the deconvolution:

$$\mathcal{L}^{-1}[q(s)] = \mathcal{L}^{-1}\left[\frac{1}{H(x, s)} \cdot T(x, s)\right] \Rightarrow q(t) = \frac{1}{h(x, t)} * T(x, t). \quad (17)$$

Therefore, observing Eq. (17), an inversion occurred between the input/output pair, that is, the solution of the problem is the estimation of the system response, which is the heat flux, whose output is the experimental temperature and the transfer function is given by $1/H(x, s)$.

3. HEAT FLUX ESTIMATION USING SIMULATED DATA FROM ONLY ONE POSITION ON FINITI PLATE

Temperature distributions for the direct problem are generated using the solution of Eq. (6) for the model X23 considering a known heat flux evolution $q(t)$. Random errors are then added to these temperatures. The temperatures with error are then used in the inverse algorithm to reconstruct the imposed heat flux.

The tests simulate two samples: a PVC sample and cemented carbide ISO K10 sample that are exposed to a constant heat flux. Noises represent 1% of the maximum temperature for the heat fluxes. The thermal properties of both materials are shown in Tab. 5.

Table 5: Thermal Properties of PVC and Cemented Carbide ISO K10

Thermal Properties	PVC	ISO K10
α (m^2/s)	1.17×10^{-7}	4.36×10^{-5} ⁽¹⁾ ⁽²⁾
k ($W/m K$)	0.15	130 ⁽¹⁾

⁽¹⁾ Engqvist *et al.* (2000) ⁽²⁾ Brito *et al.* (2009)

The simulations for the two materials were made at $x = 2$ [mm] position. Temperature evolutions are shown in Fig. 3 for each material. However, the heat flux at $x = 0$ is estimated just only using measurement of temperature until the deviation time. It can be observed in Fig. 3 that both temperature is very close during this period of time.

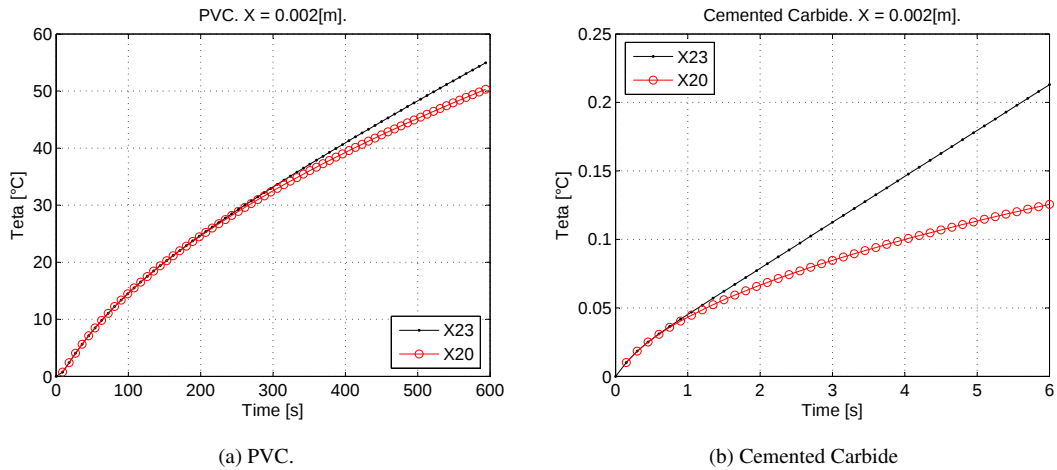


Figure 3: Temperature evolution form fininite and semi-infinite models for both materials

Figures 4a and 4b present the simulated temperature at position $x = 2$ [mm] with random errors for both materials using the finite model.

The heat flux used for this test was $q_0 = 1000 [W/m^2]$ and heat transfer coefficient $h = 5 [W/m^2 K]$.

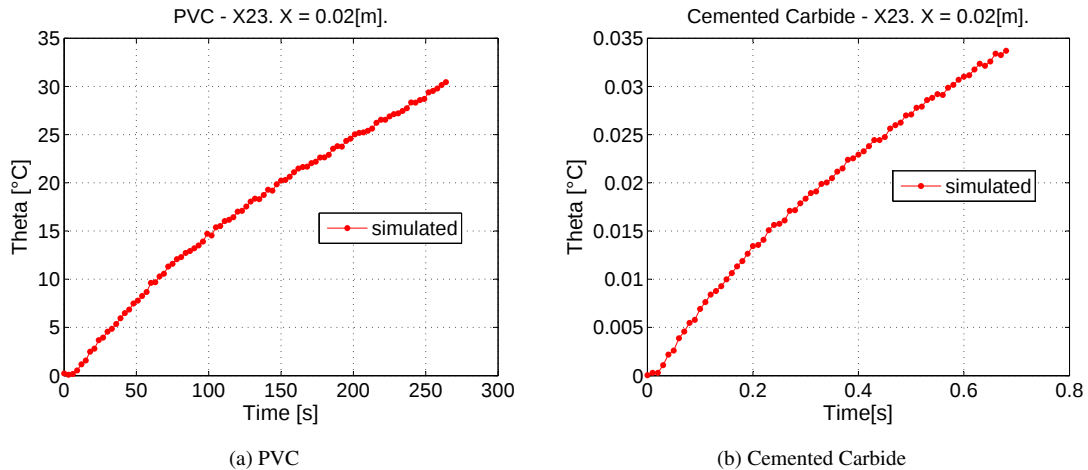


Figure 4: Temperature evolution until the deviation time with $n = 2$ at $x = 2$ [mm].

The estimated heat flux using TFBGF method and semi-infinite model, for both models, are shown in Fig. 5a and 5b,

respectively.

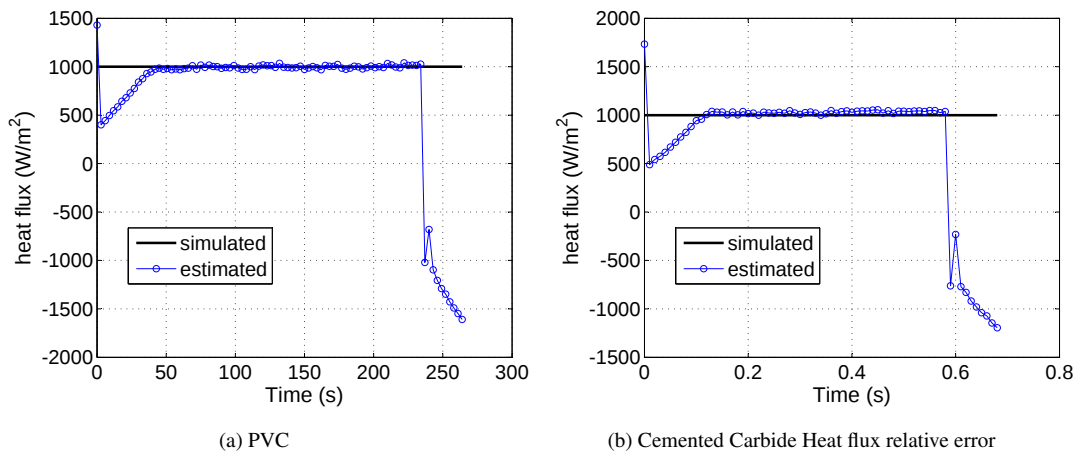


Figure 5: Heat flux estimated using TFBGF method and X20 solution.

It is noted that the heat flux was estimated satisfactorily obtaining a maximum error 5.75% for both materials. It is observed that the estimate has a transition deviate in the initial and end times. This is due to the characteristics of calculation of Fourier transforms and heat penetration and deviation time. For PVC the heat penetration time is of the order of 50.00[s] in the position 2 mm and for cemented carbide in the range of 0.0089[s].

Figures 6a and 6b shows the relative error for heat flux estimations for both materials.

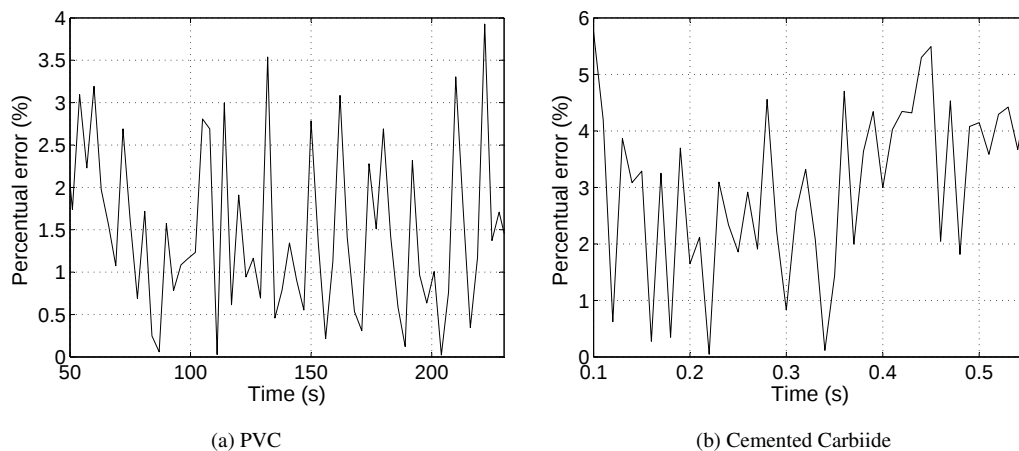


Figure 6: Heat flux relative error.

It can be observed that the largest relative error percentage for PVC was 3.93% and for cemented carbide 5.75%. Therefore it is concluded that is possible to obtain excellent estimations of heat flux using the analytic solution of semi-infinite model instead of using the finite. Advantages in this procedure is the independence of the geometry and simplicity of the model.

4. ACKNOWLEDGEMENTS

The authors would like to thanks the Brazilian Agencies CNPq, CAPES and FAPEMIG.

5. REFERENCES

- Brito, R.F., Carvalho, S.R., de Lima e Silva, S.M.M. and Ferreira, J.R., 2009. "Análise térmica em ferramenta de metal duro revestida". *V Congresso Brasileiro de Engenharia de fabricação*.
- Cole, K.D., Beck, J.V., Haji-Sheikh, A. and Litkouhi, B., 2010. *Heat Conduction Using Green's Functions*. Taylor & Francis Group, Whashington, 2nd edition.
- Cole, K.D., Beck, J.V., Woodbury, K.A. and de Monte, F., 2014. "Intrinsic verification and a heat conduction database". *Int. J. Heat and Mass Transfer*, Vol. 78, pp. 36–47.

- de Monte, F., Beck, J.V. and Amos, D.E., 2008. "Diffusion of thermal disturbances in two-dimensional cartesian transient heat conduction". *Int. J. Heat and Mass Transfer*, Vol. 51, pp. 25–26.
- Engqvist, H., Hogberg, H., Botton, G.A., Ederyd, S. and Axén, N., 2000. "Tribofilm formation on cemented carbides in dry sliding conformal contact". *Wear*, Vol. 239, pp. 219–228.
- Fernandes, A.P., 2013. *Analytical Transfer Function Applied to Solve IHCP*. Ph.D. thesis, Federal University of Uberlândia, Uberlândia.
- Fernandes, A.P., dos Santos, M.B. and Guimarães, G., 2015. "An analytical transfer function method to solve inverse heat conduction problems". *Applied Mathematical Modelling*, pp. –.

6. RESPONSIBILITY NOTICE

The authors are the only responsible for the printed material included in this paper.