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DETERMINATION OF THE OPTIMAL TRAJECTORIES ON RACE TRACKS WITH DYNAMIC AND GEOMETRIC CONSTRAINTS

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Abstract. *This work proposes a new procedure to determine the optimal trajectory on race tracks based on constrained optimization techniques, where the constraints are defined by means of the dynamic characteristics of the vehicle and the geometrical limits of the track. The vehicle is represented by an oriented particle with the capabilities of traction, braking and lateral acceleration, which are typical in a competition vehicle.*

First, the minimum-time trajectory for a 90-degree curve is obtained through a geometrical analysis of the problem. The solution is then expanded to be applied to all angles. Starting from the individual minimum-time trajectory for each curve of the track, a constrained optimization technique is employed in order to obtain a shorter curve that concatenates these optimal trajectories. The optimization model consists in determining the coefficients of a well-known Hermite cubic polynomial curve that better concatenates the individual trajectories without violating any of the constraints. The results of the concatenation procedure are presented, and the advantages and disadvantages of the proposed method are discussed.

Keywords: *constrained optimization, optimal trajectory, Hermite curve, race tracks*

1. INTRODUCTION

Currently, we are in a breakthrough scenario for the automotive area, where many companies have employed several efforts to develop more efficient and safer vehicles. One of the automotive modalities that are constantly evolving with innovative discoveries is the motor sport industry for competitions in closed circuits. Over the years, the popularity of this sport has contributed to the need to create high-performance cars and, with that, to minimize the time taken to complete the most diverse circuits. In addition, it is clear that prior knowledge of the minimum time path of these circuits implies many advances and facilities in the development process, control and study of the dynamics of racing vehicles and makes simulations and tests much more realistic. The calculation of the optimal trajectory of a race track involves several steps from different areas of knowledge. Some optimization strategies using constrained optimization techniques have been employed in the literature for this purpose. Applying constrained optimization functions from the Optimization Toolbox library of Matlab®, Carrera (2006) proposed an algorithm to obtain the optimal path, where the optimization variables were the tangential and lateral accelerations of the trajectory and the positions on the track where these accelerations were applied during the movement of the vehicle to travel a specific path. Xiong (2010) developed an iterative algorithm that determined the parameters of a spiral of Euler (Ávila, 1985) that is completely contained on the track, so the optimal trajectory is formed by spiral sections. Ribeiro (2009) presented a path optimization algorithm based on genetic algorithms where the constraints of the problem and the objective function were described in the adaptation function of the individuals in the genetic algorithm model.

In addition to constrained optimization techniques, the optimal control approach has been widely used in minimum time path problems. Cossalter *et al.* (1999) presented a model for the minimum time trajectory problem as an optimal control problem with boundary conditions, where the time travel was fixed and the distance traveled was minimized. The acceleration and force constraints were included as a penalty on the cost function. Casanova *et al.* (2000) used an optimal control model to evaluate how the inertia of the vehicle on the yaw-axis affects the minimal time path. The *Sequential Quadratic Programming* algorithm was used to solve the optimal control problem. In the work of Zhang *et* *al*., 2007, the best velocity profile problem was transformed into an optimal control problem using the inverse dynamics of the vehicular maneuvers.

This paper proposes an optimization method based on the use of Hermite curves to concatenate the optimal trajectories of each turn of the track. First, a simplified model of the race car is presented, considering its speed, acceleration and braking limits. Once the model is set, the geometric constraints of the race track are analyzed and an analytical solution is developed to calculate the optimal paths for each turn individually. From these individual optimal trajectories, an optimization model is presented, and the model consists of determining the coefficients of the minimum time Hermite curve that best concatenates them respecting the limits of acceleration. The method developed here presents a few advantages with respect to the others found in the literature. Because of the simpler parametrization of the Hermite curve, the implementation of the algorithm and the analysis of the results are much easier. Additionally, the computational cost is considerably lower. In addition, most previous work did not attempt to find the minimum time trajectory for an entire race track but, rather, focused just on separated sections. The optimal control approach is well established from a theoretical point of view; however, the numerical solution is still a computational challenge.

2. VEHICLE MODEL

The modeling of the dynamic characteristics of a racing car involves many variables and equations depending on the complexity of the model developed. In the work of Casanova *et al*., 2000, a complete model of the vehicle was developed, where the chassis was treated as a rigid body with three degrees of freedom, the yaw angle and the lateral and longitudinal displacements. The wheels were rigidly attached to the body, allowing only the rotation of each wheel relative to the vehicle chassis about its spin axis. Therefore, they added four more degrees of freedom to the system, reaching a total of seven degrees of freedom. For other applications, however, a simpler model represented by only four equations, such as the one that will be presented here, is enough. In this article, the vehicle will be modeled as an oriented particle in a more simplified way, including the traction capabilities, braking and lateral acceleration typical of a competition vehicle.

2.1 Oriented particle model

The oriented particle model adopted for the optimization process in this paper consists of describing the movement of a particle in the global reference system XY, where the position of the vehicle center of mass is represented by [x(t), y(t)] and the car orientation relative to the trajectory is given by ψ, which is defined as the angle between the vehicle longitudinal axis and the global X-axis. The tangential direction of the vehicle trajectory is represented by the vector $\vec{t}$ and the radial direction by the vector $\vec{n}$. The tangential acceleration (t-axis component) is given by a_t, and the normal acceleration (n-axis component) is given by a_n. Figure 1 illustrates the variables used here in the model of a racing car.

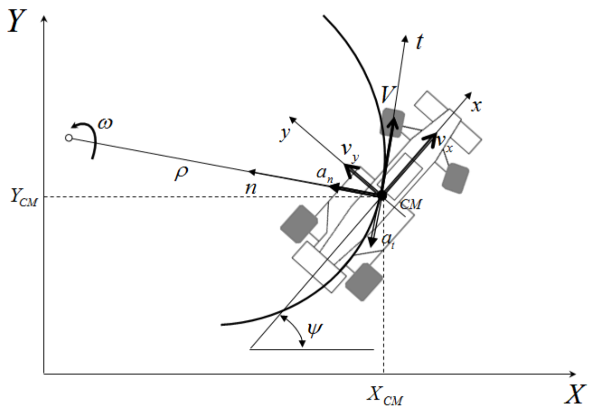

Figure 1: Oriented particle model representation

The differential equations that describe the dynamic behavior of the vehicle model are presented in Eqs. (1) to (4). First, the normal and tangential accelerations are described, followed by the car's orientation angle ψ and, finally, the coordinates of the trajectory of the vehicle's center of mass X_{CM} and Y_{CM}.

$$\frac{dV}{dt} = a_t \rightarrow V = \int a_t \, dt + V(0) \tag{1}$$

$$a_n = \frac{V^2}{\rho} = V \cdot \frac{V}{\rho} = V \cdot \omega \therefore \omega = \frac{a_n}{V} \quad \rightarrow \quad \frac{d\psi}{dt} = \omega = \frac{a_n}{V} \rightarrow \psi = \int \frac{a_n}{V} dt + \psi(0) \tag{2}$$

$$\frac{dX_{CM}}{dt} = V\cos(\psi) \rightarrow X_{CM} = \int V\cos(\psi)\,dt + X(0) \quad (3)$$

$$\frac{dY_{CM}}{dt} = V\text{sen}(\psi) \rightarrow Y_{CM} = \int V\text{sen}(\psi)\,dt + Y(0) \quad (4)$$

2.2 Acceleration limits - Friction Circle

The friction circle graphically represents the limits of acceleration that can be imposed on the vehicle so it does not lose stability and slide. The limits may vary according to the race track coefficient of friction, the maximum engine power, and the tires used, among other factors. Because all vehicles have limited power, the limit for the positive longitudinal acceleration is usually lower than the normal acceleration limits, which are bounded only by the coefficient of adhesion on the track-tire interface, causing the friction circle not to be exactly circular (Milliken *et al.*, 1995). Moreover, the negative longitudinal acceleration limit is also greater than the positive acceleration because the braking capacity of a vehicle must be greater than the traction. Based on this information, Fig. 2 shows how a friction circle is typically drawn and the values for the acceleration limits that will be used to perform the simulations in this article.

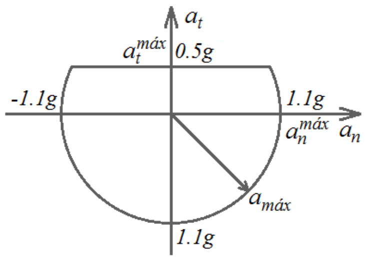

Figure 2: Typical friction circle

It is important that the accelerations obtained in the optimization process are within the friction circle. This information is responsible for establishing the physical enforceability of the obtained trajectory and maintaining the stability and control of the vehicle, especially in regard to racing vehicles, where the goal is always to remain at the limit of their capacity to achieve the best possible performance.

3. OPTIMAL PATH FOR INDEPENDENT CURVES

The optimal path for independent curves is fairly simple to obtain because it is an optimization problem with a known analytical solution (Edmonson, 2011). The optimization problem consists of obtaining the greatest radius possible within the geometric limits of the track, considering that the apex (the point closest to the internal limit) is in the geometric center of the curve. In this way, the angle that is traversed to the apex is half of the total arc. Figure 3a shows the variables for a particular problem in the case of a 90° turn.

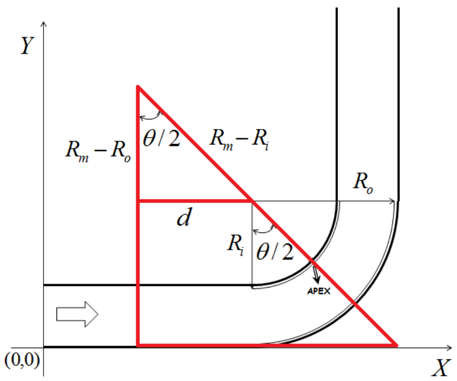

Figure 3a: Geometry of the problem

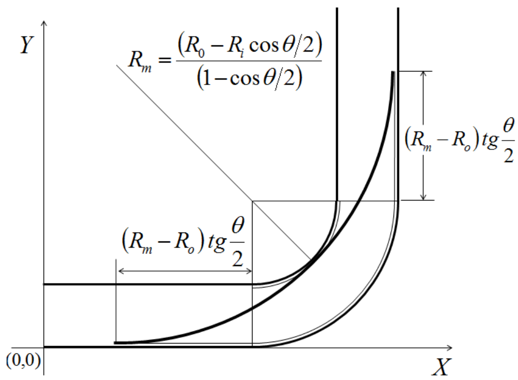

Figure 3b: Solution of the problem

The external radius of the track is given by R_o, the inner radius of the track is given by R_i and the angle of the curve is given by θ. The solution to this problem is a circular arc with the angle equivalent to the curve, which should be tangent to the apex in the minimum possible width, determined by the internal radius, and be limited by the maximum permissible width determined by the outer radius. Eq. (5) shows the calculation of the distance between the trajectory start point and the beginning of the track curve, represented by the variable *d* in Fig. 3a.

$$\mathrm{tg}\,\frac{\theta}{2} = \frac{d}{R_m - R_o} \;\therefore\; d = (R_m - R_o)\,\mathrm{tg}\,\frac{\theta}{2} \tag{5}$$

Using some trigonometric relations in right triangle, it is possible to obtain the start and end points of the optimal trajectory for the curve and the radius of the arc R_m, as given in Eq. (6). The optimal trajectory is illustrated in Fig. 3b. The solution presented in Eqs. (5) and (6) is general and can be applied to curves of any angle. For the analysis of the results and for the development of the concatenation algorithm of these trajectories, the solution presented here was applied on a test track, as shown in Fig. 4a. The independent optimal trajectories obtained for each test lane curve are shown in Fig. 4b.

$$R_m - R_o = (R_m - R_i)\cos\frac{\theta}{2} \quad \therefore \quad R_m = \frac{\left(R_o - R_i\cos\frac{\theta}{2}\right)}{\left(1 - \cos\frac{\theta}{2}\right)} \tag{6}$$

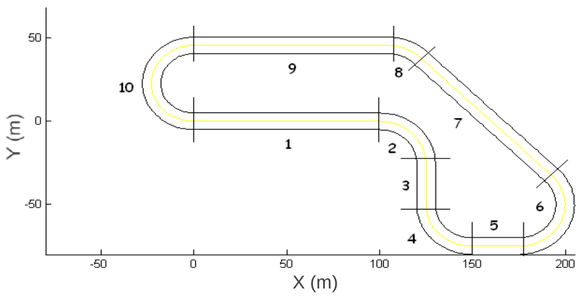

Figure 4a: Test track

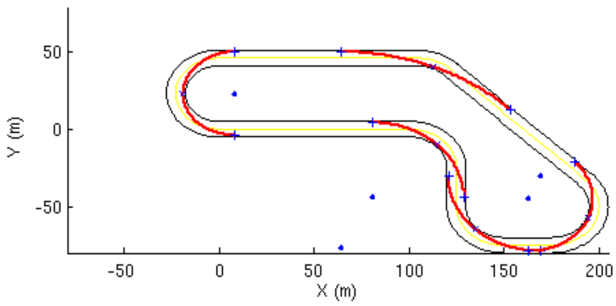

Figure 4b: Independent optimal trajectories

For the dynamic analysis of the vehicle, it is also important to have information about the speed and the accelerations along the trajectories obtained. To ensure the minimum time, it is assumed that the trajectories are held in a uniform circular motion, keeping the normal acceleration in the limit value of the friction circle, which is a_n = 1.1 g m/s² = 10.79 m/s². Eq. (7) shows the calculation of the tangential velocity and the travel time. The results obtained for this test track are presented in Fig. 8 in Section 5 of this article.

$$a_n = \frac{v^2}{R_m} \rightarrow v = \sqrt{a_n * R_m}\,; \qquad t = \frac{\Delta s}{v} = \frac{\mathrm{arc[rad]} \cdot \mathrm{R[m]}}{\mathrm{v\,[m]}} \tag{7}$$

4. CONCATENATION OF THE INDEPENDENT OPTIMAL CURVES

The aim of this work is to develop a method to determine the minimum time path of a race track by means of the concatenation of the independent optimal trajectories of each curve. The main issue here is the choice of the curve that should be used in the concatenation. There are some reasons why the Hermite curve was chosen for this purpose. First, its parametric form is based on simple third-degree polynomial expressions, accelerating the computational time for determining and analyzing the curves. Moreover, because of the simplicity of the representation of the Hermite polynomial, the velocities and accelerations along the curve are determined by a simple derivation of its equations. Once the concatenation curve is chosen, the constrained optimization algorithm is developed to determine the best Hermite curve that concatenates the optimal trajectories of each pair of curve.

4.1 Hermite curve

The Hermite curve is used in several areas of engineering, especially in computer graphics and computer vision (Azevedo *et al*., 2003). It is a cubic parametric curve that is generated to connect two points respecting the tangent vector in each one of them. Its form [x(t),y(t)] is described by the following equations:

$$[x(t), y(t)] = \left(a_x t^3 + b_x t^2 + c_x t + d_x\,,\; a_y t^3 + b_y t^2 + c_y t + d_y\right) \tag{8}$$

To determine a Hermite curve, consider two points P_1 and P_4 and their derivatives $\overrightarrow{R_1}$ and $\overrightarrow{R_4}$. From there, we determine the coefficients *a*, *b*, *c* and *d* of the third-degree polynomial parameterized in the range [0, τ], which satisfies Eq. (9).

$$\begin{aligned} x(0) &= P_{1_x}\,,\; x(\tau) = P_{4_x}\,,\; x'(0) = R_{1_x}\,,\; x'(\tau) = R_{4_x} \\ y(0) &= P_{1_y}\,,\; y(\tau) = P_{4_y}\,,\; y'(0) = R_{1_y}\,,\; y'(\tau) = R_{4_y} \end{aligned} \tag{9}$$

Using the relations from Eq. (9) in the polynomials of Eq. (8), the coefficients of x(t) are obtained as:

$$\begin{bmatrix} x(0) \\ x(\tau) \\ x'(0) \\ x'(\tau) \end{bmatrix} = \begin{bmatrix} P_1 \\ P_4 \\ R_1 \\ R_4 \end{bmatrix}_x = \begin{bmatrix} 0 & 0 & 0 & 1 \\ \tau^3 & \tau^2 & \tau & 1 \\ 0 & 0 & 1 & 0 \\ 3\tau^2 & 2\tau & 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}_x \quad \rightarrow \quad \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}_x = \begin{bmatrix} \frac{2}{\tau^3} & -\frac{2}{\tau^3} & \frac{1}{\tau^2} & \frac{1}{\tau^2} \\ -\frac{3}{\tau^2} & \frac{3}{\tau^2} & -\frac{2}{\tau} & \frac{1}{\tau} \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} P_1 \\ P_4 \\ R_1 \\ R_4 \end{bmatrix}_x \tag{10}$$

Equation (10) is similarly used to determine the coefficients of the polynomial y(t). This technique was considered appropriate for the concatenation because of the suitability of the method with the available data. After obtaining the independent optimal trajectories, the entry and exit points are known, as well as the speed on each point of the curve. Once the variables P_1, P_4, $\overrightarrow{R_1}$ and $\overrightarrow{R_4}$ are known for all curves, the Hermite curve is a function of the τ variable only, which precisely defines the curve travel time. Then, the procedure consists of determining the time τ so that the corresponding Hermite curve is the shortest possible respecting the limits of acceleration and speed of the vehicle and the geometric boundaries of the track.

For the concatenation process in curvilinear sections, the best approach would be to choose the concatenation points in order to maintain the output of the independent trajectories to their original position. This strategy is grounded in the famous race phrase "slow in, fast out", which shows the importance of prioritizing higher speed at the exit of the curve, even if the input is somewhat compromised (Edmonson, 2011). Figure 5a illustrates how the points and the vectors for the concatenation will be chosen in a curvilinear section. For the concatenation on straight sections of the race track, the Hermite curve must connect the exit point of the previous curve with the entry point of the next one, as shown in Fig. 5b. In both figures, the vectors $\overrightarrow{R_1}$ and $\overrightarrow{R_4}$ are tangent to the trajectory at the points P_1 and P_4, respectively, and their magnitudes correspond to the speed of the independent optimal trajectory.

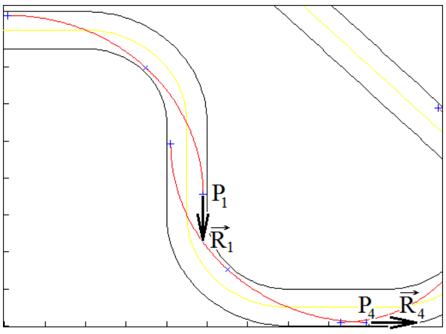

Figure 5a: Concatenation points on a curve

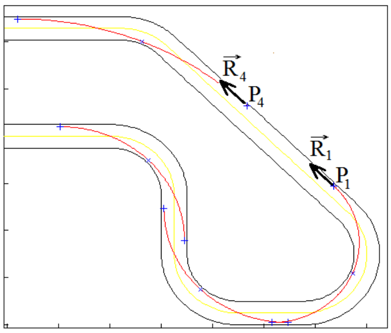

Figure 5b: Concatenation points on a straight line

4.2 Optimization problem

The modeling of the optimization problem is treated differently for curvilinear and straight sections, given the different nature of the constraints involved in each case. The geometric constraint of the curvilinear track sections should be defined taking into consideration the track center position, and the acceleration constraints involve both the tangential and normal accelerations. Conversely, in straight sections, the geometric constraint is simpler because the trajectory obtained will be parallel to the track limits, and only the tangential acceleration is restricted.

4.2.1 Concatenation of the curvilinear sections

The concatenation of the curvilinear sections on the curve is divided into two stages. The first problem consists of, given a Hermite curve, determining the time τ for which the distance between the point on the curve at this time and the center of the turn is minimal. Eq. (11) shows the optimization model for this first stage:

$$\begin{aligned} &\min_t \quad d^2 = (P(t) - C)^2 = \left(\sqrt{(x(t) - x_c)^2 + (y(t) - y_c)^2}\right)^2 = (x(t) - x_c)^2 + (y(t) - y_c)^2 \\ &\text{s.t.} \quad 0 \le t \le \tau \end{aligned} \tag{11}$$

Because this is a minimization problem, the choice to use d^2 instead of d lies in the fact that the algorithm would try to pick a point whose distance was negative and then smaller. Fig. 6a illustrates the variables of this problem, and Fig. 6b shows the result of the optimization model for several Hermite curves varying from τ = 0.05 s to τ = 5 s. The problem is solved by using the function *fminbnd* from the Matlab® optimization library.

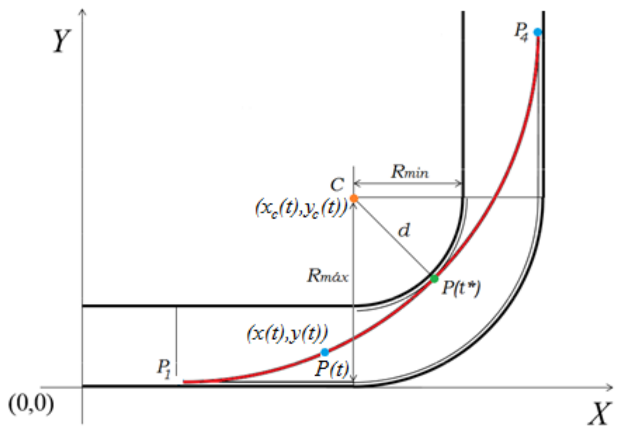

Figure 6a: Variables of the optimization model

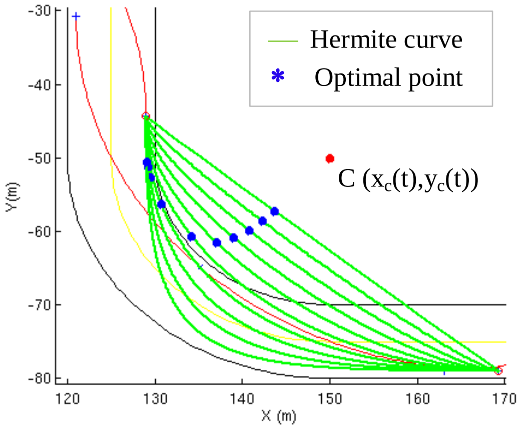

Figure 6b: Result of the optimization

The second step is to determine the shortest Hermite curve where the point P(t*) is at a minimum distance from the center of the turn and respects the limits of the normal and tangential accelerations. The problem formulation is given as:

$$
\begin{aligned}
\min_{\tau} \quad & d_h = (P(\tau^*)_{fminbnd} - {R_{min}}^2)^2 \\
\text{s.t.} \quad & P(\tau^*)_{fminbnd} - {R_{máx}}^2 \leq 0 \\
& \max(a_t) - a_{t_{max}} \leq 0 \\
& \max(a_n) - a_{n_{max}} \leq 0 \\
& \max(a_f) - a_{f_{max}} \leq 0 \\
& \max\left(\sqrt{a_t^2 + a_n^2}\right) - a_{n_{max}} \leq 0 \\
& \tau_{min} \leq \tau \leq \tau_{máx}
\end{aligned}
\qquad (12)
$$

Please notice that, according to the construction of the objective function, a curve whose distance between the point closer to C ($P(\tau^*)_{fminbnd}$) and the point C is equal to (or as close as possible to) the inner radius of the track is expected, which approximates the tangent curve to the track at this point, ensuring that the obtained curve is found to be the smallest length possible with a minimum travel time τ. The first constraint is included to make sure that the curve does not exceed the outer radius of the track in the section to be concatenated, and the last one guarantees the upper and lower bounds for the parameter τ. In this article, the bounds are defined as τ_{min} = 0.05 s and τ_{max} = 10 s. These values are chosen based on experimental tests with the algorithm by evaluating which minimum time value would not present unreasonable values of the tangential acceleration and the maximum time value that would not cause the Hermite curve to degenerate into a looping shape. Eventually, depending on the distances, these bounds have to be redefined for different race tracks. We also emphasize that, for the same reason mentioned above, the use of the square distance prevents the algorithm from finding negative solutions because this is a minimization problem.

The other constraints make up restrictions on the tangential, normal and braking accelerations (negative values of the tangential acceleration), which were placed such that the maximum value of each one of them during the trajectory does not exceed the maximum allowed value, which was already defined in Fig. 2. However, to ensure the limits of each acceleration individually is not sufficient. To guarantee that all of the accelerations are inside the friction circle, it is necessary to add one more constraints in the total acceleration, $\sqrt{a_t^2 + a_n^2}$, which should not assume values greater than the radius of the friction circle, defined by the normal acceleration limit. This optimization model, which is different from the previous one, has non-linear constraints. In this case, the *fminbnd* function cannot be used, and that is why the chosen function for solving this problem is *fmincon*, which is also available from the Matlab® optimization library.

4.2.2 Concatenation of the straight sections

The concatenation of the straight sections is simpler because there is no component of centripetal acceleration, thus reducing the number of constraints. For sections like these, the concatenation consists in determining the Hermite curve that connects the exit point from the previous curve with the entry point of the next one, respecting the speeds at both points. The objective in this case is to minimize the travel time τ of the trajectory, so the algorithm must determine the smallest value of τ for which the Hermite curve complies with the minimum and maximum tangential acceleration constraints. The mathematical model for this case is given by Eq. (13). Similar to the curved sections, the function used for the optimization was also *fmincon*.

$$\begin{aligned} &\min_{\tau} \quad \tau \\ &\mathrm{s.t.}\,\max(a_t) - a_{t_{max}} \le 0 \\ &\max(a_f) - a_{f_{max}} \le 0 \\ &\tau_{min} \le \ \tau \ \le \tau_{máx} \end{aligned} \tag{13}$$

As, by definition, the Hermite curve respects the direction of the derivatives in the initial and final points and the tangent vectors, which in this case are parallel to the track constraints, the curve obtained in the optimization would only violate the geometric constraints for very large values of τ, where the solution would be degenerated. Because the upper and lower bounds are imposed for τ in such way that the curve does not degenerate, it is known that the results of the optimization problem will be inside the track.

5. RESULTS AND CONCLUSIONS

Along the algorithm formulation process, the concatenations were divided into two types: concatenations in the curved sections and concatenations in the straight sections. For each case, an optimization model was proposed to find the best Hermite curve. For the concatenation in curves, a model was developed to search the Hermite curve such that the point closer to the track restriction center is at a minimum distance from it. The intention was to find the minimum time curve that concatenates the two sections, respecting the geometric constraints of the track and the acceleration limits of the vehicle. The points selected for concatenation were always the exit points of the independent curves. For the straight sections, as there was no intersection between the independent optimal curves, the concatenation was performed with the output of the current curve and the entrance of the next one. In this case, the concatenation curve should approach a straight line, and the proposed model found the curve with the minimum time that respects the tangential acceleration limits. The algorithm implemented in Matlab® identified the type of concatenation to be made using the relation between the directions of the derivatives at the start and end points of the concatenation and applied the corresponding model for each case. The final results for the test track presented in Section 3 are shown in Fig. 7.

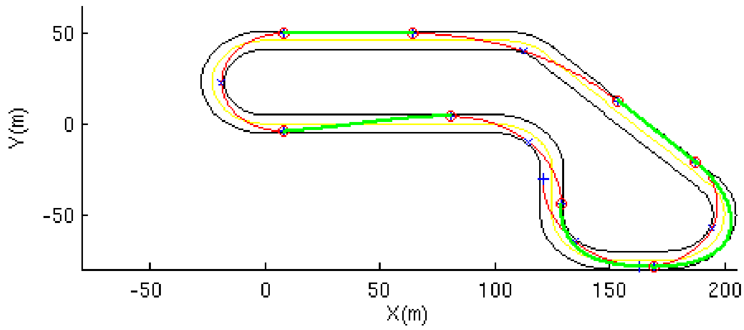

Figure 7: Complete results for the test track

It is observed that the geometric limits of the track were respected in all of the concatenations in the curved or straight sections, as predicted by the model. In section 1, as the derivatives had the same direction but the points were not at the same side of the track, the concatenation curve should represent a smooth changing of sides, as expected. The dynamic characteristics of each independent and optimal curve of the concatenated sections are summarized in a table generated by the algorithm and presented in Fig. 8.

--> Summary for Independent Curves

curve	s(m)	v(m/s)	an (m/s^2)	t(s)
2	75.89	22.83	10.79	3.324
4	75.89	22.83	10.79	3.324
6	80.01	19.14	10.79	4.18
8	99.04	22.09	3.87	4.482
10	84.35	17.02	10.79	4.956

--> Summary for Concatened Sections

section	s(m)	t(s)	at_max(m/s^2)	af_max(m/s^2)	an_max(m/s^2)
3	61.45	3.4	5.934	-8.859	9.863
5	85.12	5.395	2.551	-8.257	8.087
7	47.81	2.182	4.905	-2.194	0
9	56.07	2.498	4.905	-8.921	0
1	73.2	3.35	4.905	-1.435	0

Limit values for accelarations: at_max = 4.905m/s^2; af_max = -10.791m/s^2; an_max = 10.791m/s^2
Algorithm Time = 29.31 s = 0.4885 min

Figure 8: Dynamic results for the test track

Notice, in Fig. 8, that there was a problem in the concatenated section number 3, and it can be concluded that there is no Hermite curve that concatenates the chosen points respecting all friction circle limits (especially the tangential acceleration), as shown in Fig. 9. In other words, this means that the vehicle would lose stability and slide in the proposed curve 3. One possible solution to this problem would be to change the entry point until the solution becomes feasible or to reduce the normal acceleration in the independent curve before. The accelerations of the section 5 are all inside the friction circle; however, because the turn is very sharp, the curve almost reached the outer radius of the track in order to keep the normal acceleration within the limits.

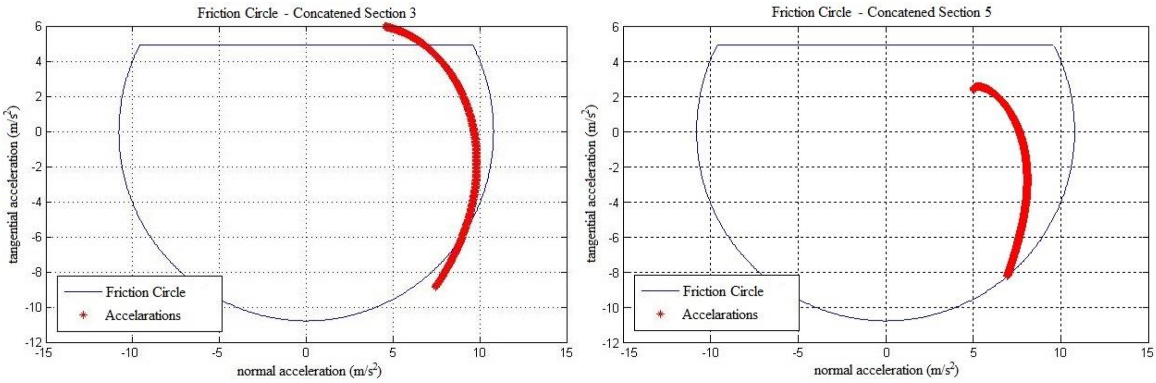

Figure 9: Friction circle results for concatenated curves 3 and 5

Another change that was required was the speed of the independent curve number 8, which had to be reduced so the vehicle could reach the input speed of the curve in a timely manner, considering the small space between curves 6 and 8. This change, however, did not arise from a limitation of the technique used but from the fact that the chosen vehicle was not capable of accelerating so much in that limited amount of time. Furthermore, because the independent optimal curves do not take into account these factors a priori, the reduction is completely acceptable. As for the computational expense, the algorithm took 29.31 seconds to finish the entire concatenation process (using a standard i5 notebook with a Linux operating system), which is considerably efficient given the amount of processing that the optimization model requires. In conclusion, the novel technique for calculating the minimal time path of a race track presented here is efficient and simple to be implemented. Further work will study ways to vary the entry point or the apex point of the concatenated curve in order to improve the accelerations during the trajectories. Additionally, an optimization model using the optimal control technique based on the solution of a boundary value problem is currently under development, where the constraints due to the vehicle dynamics and the track geometrical limits will be included as a penalty on the cost function.

6. ACKNOWLEDGMENTS

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8. RESPONSIBILITY NOTICE

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