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STUDY OF THE PERIODICITY OF THE DUFFING OSCILLATOR WITH TWO DEGREES OF FREEDOM

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Abstract. *Nonlinear systems are becoming each more and more important to the conception of the mechanics projects, because the necessity of the machines that works with great reliability, with more precision and in aggressive environments. In this context, the study about the characteristics of the nonlinear systems become fundamental to the design of the projects. In this work is used the Duffing oscillator because the great number of the physics systems that can be approximated by their equation of motion, it's made through the Wavelet transform and Fourier transform to show the periodicity of the system or the chaotic behavior.*

Keywords: *Duffing Oscillator, Chaos, Nonlinear Dynamics.*

1. INTRODUCTION

The nature is fundamental nonlinear, but because the difficult to find an analytic solution for nonlinear differential equations this models was abandoned instead the use of linearized equation, from (Savi, 2004). But the necessity of machines that works with great reliability, with more precision and in aggressive environments and the effort to predict events like the weather or the position of satellites became essential for the modern society and its importance still increasing. Then, the study of their behavior make possible design projects nearest the nature and consequently to obtain durable machines, more economics and efficiently.

The Duffing equation is very important to the study of the nonlinear systems behavior. Because many physical phenomena are modeled by this equation (Kovacic and Brennan, 2011): the electric circuit with nonlinear inductance, the behavior of a spring for big displacements, nonlinear cable vibration, vibrations in a engage magnetic-elastic beam, magnetic beam, beam with nonlinear stiffness due to in plane tension, large deflection of a beam with nonlinear stiffness, piezoceramics under strong electric fields, simple pendulum for big angles, and others.

2. MODEL

The Duffing equation in its many forms is used to describe a lot nonlinear systems, for many cases is possible get this equation as an approach description of the system to analyze quantitatively and qualitatively. In some cases both, the qualitative analysis could be done for small amplitudes of exciting. For many others this is the first step to move the system from the linear to nonlinear analyze (Jordan and Smith, 2007).

The simplest system that could approach from Duffing equation is the mass-spring-shock absorber. Where the spring is nonlinear $k(u) = ku + \gamma u^3$; and the shock absorber as a linear part *c*. The variable *u* is used to derive the equation of the system to show position displacement of the mass in relation of the equilibrium position; and $f(t) = F\cos(\Omega t)$ as a periodic force in the system (Enns, 2011).

In this work is used the Duffing system with two degrees of freedom as the following picture:

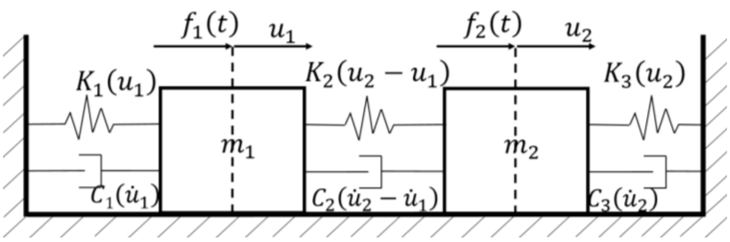

Figure 1. Duffing Oscillator with two degrees of freedom.

First order differential equations to the Duffing system with two degrees of freedom:

$$\dot{x}_1 = y_1 \tag{1}$$

$$\dot{y}_1 = \left(\frac{1}{m_1}\right)\left(F_1 \cos(\Omega_1 t) - (c_1 + c_2)y_1 + c_2 y_2 - (k_1 + k_2)x_1 + k_2 x_2 - \gamma_1 x_1^3 + \gamma_2 (x_2 - x_1)^3\right) \tag{2}$$

$$\dot{x}_2 = y_2 \tag{3}$$

$$\dot{y}_2 = \left(\frac{1}{m_2}\right)\left(F_2 \cos(\Omega_2 t) + c_2 y_1 - (c_2 + c_3)y_2 + k_2 x_1 - (k_2 + k_3)x_2 - \gamma_2 (x_2 - x_1)^3 - \gamma_3 x_2^3\right) \tag{4}$$

Where: $x_1 = u_1$, $x_2 = u_2$, $y_1 = \dot{y}_1$ $y_2 = \dot{u}_2$.

3. SIMULATION

For to realize the numeric integration of Duffing equation with two degrees of freedom the Runge-Kutta of fourth order was chosen. This choice is based in your widespread use and a good resolution without a great computational cost. The elaboration of the script to this work was done in MATLAB because its toolboxes makes easier to create programs, in special on the case of Wavelet Transform and Fourier Transform.

All the simulations considered the initial conditions equal zero and the amplitude of the force to body 1 was 7,5 N and to body 2 equal 4 N. The masses 1 and 2 ware considered equal 1 Kg. The value of some parameter are important to say, as the number of forced periods to each simulation was 1000 periods, and the half of this was considered the transient part. Each period of force was shared in 600 parts and only the first was collected to create the Poincaré Section.

To analyze the periodicity of one dynamic system is important to know what periodicity is. The analyzed system in this work is a forced dynamic system, whence, exist a forced frequency that make the system behavior depends of it. Then, when the system no show a chaotic behavior, there behavior will be periodic and this name is because the cyclic form of the trajectory of the body.

The periodicity could be describe as a number of turn which the trajectory of the move equation solution around the fixe point in a close way during a forced period. Fixe points define the form of a vector camps around them where the trajectories of the solution of the differential equation progress. The presence of a vector camp means that exist a specific energy for each initial condition, in the case of mass-spring-shock absorber a mechanical energy. With the time progress, the trajectories stay around the fixe points because they are limited by the energy in the system.

4. RESULTS AND ANALYSI

4.1 Variation of the periodity by incrising a parameter

The analyze done in this section consider different possibilities to the value of k_2 and consequently obtain different periodicities for little part in parameter variation in these work k_2 equal *0,325 N/m, 0,512 N/m* and *0,727 N/m.* The following table show the groups of parameters used to obtain the simulations.

Table 1. Parameters of the system.

k_1 (N/m)	k_3 (N/m)	c_1 (N.s/m)	c_2 (N.s/m)	c_3 (N.s/m)	γ_1 (N/m³)	γ_2 (N/m³)	γ_3 (N/m³)
-0,02	-0,02	0,05	0	0,05	1	0	1

- For k_2=*0,325 N/m*:

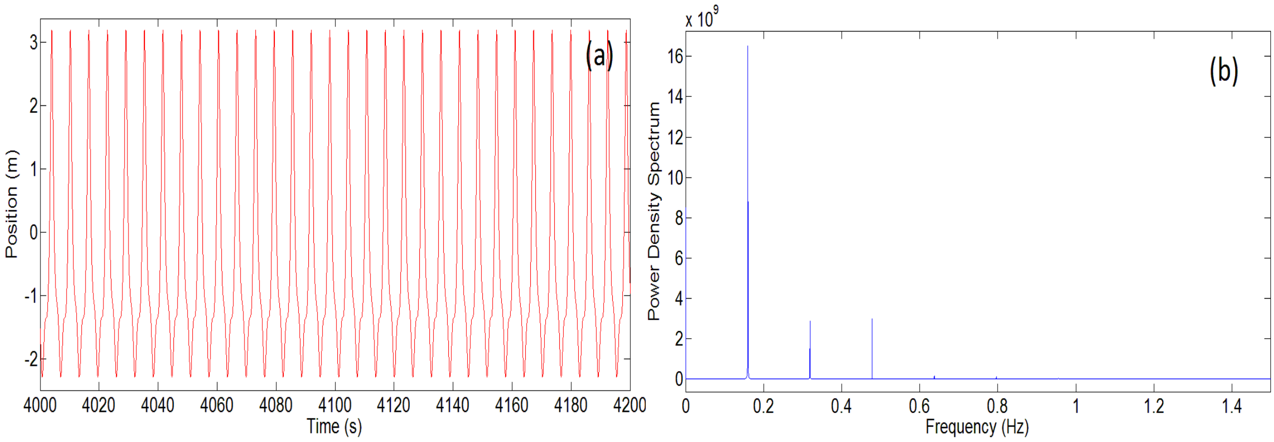

Figure 2. (a) Position time evolution of body 1 in the permanent region. (b) Power density spectrum for body 1 in permanent region.

In the "Figure 2-(b)" have three peaks; these frequencies made as a sine or a cosine the shape of the "Fig. 2-(a)" with just one maximum point per period. It's because the Fourier Transform responsible for make the "Fig. 2-(b)", is based in the approach of a sign by a series of sine or cosine. In a first moment could believe which in the temporal variation should find three peaks because the sign is made of three frequencies.

However, these three frequencies are part of a harmonic sequence, e.g., the value of some frequencies is an integer multiple of the first frequency. Then, while the time developed all the terms of the series maintain in phase. This fact show why the time evolution of the sign is different from a single sine or cosine and not have other peaks.

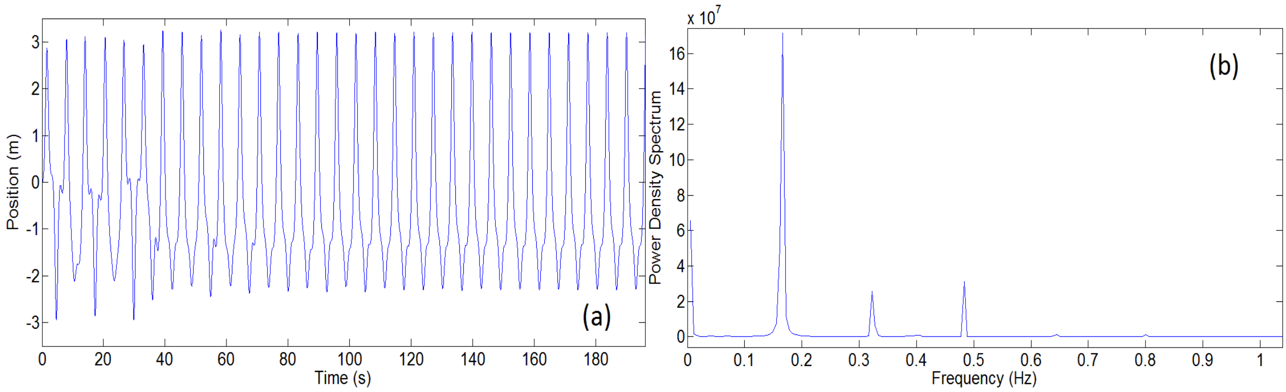

Figure 3. (a) Position time evolution of body 1 in the transient region. (b) Power density spectrum of body 1 in transient region.

"Figure 3-(a)" show a time evolution of body 1 in transient region, where the sign change its shape along this part. Consequently, in the "Fig. 3-(b)" the peaks got a bigger base than in permanent sign, is possible to see that these sign follow the same harmonic series and increase of the base is a result of the presence of transient frequencies in the initial moments of the simulation. Then exist an evidence that the Fourier Transform is not sufficient to analyze the transient part. However, the Wavelet Transform "Fig. 4-(a) and (b)" show that frequencies until 4 Hz had a reduction of amplitude before 40 seconds and is possible to see frequencies bigger than 4 Hz that not exist in the permanent "Fig. 5-(a) and (b)".

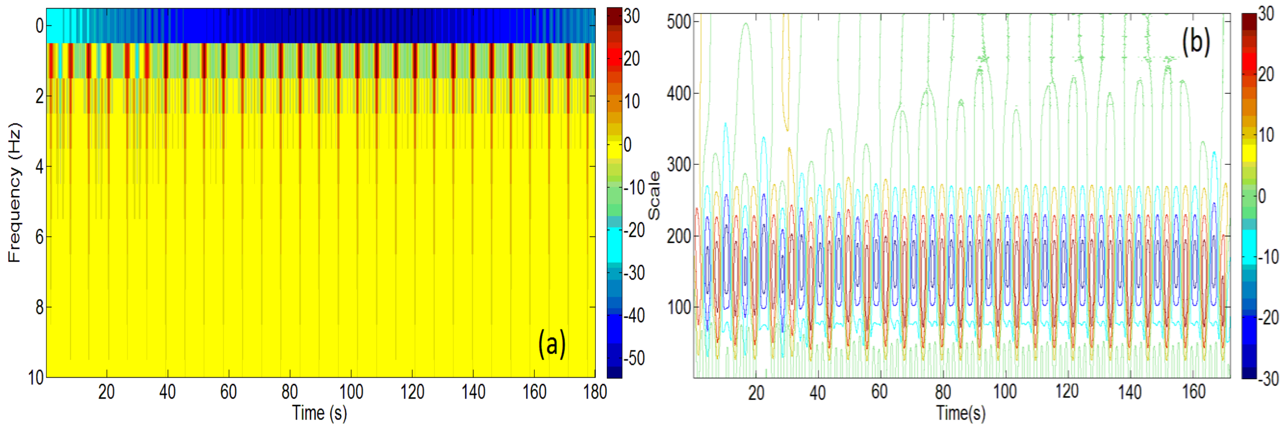

Figure 4. (a) Time-frequency diagram of body 1 in transient region. (b) Time-scale diagram of body 1 in transient region.

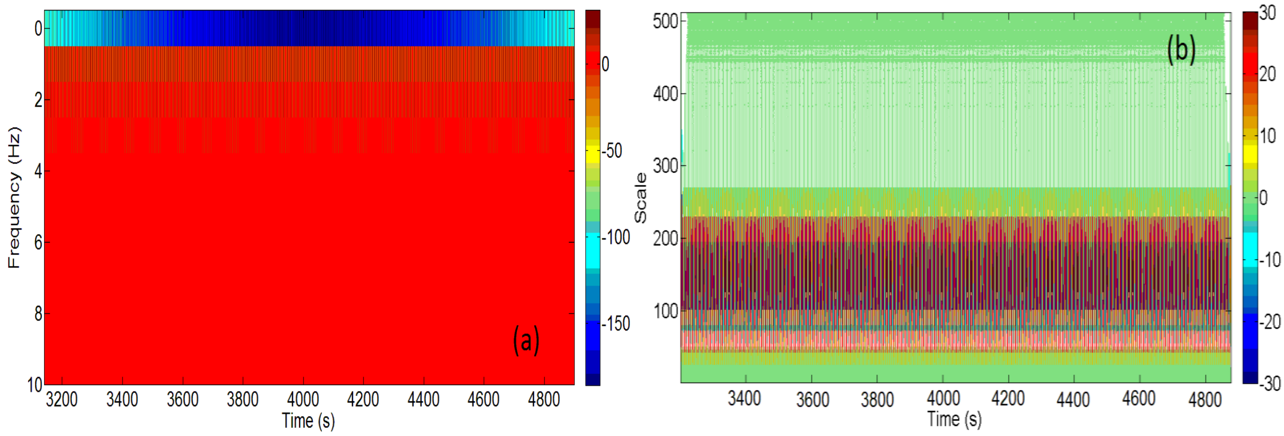

Figure 5. (a) Time-frequency diagram of body 1 in permanent region. (b) Time-scale diagram of body 1 in permanent region.

- For $k_2 = 0{,}512\ N/m$:

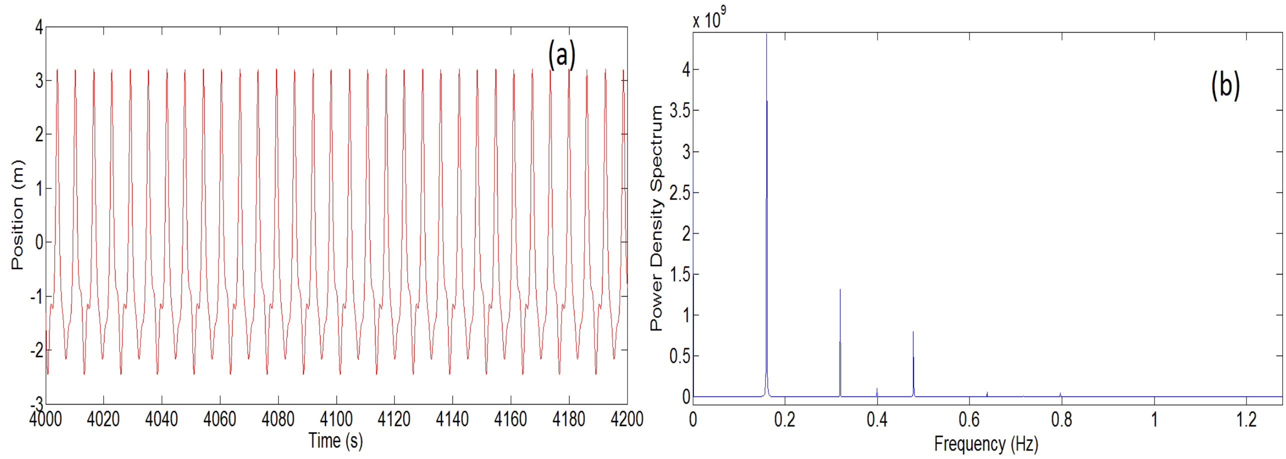

Figure 6. (a) Position time evolution of body 1 in permanent region. (b) Power density spectrum of body 1 in permanent region.

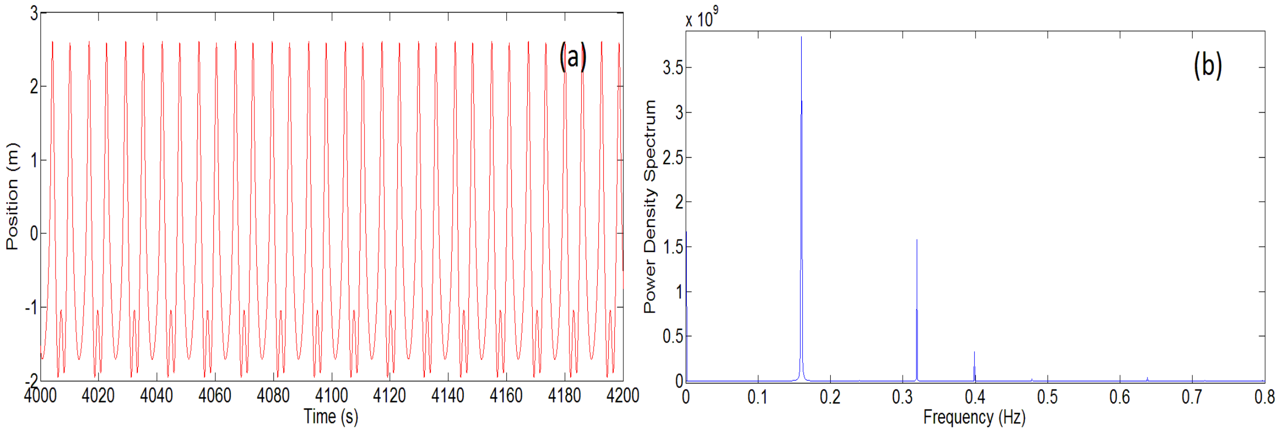

Figure 7. (a) Position time evolution of body 2 in permanent region. (b) Power Density Spectrum of body 2 in the permanent region.

“Figures 6-(a-b) and 7-(a-b)” show in the Fourier Transform that the sign have groups of peaks which form a harmonic series with exception of the peak of 0,4 Hz for the bodies 1 and 2. Because this frequency out of the series, because of these its simulation have periodicity two.

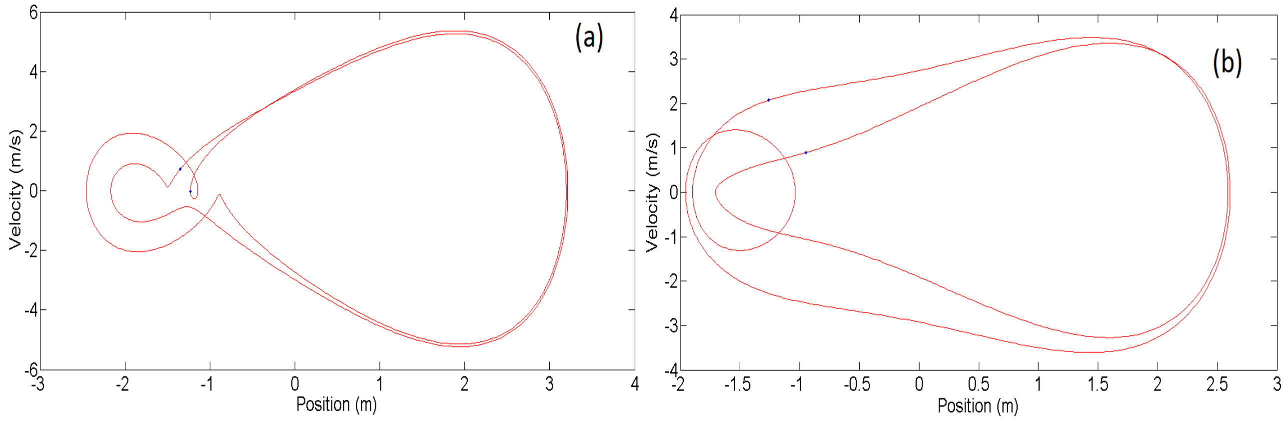

Figure 8. (a) Phase space of body 1 (the two blue points came from the Poincaré Section). (b) Phase space of body 2 (the two blue points came from the Poincaré Section).

Watching the Fourier Transform to the sign of the body 2 is evident which the amplitude of the frequency 0,4 Hz is bigger than the amplitude of the same frequency for the body 1. And this frequency is nearest of the harmonic series of the body 2 than of the body 1, fact that help understand why in phase space of the body 1 the trajectories are so different each one to other. The Poincaré Section is a technic that show exactly the periodicity of the system, but without other information. The Fourier Transform, for example, show the periodicity and the frequencies in the sign, but this periodicity is not exactly. Because in some cases is necessary to pay attention if the peaks not make a harmonic series, fact that is difficult for signs with many frequencies.

- For k_2=*0,727 N/m*:

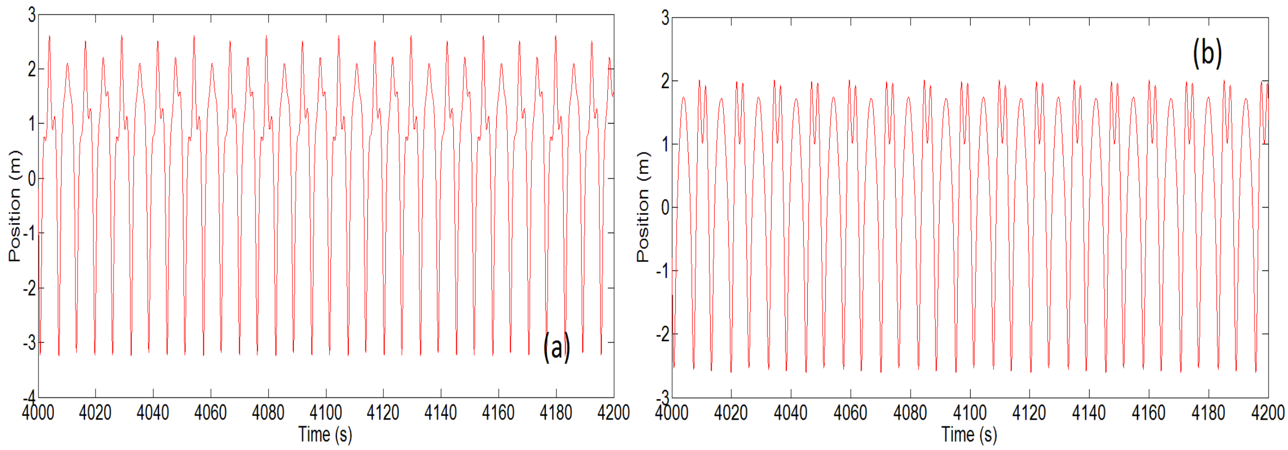

Figure 9. (a) Position time evolution of body 1. (b) Position time evolution of body 2.

“Figure 9-(a)” is possible to see the periodicity of the system with four peaks in sequence repeating along the sign. However, in “Fig. 9-(b)” the periodicity is four too, but is very difficult to separate four peaks repeating. This fact show that importance of analyze of the sign in the Poincaré Section or in the Fourier Transform, because get this information is sometimes too hard in the time domain.

4.2 Analyze of the periodicity in the case of chaos

This section will be understood the chaos from the perspective of periodicity, because one of the most fundamental characteristic of chaotic behavior is no periodicity. With the Wavelet Transform could describe the frequencies on the sign in each part and their intensity what help to understand the changes that happens in a chaotic sign.

Table 2. Parameters of Simulation.

k_1 (N/m)	k_2 (N/m)	k_3 (N/m)	c_1 (N.s/m)	c_2 (N.s/m)	c_3 (N.s/m)	γ_1 (N/m³)	γ_2 (N/m³)	γ_3 (N/m³)
-0,02	-0,02	-0,02	0,05	0,05	0,05	1	0	1

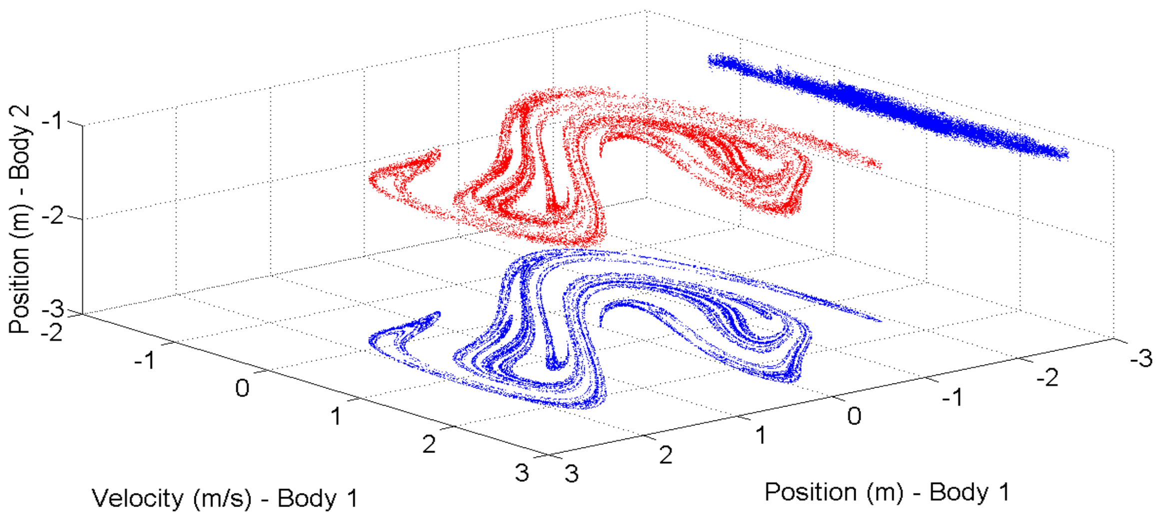

Figure 10. Tridimensional Poincaré Section.

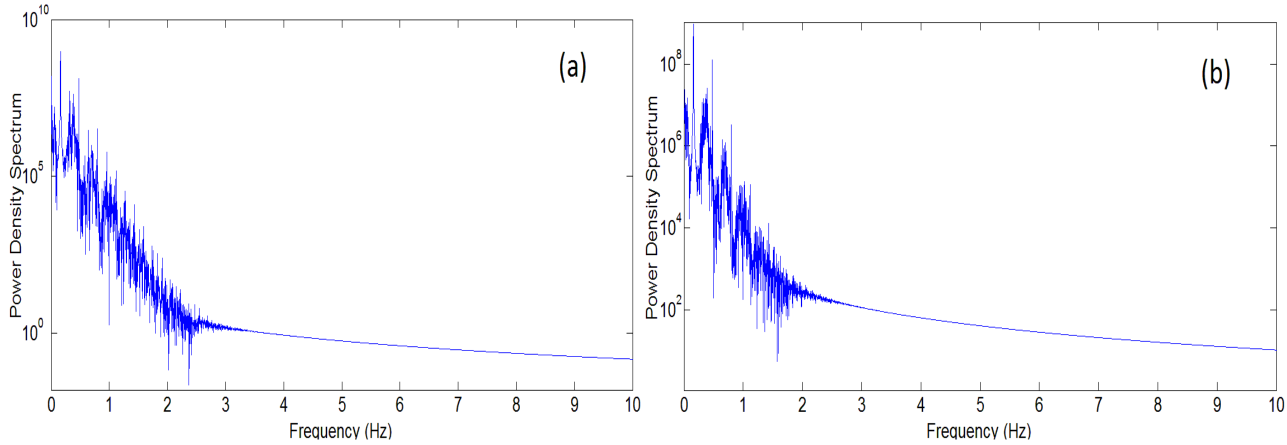

Figure 11. (a) Power density spectrum of body 1 in permanent region. (b) Power density spectrum of body 1 in transient region.

"Figures 11-(a) and (b)" demonstrate a large band of frequencies that form together the sign, e.g., do not have a defined frequency of vibration. Because of that system no show any periodicity on the Poincaré Section. This characteristic is fundamental propriety of the chaotic behavior. Comparing these two pictures it's not possible to know what are the transient frequencies in the sign. It's not happen on periodic case.

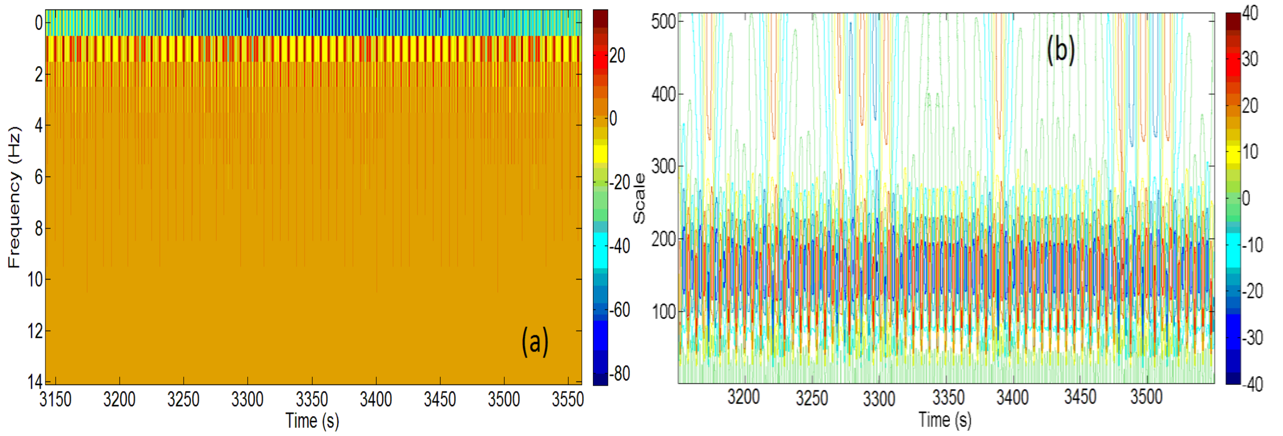

Figure 12. (a) Time-frequency diagram of body 1 in permanent region. (b) Time-scale diagram of body 1 in transient region.

"Figure 12-(a)" show variation on the frequencies between 0,5 and 1,5 Hz in their pattern during the time. That variations could be noted by the change in the color pattern, this fact denote an amplitude change. Then, the contribution of these frequencies to create the original sign is not the same along the time. Frequencies bigger than 2 Hz change a lot their pattern what consequently create large bands of frequencies in this part. "Figure 12-(b)" have changes of pattern for high scales (more than 300, in this case). The higher scales show low frequencies, this fact represents an amplitude variation for this frequencies. It's not demonstrate in "Fig. 12-(a)".

Following have frequency and scale diagrams per time for the transient region of the sign, these pictures show changes of frequency along the time. However as descripted before, it's not possible to know exactly what frequencies are present in transient region and aren't in permanent region.

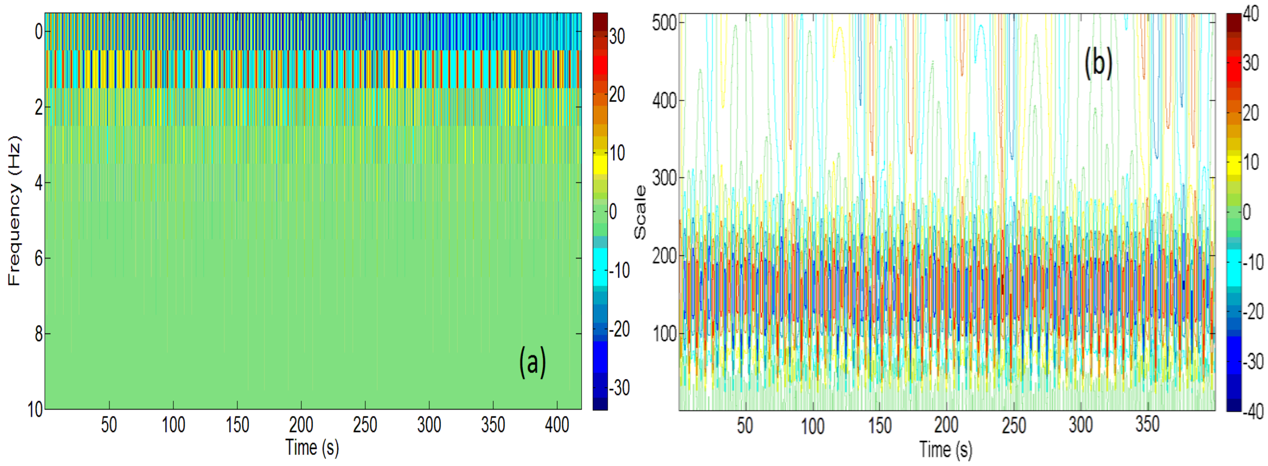

Figure 13. (a) Time-frequency diagram of body 1 in transient region. (b) Time-scale diagram of body 1 in transient region.

5. CONCLUSIONS

When is used Fourier Transform to a periodic sign normally the number of peaks are the same of the periodicity. However, in this work was observed that it's not always happen, because in some cases one or more period is divided in many sub frequencies on a harmonic series. It's possible because these frequencies maintain in phase.

The Poincaré Section is the best option to observe the periodicity of the system this technic show exactly the number of points equal the system periodicity. But when is necessary more details about the sign behavior others technics might be used. For example, to know the manly frequency on the sign or define with more precision where starts the permanent region, for these cases the Wavelet Transform and the Fourier Transform are necessary. The results on this work permit to say which these technics don't should be interpreted alone, but together for made a large vision of the system characteristics.

The study of Duffing equation by these technics is very important to understand the general system behavior. Using this information is possible to do a structural project, design a piece or create a control system in some element. One project based on Duffing equation have closer behavior from nature than a linear model, is more reliable and have durable components.

6. REFERENCES

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