23rd ABCM International Congress of Mechanical Engineering
December 6-11, 2015, Rio de Janeiro, RJ, Brazil

STOCHASTIC MODELING OF COMPOSITE STRUCTURES COUPLED WITH PIEZOELECTRIC SHUNTED RESISTIVE CIRCUITS

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Abstract. *This work is devoted to the stochastic finite element modeling of a composite structure containing piezoelectric element coupled with a shunt resistive circuit with the aim of passive vibration attenuation. The deterministic electromechanical problem is modeled by combining the First-Order Shear Deformation Theory, in order to approximate the mechanical displacement fields, with the so-called Layerwise Theory used to model the discrete electric fields within the composite element. It must be emphasized that the deterministic finite element modeling procedure was performed taking into the parameterization process of the design variables of interest to be further assumed as random variables in a straightforward way. Thus, by using Stochastic Finite Element Method, uncertain variables were modeled as Gaussian stochastic homogeneous fields and discretized in accordance with the Karhunen-Loève series expansion method. By intervening directly in the process of integration, was possible to find the exact mass and stiffness matrices of the composite structure. The modeling procedure is confined to the frequency domain, and the dynamic responses are characterized by frequency response functions. To solve the stochastic problem, Monte Carlo simulation was used as stochastic solver combined with the sampling method called Latin Hypercube.*

Keywords: *stochastic modeling, composite materials, passive vibration control, piezoelectric shunted circuits, uncertainty propagation*

1. INTRODUCTION

Experience in Engineering has shown that the introduction of uncertainties in the traditional deterministic model of a system becomes increasingly necessary since these uncertainties directly influence the performance, durability, safety and reliability of the structure (Koroishi et al, 2012).

A powerful tool in stochastic computational mechanics is the Stochastic Finite Element Method (SFEM), which is an extension of the classical finite elements approach for stochastic context, by solving stochastic problems (statics and dynamics) that involve finite elements which design parameters are random. Therefore, the SFEM allows a combination of classical finite element analysis and statistical analysis (De Lima, Rade and Bouhaddi, 2010).

In this work, attention is given to the use of composite materials because of the present engineering context where their use instead of conventional materials is a reality. This fact is mainly based in the question of the superiority of mechanical properties that can be achieved when performing the design of a composite structure. As an example, can be mentioned the relationship resistance per weight of these materials that is far superior compared to traditional metal materials such as steel and aluminum (Faria, 2006; Callister Jr. and Rethwisch, 2009).

The piezoelectrics have been widely used in research work and experience as acting components in the control structures vibrations. In the case of active control systems, they require complex amplifiers, one associated detection as well as electronic control systems. All this apparatus is unnecessary in applications of so-called shunt circuits where the only external element is a passive electrical circuit, which is the shunt circuit. (Hagood and Von flotow, 1991).

In order to model the electromechanical problem two theories are combined. The mechanical displacements field is considered in a condensed form in a single layer and the equivalent electric potential distributed by layers. Thus, for the approach of mechanical displacements field, the First-order Shear Deformation Theory (FSDT), which fits into the context of Equivalent-single layer theories (ESL) is used. The modeling of electricals degrees of freedom is performed by using Layerwise Theory that is inserted into the group of three-dimensional theory of elasticity (Reddy, 1997).

The determinist electromechanical model was parameterized and the major stochastic variables such as the layers' thickness and fiber directions angles were factored out of mass and stiffness matrices. By doing this, the uncertainties could be introduced in a direct way into the composite structure design variables. These random variables and also the resistance circuit parameter were then modeled as stochastic homogeneous Gaussian fields. This work uses the Karhunen-Loève (KL) method that fall into the group of Series Expansion Methods. These methods consist of the coupling between the developing in series of the random field with a spectral analysis to select the most important terms. In this way, there is a direct intervention in the integration process to obtain the exact mass and stiffness matrices (Ghanem et Spanos, 1991). In sequence, is possible to evaluate the variability of response because of the uncertainty propagation into the model, by using the Monte Carlo Simulation (MCS) combined with Latin Hypercube (LHC) sampling as stochastic solver.

2. DETERMINISTIC FINIT ELEMENT FORMULATION

2.1 Mechanical problem modeling

The displacement fields for FSDT theory can be represented as follows (Reddy, 1997):

$$\boldsymbol{U}(x,y,z,t) = \boldsymbol{A}_u(z)\boldsymbol{u}(x,y,t) = (\boldsymbol{A}_0 + z\boldsymbol{A}_1)\boldsymbol{u}(x,y,t) \tag{1}$$

where $\boldsymbol{U}(x,y,z,t)$ denotes the vector of mechanical displacements field, $\boldsymbol{A}_u(z)$ is the matrix containing the factored-out term z and $\boldsymbol{u}(x,y,t)$ is the vector that contains de five degrees of mechanical freedom.

Assuming small displacements the relation between strains and displacements, can be obtained:

$$\boldsymbol{\varepsilon}(x,y,z,t) = \boldsymbol{D}(z)\boldsymbol{u}(x,y,t) = (\boldsymbol{D}_0 + z\boldsymbol{D}_1)\boldsymbol{u}(x,y,t) \tag{2}$$

where $\boldsymbol{D}(z)$ is a matrix array of differential operators.

A global coordinate system was adopted, with the coordinates (x,y,z) common to the whole structure. In the following equations a local coordinate system (ξ,η,z) was used due to the fact that at this moment is desired to find the mass and stiffness elementary matrices and, the integration to obtain these matrices need to be done at the local system. After, the Jacobian J is used for linear coordinate transformation.

Due to the nature of the orthotropic composite materials, the mechanical properties depend strongly on the guidelines of the fibers in the laminate. The mechanical properties of matrix materials may be transformed by a rotation angle θ around the z-axis using a transformation matrix $\boldsymbol{T}$ (for mechanical properties) or $\boldsymbol{Q}$ (for electrical properties), which can be found in the work of Faria (2006). As example the rotation of the mechanical properties matrix: $\boldsymbol{C}_t = \boldsymbol{TCT}^T$.

For mechanical modeling is adopted a finite element of rectangular flat plate from Serendipity family (Reddy, 1997) that contains eight nodes which no central node. Thus, due to the use of FSDT theory it is known that it takes five mechanical variables in defining its displacement field, which can be expressed in terms of its 40 mechanical nodal variables corresponding to $\boldsymbol{u}_e = \{u_{ij}, v_{ij}, w_{ij}, \psi_{x_{ij}}, \psi_{y_{ij}}\}$ with $i=1...8$ and $j=1...5$. The correspondence between the total displacement of the element with the contribution of each node is done through the Shape Functions, $\boldsymbol{N}_u(\xi,\eta)$, that represent the specificities of each type of element, and, they are already represented in terms of local coordinates and defined in (Faria, 2006).

$$\boldsymbol{u}(\xi,\eta,t) = \boldsymbol{N}_u(\xi,\eta)\boldsymbol{u}_e(t) \tag{3}$$

The vectors of mechanical displacements and mechanical strains can be expressed in terms of the Shape Functions and the nodal mechanical variables, respectively, as follows (Reddy, 1997):

$$\boldsymbol{U}(x,y,z,t) = \boldsymbol{A}_u(z)\boldsymbol{N}_u(\xi,\eta)\boldsymbol{u}_e(t); \boldsymbol{\varepsilon}(x,y,z,t) = \boldsymbol{D}(z)\boldsymbol{N}_u(\xi,\eta)\boldsymbol{u}_e(t) \tag{4}$$

The elementary mass matrix is obtained from the kinetic energy at the elementary level, and is given by (Faria, 2006):

$$\boldsymbol{M}_{uu}^e = \sum_{k=1}^{n} \int_{V_e} \rho_k \boldsymbol{N}_u^T \boldsymbol{A}_u^T \boldsymbol{A}_u \boldsymbol{N}_u dV_e = \sum_{k=1}^{n} \sum_{i=1}^{3} t_k^i \rho_k \boldsymbol{M}_{ui}^{(k)} \tag{5}$$

where ρ_k is the material's density of the k^{th} layer, V_e is the finite element volume, $t_k^i = (z_{k+1}^i - z_k^i) = \left[k^i - (k-1)^i\right] h_k^i$ com $i=1...3$ and h_k is the layer's thickness.

So, it should be observed that the elementary mass matrix was already modeled in a parametrized way where the parameters, material density and layers thickness, were factored-out.

The elementary stiffness matrix is obtained from the potential energy of mechanical deformation and is given by (Faria, 2006)

$$\boldsymbol{K}_{uu}^{e} = \int_{V_e} \boldsymbol{B}_u^T \boldsymbol{C}_t \boldsymbol{B}_u J dV_e \tag{6}$$

where $\boldsymbol{B}_u = \boldsymbol{D}(z)\boldsymbol{N}_u(\xi,\eta)$ and $\boldsymbol{C}_t$ is the mechanical properties matrix already considering the fibers directions.

The elementary stiffness matrix was parameterized in terms of the layers thickness and the fibers directions angles. It was also separated in terms of bending and shear effects as shown in sequence (Zambolini-Vicente, 2014):

$$\boldsymbol{K}_{ub}^{e} = \sum_{k=1}^{n}\sum_{i=1}^{n} t_k^i \left(c_k^4\, {}_1\boldsymbol{K}_{ubi}^{(k)} + c_k^2\, {}_2\boldsymbol{K}_{ubi}^{(k)} + s_k c_k^3\, {}_3\boldsymbol{K}_{ubi}^{(k)} + {}_4\boldsymbol{K}_{ubi}^{(k)} + s_k c_k\, {}_5\boldsymbol{K}_{ubi}^{(k)} + s_k^2 c_k^2\, {}_6\boldsymbol{K}_{ubi}^{(k)} \right) \tag{7a}$$

$$\boldsymbol{K}_{us}^{e} = \sum_{k=1}^{n}\sum_{i=1}^{n} t_k^i \left(c_k^2\, {}_1\boldsymbol{K}_{usi}^{(k)} + s_k c_k\, {}_2\boldsymbol{K}_{usi}^{(k)} + s_k^2\, {}_3\boldsymbol{K}_{ubi}^{(k)} \right) \tag{7b}$$

where $\boldsymbol{K}_{uu}^{e} = \boldsymbol{K}_{ub}^{e} + \boldsymbol{K}_{us}^{e}$, $s_k = sin(\theta_k)$, $c_k = cos(\theta_k)$ and θ_k is the angle of the fiber direction of the k^{th} layer.

In order to obtain the mass and stiffness global matrices, classical assembly finite element techniques are used, knowing the system connectivity of the nodes. So, the following equation of motion in matrix form can be obtained:

$$\boldsymbol{M}_{uu}\ddot{\boldsymbol{u}}(t) + \boldsymbol{K}_{uu}\boldsymbol{u}(t) = \boldsymbol{f}(t) \tag{8}$$

where $\boldsymbol{M}_{uu}$ and $\boldsymbol{K}_{uu}$ are the mass and stiffness global matrices, $\boldsymbol{u}(t)$ is the vector of global degrees of freedom and $\boldsymbol{f}(t)$ is the vector of generalized efforts.

2.2 Eletromechanical problem modeling

For modeling laminated composite structures containing piezoelectric elements FSDT is used to approximate the mechanical displacements fields and the electric potentials are distributed discretely along the piezoelectric layers and approximated as follows (Saravanos and Heyliger, 1995; Chee 2000):

$$\Phi_{(k)}(x,y,z,t) = L_{kd}(z)\Phi_k(x,y,t) + L_{ku}(z)\Phi_{k+1}(x,y,t) \tag{9}$$

where $\Phi_{(k)}$ is the electric potential of a k^{th} layer, Φ_k and Φ_{k+1} are the layerwise functions in plane, given by the electric potential of the lower and upper interfaces, respectively; $L_{kd}(z) = (z_{k+1}\text{-}z)/(z_{k+1}\text{-}z_k)$ and $L_{ku}(z) = (z\text{-}z_k)/(z_{k+1}\text{-}z_k)$ are the transverse layerwise functions for the lower and upper interfaces, respectively.

The electric potential of a k^{th} layer can also be written in matrix form and in terms of the Shape Functions $\boldsymbol{N}_u(\xi,\eta)$ and nodal electrical potentials $\boldsymbol{\varphi}_{ek}$ (Saravanos and Heyliger, 1995):

$$\Phi_{(k)}(\xi,\eta,z,t) = \boldsymbol{L}_k(z)\boldsymbol{N}_u(\xi,\eta)\boldsymbol{\varphi}_{ek}(t) = \boldsymbol{N}_{\Phi}(\xi,\eta,z)\boldsymbol{\varphi}_{ek}(t) \tag{10}$$

The electric field is given by the negative electrical potential gradient and can also be written in terms the Shape Functions $\boldsymbol{N}_u(\xi,\eta)$ and nodal electrical potentials $\boldsymbol{\varphi}_{ek}$ as follows (Faria, 2006):

$$\boldsymbol{E}_{(k)}(\xi,\eta,z,t) = -\nabla\Phi_{(k)}(\xi,\eta,t) = -\nabla\boldsymbol{N}_{\Phi}(\xi,\eta,z)\boldsymbol{\varphi}_{ek}(t) = -\boldsymbol{B}_{\Phi}(\xi,\eta,z)\boldsymbol{\varphi}_{ek}(t) \tag{11}$$

The piezoelectric-elastic-dielectric matrix that shows the electromechanical coupling and the relations between mechanical stress, $\boldsymbol{\sigma}$, mechanical strain, $\boldsymbol{\varepsilon}$, electrical displacements, $\boldsymbol{D}$, and electric field, $\boldsymbol{E}$, are given as follows (Chee, 2000):

$$\begin{Bmatrix} \boldsymbol{\sigma} \\ \boldsymbol{D} \end{Bmatrix} = \begin{bmatrix} \boldsymbol{C}_t & -\boldsymbol{e}_t^T \\ \boldsymbol{e}_t & \boldsymbol{\chi}_t \end{bmatrix} \begin{Bmatrix} \boldsymbol{\varepsilon} \\ \boldsymbol{E} \end{Bmatrix} \tag{12}$$

where $\boldsymbol{e}_t$ are the dielectric constants and $\boldsymbol{\chi}_t$ is the electrical permittivity matrix, both already considering the fibers directions.

For the electromechanical system, the elementary stiffness matrices are now obtained by means of deformation energy, involving the electrical contribution as follows (Faria, 2006):

$$\boldsymbol{U}_e = \int_{V_e} \left(\boldsymbol{\varepsilon}^T \boldsymbol{\sigma} - \boldsymbol{E}^T \boldsymbol{D} \right) dV_e \tag{13}$$

By combining the Eqs. (12) with Eq. (13) it is possible to obtain the mechanical, electromechanical and electrical elementary stiffness matrices, respectively, with the following integrations (Faria, 2006):

$$\boldsymbol{K}_{uu}^e = \sum_{k=1}^{n} \int_{z=z_k}^{z_k+1} \int_{\xi=-1}^{\xi=1} \int_{\eta=-1}^{\eta=1} \left(\boldsymbol{B}_u^T \boldsymbol{C}_t \boldsymbol{B}_u \right) J d\eta d\xi dz \tag{14}$$

$$\boldsymbol{K}_{u\Phi}^e = \sum_{k=1}^{n} \int_{z=z_k}^{z_k+1} \int_{\xi=-1}^{\xi=1} \int_{\eta=-1}^{\eta=1} \left(\boldsymbol{B}_u^T \boldsymbol{e}_t \boldsymbol{B}_\Phi \right) J d\eta d\xi dz \tag{15}$$

$$\boldsymbol{K}_{\Phi\Phi}^e = \sum_{k=1}^{n} \int_{z=z_k}^{z_k+1} \int_{\xi=-1}^{\xi=1} \int_{\eta=-1}^{\eta=1} \left(-\boldsymbol{B}_\Phi^T \boldsymbol{\chi}_t \boldsymbol{B}_\Phi \right) J d\eta d\xi dz \tag{16}$$

The elementary stiffness mechanical matrix can be parametrized in terms of the layers thickness and the fibers directions angles as was shown in Eqs. (7). The parametrization of the elementary stiffness electromechanical and electrical matrices was done as follows (Zambolini-Vicente, 2014):

$$\boldsymbol{K}_{u\Phi}^e = \frac{1}{t_k} \left(t_k \boldsymbol{K}_{u\Phi}^{00} + t_k c_k^2 \boldsymbol{K}_{u\Phi}^{01} + t_k s_k c_k \boldsymbol{K}_{u\Phi}^{02} + t_k s_k^2 \boldsymbol{K}_{u\Phi}^{03} + t_k^2 \boldsymbol{K}_{u\Phi}^{10} + t_k^2 c_k^2 \boldsymbol{K}_{u\Phi}^{11} + t_k^2 s_k c_k \boldsymbol{K}_{u\Phi}^{12} + t_k^2 s_k^2 \boldsymbol{K}_{u\Phi}^{13} \right) \tag{17}$$

$$\boldsymbol{K}_{\Phi\Phi}^e = \frac{1}{t_k^2} \left(t_k \boldsymbol{K}_{\Phi\Phi}^{00} + t_k c_k^2 \boldsymbol{K}_{\Phi\Phi}^{01} + t_k s_k c_k \boldsymbol{K}_{\Phi\Phi}^{02} + t_k s_k^2 \boldsymbol{K}_{\Phi\Phi}^{03} + t_k^2 c_k^2 \boldsymbol{K}_{\Phi\Phi}^{11} + t_k^2 s_k c_k \boldsymbol{K}_{\Phi\Phi}^{12} + t_k^2 s_k^2 \boldsymbol{K}_{\Phi\Phi}^{13} + t_k^3 c_k^2 \boldsymbol{K}_{\Phi\Phi}^{21} + t_k^3 s_k c_k \boldsymbol{K}_{\Phi\Phi}^{22} + t_k^3 s_k^2 \boldsymbol{K}_{\Phi\Phi}^{23} \right) \tag{18}$$

In order to obtain the mass and stiffness global matrices, classical assembly finite element techniques are used, knowing the system connectivity of the nodes. So, the following equations of motion in matrix form can be obtained:

$$\begin{bmatrix} \boldsymbol{M}_{uu} & 0 \\ 0 & 0 \end{bmatrix} \begin{Bmatrix} \ddot{\boldsymbol{u}}(t) \\ \ddot{\boldsymbol{\Phi}}(t) \end{Bmatrix} + \begin{bmatrix} \boldsymbol{K}_{uu} & \boldsymbol{K}_{u\Phi} \\ \boldsymbol{K}_{\Phi u} & \boldsymbol{K}_{\Phi\Phi} \end{bmatrix} \begin{Bmatrix} \boldsymbol{u}(t) \\ \boldsymbol{\Phi}(t) \end{Bmatrix} = \begin{Bmatrix} \boldsymbol{f}(t) \\ \boldsymbol{q}(t) \end{Bmatrix} \tag{19}$$

where $\boldsymbol{K}_{u\Phi}$ and $\boldsymbol{K}_{\Phi\Phi}$ are the global electromechanical and electrical stiffness matrices, respectively, $\boldsymbol{K}_{\Phi u} = \boldsymbol{K}_{u\Phi}^T$, $\boldsymbol{\Phi}(t)$ are the global electric potentials, $\boldsymbol{q}(t)$ is the global electric charges.

2.3 Inclusion of the shunt circuit in the model

The Equations (19) can be represented in the Fourier domain, neglecting initial conditions, leading to the following equations of motion (Zambolini-Vicente, 2014; Ribeiro and De Lima, 2014):

$$\left(\boldsymbol{K}_{uu} - \omega^2 \boldsymbol{M}_{uu} \right) \boldsymbol{U}(\omega) + \boldsymbol{K}_{u\Phi} \boldsymbol{\Phi}(\omega) = \boldsymbol{F}(\omega) \tag{20}$$

$$\boldsymbol{K}_{\Phi u} \boldsymbol{U}(\omega) + \boldsymbol{K}_{\Phi\Phi} \boldsymbol{\Phi}(\omega) = \boldsymbol{Q}(\omega) \tag{21}$$

The vector of electrical charges flowing through the shunt circuit is given by:

$$\boldsymbol{Q}(\omega) = (1/j\omega) \boldsymbol{Z}^{-1}(\omega) \boldsymbol{L} \boldsymbol{\Phi}(\omega) \tag{22}$$

where $\boldsymbol{Z}$ is the electrical matrix impedances and $\boldsymbol{L}$ allows to select from among the independent electrical potentials, those corresponding to the electrodes where the shunt circuits are connected.

By combining the Eq. (21) with the Eq. (22):

$$\boldsymbol{K}_{\Phi u}\boldsymbol{U}(\omega)+\left(\boldsymbol{K}_{\Phi\Phi}-\frac{\boldsymbol{Z}^{-1}(\omega)}{j\omega}L\right)\boldsymbol{\Phi}(\omega)=0 \tag{23}$$

At this point the frequency response function (FRF) of the electromechanical system can be found by combining the Eq. (20) and the Eq. (23) only in terms of the mechanical degrees of freedom, as given in sequence:

$$\boldsymbol{H}(\omega)=\left[\boldsymbol{K}_{uu}-\boldsymbol{K}_{u\Phi}\left(\boldsymbol{K}_{\Phi\Phi}-\frac{1}{j\omega}\boldsymbol{Z}^{-1}(\omega)\right)^{-1}\boldsymbol{K}_{\Phi u}-\omega^2\boldsymbol{M}_{uu}\right]^{-1} \tag{24}$$

3. STOCHASTIC FINIT ELEMENT FORMULATION

The uncertainties in the deterministic parameterized electromechanical model, represented by the design parameters that were factored-out of the matrices will, at this point, be taken as random. The random variables are modeled as stochastic fields, $H(\boldsymbol{x},\boldsymbol{y},\theta)$,and the discretization of these fields is done by the Karhunen-Loève (KL) decomposition, which is a continuous representation for random fields expressed as the superposition of orthogonal random variables weighted by deterministic spatial functions. According to this technique, a random field can be viewed as a spatial extension of a random variable that describes the spatial correlation of a structural parameter that fluctuates randomly (Ghanem and Spanos, 1991; Koroishi et al, 2012).

A bi-dimensional random field $H(\boldsymbol{x},\boldsymbol{y},\theta)$, where $\boldsymbol{x}$ and $\boldsymbol{y}$ denotes the spatial dependence of the field, θ represents a random field, is defined by its mean, $\mu(\boldsymbol{x},\boldsymbol{y})$ and its covariance $C\left[(x_1,y_1),(x_2,y_2)\right]$. For a two-dimensional homogeneous Gaussian random field $H(\boldsymbol{x},\boldsymbol{y},\theta)$ with a symmetric and positive-definite covariance function over a domain $\Omega\in R^d$, it is possible to find a unique projection of $H(\boldsymbol{x},\boldsymbol{y},\theta)$ on an orthonormal truncated random basis as follows (Ghanem and Spanos, 1991; De Lima, Rade and Bouhaddi, 2010):

$$H(\boldsymbol{x},\boldsymbol{y},\theta)=\mu(\boldsymbol{x},\boldsymbol{y})+\sum_{r=1}^{\infty}\sqrt{\lambda_r}\xi_r(\theta)f_r(\boldsymbol{x},\boldsymbol{y})\approx\hat{H}(\boldsymbol{x},\boldsymbol{y},\theta)=\mu(\boldsymbol{x},\boldsymbol{y})+\sum_{r=1}^{M}\sqrt{\lambda_r}\xi_r(\theta)f_r(\boldsymbol{x},\boldsymbol{y}) \tag{25}$$

where $\{\xi_r, r\in N^*\}$ are the zero-mean orthogonal variables, the deterministic functions $f_r(\boldsymbol{x},\boldsymbol{y})$ and the scalar values λ_r are, respectively, the eigenfunctions and the eigenvalues of the covariance function $C\left[(x_1,y_1),(x_2,y_2)\right]$.

The KL expansion is defined with reference to a particular geometric domain Ω, so that in the case of modeling an uncertain parameter of a structural model by means of a random field, this geometry at least includes the domain of the structure under consideration.

Ghanem and Spanos (1991) brought some analytical solutions to the eigenproblem, but they are valid just for simple geometric configurations. In this way, the analytical solution to the eigenproblem proposed, for the KL expansion into the domains $\Omega_x=(x_1,x_2)=[0,a]$ and $\Omega_y=(y_1,y_2)=[0,b]$ is given as follows:

$$C\left[(x_1,y_1),(x_2,y_2)\right]=exp\left(\frac{|x_1-x_2|}{L_{cov,x}}-\frac{|y_1-y_2|}{L_{cov,y}}\right) \tag{26}$$

where $L_{cov,x}$ and $L_{cov,y}$ represent the correlation lengths in x and y directions, respectively. Talking into account the separability property of the covariance function, the two-dimensional problem is decoupled into two one-dimensional independent eigenproblems with solution $f_r(\boldsymbol{x},\boldsymbol{y})=f_i(\boldsymbol{x})f_j(\boldsymbol{y})$ and $\lambda_r=\lambda_i\lambda_j$. The couples $(\lambda_i, f_i(\boldsymbol{x}))$ are obtained by solving the KL decomposition of a first-order field of variance equal to one, with a correlation length $L_{cov,x}$ on a domain $[0,a]$, while the couples $(\lambda_j, f_j(\boldsymbol{y}))$ are obtained by solving the same problem, but with a correlation length $L_{cov,y}$ on a domain $[0,b]$. To calculate the pairs of eigenvalues and eigenvectors, the procedure is described below succinctly. The first case is done for i and j odd, with $i\geq 1$ and $j\geq 1$ (Ghanem and Spanos, 1991):

$$\lambda_i = \frac{2c_1}{\omega_i^2 + c_1^2}\ ,\ \ f_i\left(\boldsymbol{x}\right) = \alpha_i\, cos\left(\omega_i \boldsymbol{x}\right) \tag{27}$$

$$\lambda_j = \frac{2c_2}{\omega_j^2 + c_2^2}\ ,\ \ f_j\left(\boldsymbol{y}\right) = \alpha_j\, cos\left(\omega_j \boldsymbol{y}\right) \tag{28}$$

where $\alpha_i = 1\Big/\sqrt{a + sin\left(2\omega_i a\right)/2\omega_i}$, $\alpha_j = 1\Big/\sqrt{b + sin\left(2\omega_j b\right)/2\omega_j}$, $c_1 = 1/L_{cov,x}$, $c_2 = 1/L_{cov,y}$.

Considering the domains $\left[\left(i - \frac{1}{2}\right)\frac{\pi}{a}, i\frac{\pi}{a}\right]$ and $\left[\left(j - \frac{1}{2}\right)\frac{\pi}{b}, j\frac{\pi}{b}\right]$, the roots ω_i and ω_j are the solutions of the following transcendental functions:

$$\omega_i + c_1\, tg\left(\omega_i a\right) = 0\ ;\ \ \omega_j + c_2\, tg\left(\omega_j b\right) = 0 \tag{29}$$

By using the KL expansion is possible to model the elementary random mass and the stiffness matrices of the electromechanical structure. It can be observed that both mass and stiffness matrices have a deterministic portion, which is calculated as was shown in the section before, and one stochastic portion that can be better explained in the following equations (De Lima, Rade and Bouhaddi, 2010; Koroishi et al, 2012)

$$\boldsymbol{M}_{uu}^e\left(\theta\right) = \sum_{k=1}^{n}\int_{\Omega_e} H\left(\boldsymbol{x},\boldsymbol{y},\theta\right)\boldsymbol{N}_u^T\boldsymbol{N}_u d\Omega_e = \boldsymbol{M}_{uu}^e + \boldsymbol{M}_r^e\left(\theta\right) = \boldsymbol{M}^e + \sum_{k=1}^{n}\xi_r^k\alpha_i\alpha_j\sqrt{\lambda_i}\sqrt{\lambda_j}\int_{\Omega_y} f_j\left(\boldsymbol{y}\right)\int_{\Omega_x} f_i\left(\boldsymbol{x}\right)\boldsymbol{N}_u^T\boldsymbol{N}_u d\Omega_x d\Omega_y \tag{30}$$

$$\boldsymbol{K}_{uu}^e\left(\theta\right) = \sum_{k=1}^{n}\int_{\Omega_e} H\left(\boldsymbol{x},\boldsymbol{y},\theta\right)\boldsymbol{B}_u^T\boldsymbol{C}_t\boldsymbol{B}_u d\Omega_e = \boldsymbol{K}_{uu}^e + \boldsymbol{K}_{uur}^e\left(\theta\right) = \boldsymbol{K}_{uu}^e + \sum_{k=1}^{n}\xi_r^k\alpha_i\alpha_j\sqrt{\lambda_i}\sqrt{\lambda_j}\int_{\Omega_y} f_j\left(\boldsymbol{y}\right)\int_{\Omega_x} f_i\left(\boldsymbol{x}\right)\boldsymbol{B}_u^T\boldsymbol{C}_t\boldsymbol{B}_u d\Omega_x d\Omega_y \tag{31}$$

$$\boldsymbol{K}_{u\Phi}^e\left(\theta\right) = \sum_{k=1}^{n}\int_{\Omega_e} H\left(\boldsymbol{x},\boldsymbol{y},\theta\right)\boldsymbol{B}_u^T\boldsymbol{e}_t\boldsymbol{B}_\Phi d\Omega_e = \boldsymbol{K}_{u\Phi}^e + \boldsymbol{K}_{u\Phi r}^e\left(\theta\right) = \boldsymbol{K}_{u\Phi}^e + \sum_{k=1}^{n}\xi_r^k\alpha_i\alpha_j\sqrt{\lambda_i}\sqrt{\lambda_j}\int_{\Omega_y} f_j\left(\boldsymbol{y}\right)\int_{\Omega_x} f_i\left(\boldsymbol{x}\right)\boldsymbol{B}_u^T\boldsymbol{e}_t\boldsymbol{B}_\Phi d\Omega_x d\Omega_y \tag{32}$$

$$\boldsymbol{K}_{\Phi\Phi}^e\left(\theta\right) = \sum_{k=1}^{n}\int_{\Omega_e} H\left(\boldsymbol{x},\boldsymbol{y},\theta\right)\boldsymbol{B}_\Phi^T\boldsymbol{\chi}_t\boldsymbol{B}_\Phi d\Omega_e = \boldsymbol{K}_{\Phi\Phi}^e + \boldsymbol{K}_{\Phi\Phi r}^e = \boldsymbol{K}_{\Phi\Phi}^e + \sum_{k=1}^{n}\xi_r^k\alpha_i\alpha_j\sqrt{\lambda_i}\sqrt{\lambda_j}\int_{\Omega_y} f_j\left(\boldsymbol{y}\right)\int_{\Omega_x} f_i\left(\boldsymbol{x}\right)\boldsymbol{B}_\Phi^T\boldsymbol{\chi}_t\boldsymbol{B}_\Phi d\Omega_x d\Omega_y \tag{33}$$

where k is the number of the structure layers.

At this point is interesting to calculate the frequency response of the system. At first, the global random mass, $\boldsymbol{M}\left(\theta\right)$, and random stiffness matrices, mechanical, $\boldsymbol{K}_{uu}\left(\theta\right)$, electromechanical, $\boldsymbol{K}_{u\Phi}\left(\theta\right)$, and electrical, $\boldsymbol{K}_{\Phi\Phi}\left(\theta\right)$, are obtained by using the standard Finite Element matrix assembling. So, the frequency domain random equation of motion for the stochastic structure subjected to a deterministic harmonic excitation can be expressed as:

$$\boldsymbol{H}\left(\omega,\theta\right) = \left[\boldsymbol{K}_{uu}\left(\theta\right) - \boldsymbol{K}_{u\Phi}\left(\theta\right)\left(\boldsymbol{K}_{\Phi\Phi}\left(\theta\right) - \frac{1}{j\omega}\boldsymbol{Z}^{-1}\left(\omega,\theta\right)\right)^{-1}\boldsymbol{K}_{\Phi u}\left(\theta\right) - \omega^2\boldsymbol{M}\left(\theta\right)\right]^{-1} \tag{34}$$

By choosing at first the probability distributions of the uncertainty variables, the stochastic problem in Eq. (22) can be solved by using the MCS in combination with LHC sampling method as stochastic solver.

4. SIMULATIONS

Simulations were done with a cantilever beam of composite material containing a piezoelectric element coupled to a shunt resistive circuit that is illustrated in Fig. 1a. In this figure are presented the geometrical characteristics of the system to be investigated and the finite element mesh, composed by six finite elements. The dimensions are given in meters. The beam has a total length $0{,}306m$, wherein all layers have the same thickness, $0{,}002m$, including the PZT (Lead Zirconate Titanate), and the same width, $0{,}0255m$. The configuration $\left[0^{\circ}/90^{\circ}/90^{\circ}/0^{\circ}\right]$ was investigated to the guidelines of the composite fibers.

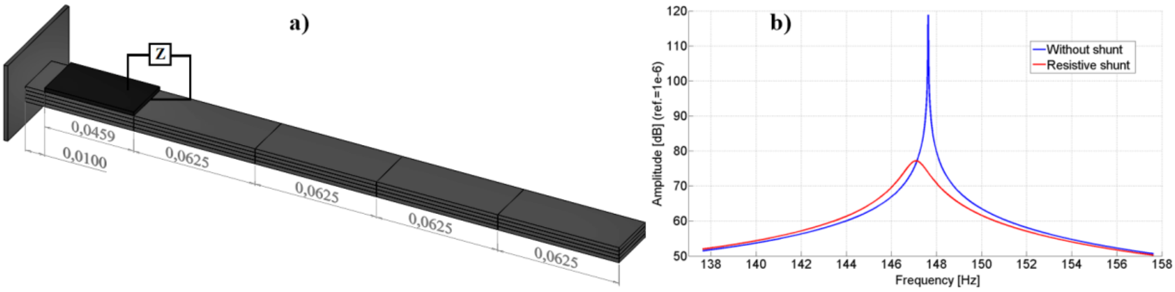

Figure 1. a) Cantilever composite beam containing PZT coupled with piezoelectric shunted resistive circuit; b) Deterministic frequency response functions of the first mode.

The first four layers of the test-structure showed in Fig. 1a are made by composite material with mass density $\rho = 1566\ Kg/m^3$ and, the last layer by PZT material with mass density $\rho = 7700\ Kg/m^3$. The Tab. 1 shows the mechanical properties of the composite material and PZT where the G and E, that are the shear and Young's Modulus were multiplied by a complex factor, $\beta = 5x10^{-8}$, to reflect the inherent damping in the structure. In sequence, the Tab. 2 shows the electrical and electromechanical properties of the PZT:

Table 1. Mechanical properties

Material	E_1(Pa)	E_2(Pa)	E_3(Pa)	G_{12}(Pa)	G_{13}(Pa)	G_{23}(Pa)	v_{12}	v_{13}	v_{23}
Composite	$1{,}72x10^{11}$	$6{,}89x10^{9}$	E_2	$3{,}45x10^{9}$	G_{12}	$1{,}38x10^{9}$	0,25	0,25	0,30
PZT - G1195	$6{,}90x10^{10}$	E_1	E_1	$2{,}59x10^{10}$	G_{12}	G_{12}	0,33	0,33	0,33

Table 2. Electrical and electromechanical properties of PZT

Material	e_{15}(C/m^2)	e_{24}(C/m^2)	e_{31}(C/m^2)	e_{32}(C/m^2)	e_{33}(C/m^2)	χ_{11}(F/m)	χ_{22}(F/m)	χ_{33}(F/m)
PZT - G1195	0,00	0,00	$-18{,}30$	$-9{,}01$	$-9{,}01$	$1{,}59x10^{8}$	χ_{11}	χ_{11}

For purposes of passively attenuation of vibration and noise levels, the electrical shunt resistive circuit was used, in order to attenuate the amplitude of vibration of the first mode of the dynamic structure. The resistance value was calculated as being optimal according to Hagood and von Flotow (1991), which was used in the simulations of deterministic and stochastic problems, with value of $R = 96907{,}00\Omega$.

Thus, by using the value of the calculated resistance, the deterministic frequency response functions (FRFs), for the first mode, both of the composite beam with only PZT without circuit, as the composite beam with PZT coupled by resistive shunt were simulated. The results of these simulations are shown in Fig. 1b.

Immediately can be noted the large capacity of attenuation of the structure vibration levels provided by the resistive shunt circuit. The amplitude was reduced by approximately $40dB$. Also is observed that the use of resistive circuit leads to a slight offset in frequency in the mode of interest, similarly to the results obtained by assigning viscoelastic properties to the system. Similar results can be found in Ribeiro and De Lima (2014).

At this point, the design parameters that were factored-out of the mass and stiffness matrices and the resistance of the circuit will be taken as random. The random variables were modeled as stochastic fields and the discretization of these fields was done by the Karhunen-Loève (KL) decomposition. The stochastic domain was carried out according to several studies in the literature, such as Ghanem and Spanos (1991), De Lima, Rade and Bouhaddi (2010), Koroishi et al (2012) where it and the correlation length were defined in accordance of the finite element shaft. Furthermore, according to literature four terms in the KL series expansion are already enough, but ten terms have the convergence guarantee. So, ten terms were used. By observing many test-simulations, it has been verified that by using ten terms in

KL expansion, the solutions always converge for *500* Latin Hypercube samples or less. In order to calculate the samples, a distribution of three times the standard deviation was used.

The first set of simulations is shown in sequence. It considers only the resistance parameter as variable. The other parameters, such as fiber direction angles and layer thicknesses were, at this moment, considered deterministic.

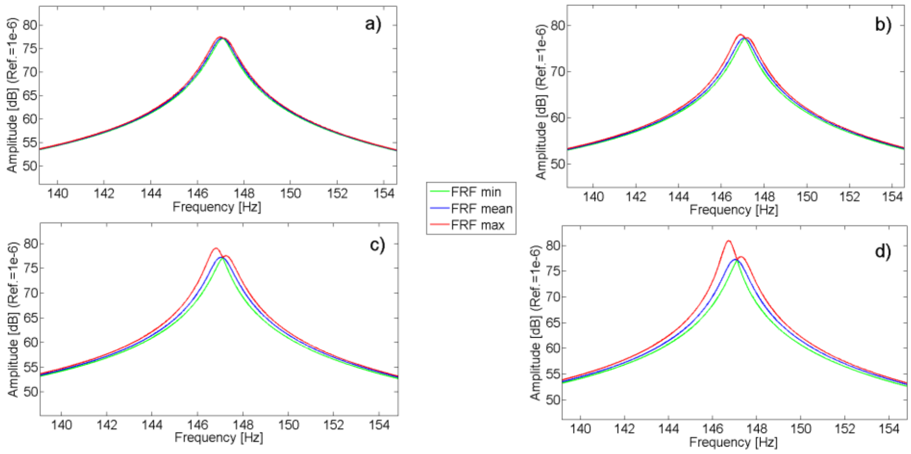

Figure 2. Envelopes of random FRFs of the structure varying only the resistance: (a) five percent, (b) ten percent, (c) fifteen percent, (d) twenty percent.

From Figure 2 is noted that the envelopes of random FRFs, that are delimited by the statistical extremes, increase with increasing level of dispersion around the resistive parameter, also occurring a consequent increase in the confidence region. Furthermore, the average stochastic FRF is found, for all scenarios, inside the envelope. Another important aspect worth mentioning is the inherent robustness of the shunt resistor since even for the worst case, given by the Fig. 2d. This circuit continues to perform the attenuation of the vibration levels of the first mode in a satisfactory way.

The next set of simulations considers the resistance parameter and the structural parameters, such as fiber direction angles and layer thicknesses as being random. The stochastic frequency response functions are shown in sequence:

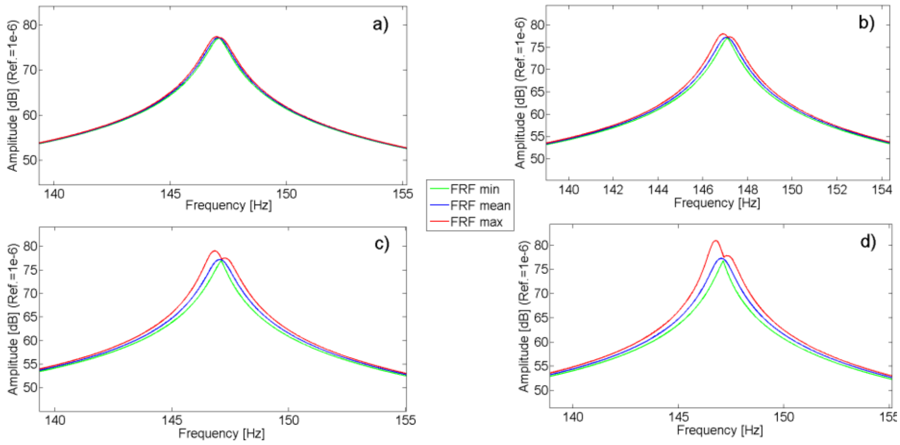

Figure 3. Envelopes of stochastic FRFs of the structure varying the resistance and the structural parameters: (a) five percent, (b) ten percent, (c) fifteen percent, (d) twenty percent.

For this second set of simulations, it can be noted the same behavior of the random FRFs obtained for the simulations in Fig. 1 where only resistance was considered as a random parameter. The increased dispersion of uncertain parameters leads to an increase in the dispersion of FRFs. Furthermore, the average stochastic FRF is inside the envelope formed by the minimum and maximum stochastic FRFs.

By comparing the random FRFs of the coupled electromechanical resistive shunt system given in Fig. 2 with Fig. 3, it is important to note that if besides the resistance value, the thicknesses of layers and fibers directions as well as being considered random variables, few changes were observed in the system responses. Thus, it can be inferred that uncertainties arising from the resistance, in this case, have the greatest influence in the variability and consequent rise in the confidence region of the FRF structure if compared with the uncertainties associated with the thickness and fibers direction. The little relevance of the structural parameters on the variability of the system responses can be explained by their small dimensions if compared with the magnitude of the resistive parameter. Furthermore, it can also be noted that for the worst case, Fig. 3d, the resistive shunt circuit continued to make the attenuation of the vibration levels of the first mode in a satisfactory way, thus demonstrating its robustness.

5. CONCLUDING REMARKS

It was noted the large capacity attenuation of vibration amplitude levels of resistive shunt circuits. Note also that the use of resistive shunt circuit coupled to the structure moves even a little, in the case of this work, the resonance frequency, resembling the results obtained by assigning viscoelastic properties to the system.

In all simulations, it was observed that with increasing level of dispersion of the parameters considered as being uncertain, was increased also the variability of the FRFs, making larger the envelope of the stochastic response functions, increasing the interval reliable. Moreover, it is observed that the average stochastic FRF is, in all cases, within the envelope formed by the minimum and maximum stochastic FRFs.

In the case of electromechanical system coupled with resistive shunt, it was observed that among the variables considered as random, it was inferred that the uncertainties arising from the resistance, have the greatest influence in the variability and consequent increase in the confidence interval of FRFs structure if compared to the uncertainties associated with the thickness of the layers and the fiber directions angles. The little relevance of the structural parameters on the variability of the system responses damped passively via resistive shunt can be explained by the small dimensions of the structure compared to the order of greatness of the resistor. It is also noted the robustness added by the resistive circuit to the structure more PZT, since, even for the worst case, inserting the maximum dispersion in the resistance value of this circuit, it continues to perform the attenuation of vibration levels of the first mode satisfactorily.

6. ACKNOWLEDGEMENTS

The authors are grateful to the following organizations: (i) School of Mechanical Engineering of the Federal University of Uberlandia; (ii) CAPES Foundation (iii)The National Institute of Science and Technology of Smart Structures in Engineering (INCT-EIE).

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8. RESPONSIBILITY NOTICE