23rd ABCM International Congress of Mechanical Engineering
December 6-11, 2015, Rio de Janeiro, RJ, Brazil

Brittle to ductile transition in democratic fiber bundle model

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Abstract. *Fiber bundle model (FBM) has been used for years as a prototypical description of brittle fracture in systems with random failure thresholds. The model describes a parallel connection of breakable units with piece-wise quadratic elastic energy. The democratic or mean-field version of the FBM assumes that after breaking of a single unit the loads are redistributed equally. The democratic FBM has been previously studied only in a soft loading setting. The ensuing behavior is always brittle and the main result was that near the global fracture threshold the model exhibits a power law distribution of failure avalanches. To capture ductile behavior in the same prototypical framework, we augment the FBM by adding internal (series to fibers) and external (series to the bundle) springs and study the resulting system in a hard device. By changing the stiffness of the series springs we were able to simulate a crossover from brittle to ductile response. While the former is characterized by intermittent avalanche distribution near the global failure threshold, the latter shows predominantly Gaussian statistics. Near the brittle-to-ductile transition point correlation length diverges and the whole transition bears a strong resemblance to the athermal phase change of the second order (critical point). Our observations regarding the structure of fluctuations can be used for the development of reliable acoustic-emission-based precursors of global failure and for non-destructive acoustic monitoring of the remaining life of fracturing samples.*

Keywords: *Fracture, Criticality, fiber bundle model, brittle, ductile*

1. Introduction

The history of fiber bundle model (FBM), which is a prototypical description of brittle fracture in systems with random failure thresholds, goes back to the 1920s (Peirce, 1926). The model describes a parallel connection of breakable units with piece-wise quadratic elastic energy. The democratic or mean-field version of the FBM assumes that after breaking of a single unit the loads are redistributed equally. The democratic FBM was previously studied only in a soft device setting and at first the main emphasis was on the distribution of global failure thresholds under the condition that we know the probability distribution of individual thresholds representing quenched disorder in the system (Daniels, 1945).

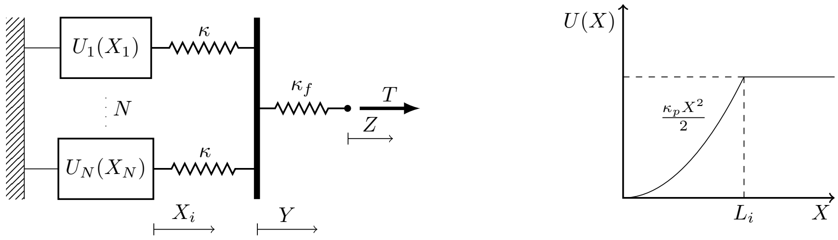
More recently it has been shown that near the global fracture threshold the FBM exhibits a power law distribution of failure avalanches. In the last thirty years the FBM received increasing attention in the physics community because of the appreciation that its critical (scale free) behavior near the global failure threshold belongs to the mean-field universality class which it describes with remarkable analytical transparency (Sornette, 1989)). Extensive computer simulations conducted under the assumption of the soft device loading, revealed that for a broad class of disorder distributions the fracture described by the FBM model is brittle exhibiting universal scaling behavior with a disorder insensitive mean field exponents. Various 'non-democratic' assumptions implying local load sharing have also been studied in the same framework (Pradhan *et al.*, 2010).

In this research communication, to capture ductile behavior in the same prototypical framework, we augment the democratic FBM by adding seemingly innocent internal (series to fibers) and external (series to the bundle) springs and study the resulting system in a hard device. The main advantage of the new model is that now, by changing the stiffness of the external series springs, we are able to simulate the crossover from brittle to ductile response. Ductile response is usually associated with the stable development of small avalanches (micro-bursts) representing debonding events at the microscopic level. Instead, brittle response is associated with large system size events representing crack-propagating instabilities. In our model the two types of fracture are distinguished by their statistical distribution of bond-breaking avalanches. More specifically, we show that brittle fracture is characterized by intermittent avalanche distribution near the global failure threshold with mean field exponents as in the case of soft device, while ductile fracture is characterized by predominantly Gaussian statistics of bursts. Near the brittle-to-ductile transition point correlation length diverges and the whole transition bears a strong resemblance to the athermal phase change of the second order (critical point). We show that the location of the brittle-ductile transition in the FBM is controlled as much by the elastic stiffness of the springs, as by the variance of the quenched disorder.

In contrast to the standard FBM, our augmented model exhibits various types of scaling. The first type, which can be interpreted as resulting from spinodal instability, is observed in brittle regimes near the ultimate failure threshold.

These regimes cannot be called critical because of the ubiquitous presence of the system size events corresponding to the development of the global crack. On the background of this super-critical or SNAP behavior we observe scaling with mean field exponents and system size cut off as in the case of the soft device (Hemmer and Hansen, 1992). The complete criticality or crackling behavior, is observed exactly at the brittle-to-ductile transition, which can be then identify with the classical thermodynamic critical point. We show that the critical exponents in this regime remain of the mean-field type which is natural in view of the infinitely-long-range interactions in the democratic model. In the weakly ductile regimes we observe the coexistence of limited scaling with Gaussian POPing and in this case the scaling region cut off has nothing to do with the system size. Finally, the fully developed ductile regime does not show any traces of intermittency in the distribution of fracture bursts and ductile fracture can be interpreted as a succession of uncorrelated events. These and other related observations can be used for the development of reliable acoustic-emission-based precursors of global failure and for non-destructive acoustic monitoring of the remaining life of fracturing samples.

In contrast to most engineering studies of fracture, we interpret in this communication the phenomenon of bond breaking as a reversible process. The reversible model is inspired by the studies of muscle contraction (Marcucci and Truskinovsky, 2010; Caruel *et al.*, 2013) where parallel connections of the FBM type also play a prominent role and by the theories of cell adhesion where FBM type models of interacting integrin binders have been considered as well (Schwarz and Safran, 2013). The reversibility of the model turns out to be irrelevant in the FBM setting as far as the criticality and scaling is concerned. Our model, of course, remains applicable to irreversible fracture if we consider monotone loading only.



(a) Mechanical Model (b) Potential Energy

Figure 1. Mechanical model of the bundle structure and the potential energy of a single bond.

2. The model

Consider N parallel bonds that are fixed at one side and linked to a rigid back bone in the other side, see Fig. (1)(a). Each bond combines two springs in series, a linear one with a fixed elastic modulus κ and a nonlinear one whose elasticity is represented by a potential shown in Fig. (1) (b). The nonlinear springs show Hookean elasticity with the same elastic modulus κ_p up to the point of failure characterized by random thresholds L_i

$$U_i(X) = \begin{cases} \dfrac{\kappa_p X^2}{2} & if\, X \leq L_i \\ \dfrac{\kappa_p L_i^2}{2} & if\, X > L_i \end{cases} \tag{1}$$

For values of X_i smaller than L_i the bond is considered to be closed and when the variable X_i reaches the threshold the bond breaks dissipating (or transforming into a non mechanical form) the energy $\kappa_p L_i^2/2$.

If we now assume that the bundle is loaded in a hard device through a spring with stiffness κ_f, we can write the total energy of the system in the form

$$E = \sum_i U_i(X_i) + \frac{\kappa}{2}\sum_i (Y - X_i)^2 + \frac{\kappa_f}{2}(Z - Y)^2 \tag{2}$$

where Y is a position of the backbone and Z is the controlling parameter representing total elongation, see Fig. (1) (a).

We assume that the bonds are initially closed and that the system is stretched until complete failure. As each individual bond breaks the load distributes itself equally among the surviving bonds, while the broken bonds carry no more load. When the stress is redistributed on the surviving bonds it may compel further bonds to break because the stress on the intact bonds increases. This start an iterative process producing bursts or avalanches that continues until an equilibrium is reached or to the complete breakdown of the bundle.

It will be convenient to work with dimensionless variables. We define the average threshold as $\bar{L}$ and then introduce $x_i = X_i/\bar{L}$, $y = Y/\bar{L}$, $z = Z/\bar{L}$ and $l_i = L_i/\bar{L}$. The remaining non-dimensional parameters of the problem are

$$\lambda = \frac{\kappa}{\kappa_p} \tag{3}$$

and

$$\lambda_f = \frac{\kappa_f}{N\kappa_p}. \tag{4}$$

Additional dimensionless parameters will be associated with the probability distributions of thresholds l_i which we denote by $p(x)$. By normalization we know that $\int_0^\infty p(x)dx = 1$ and $\int_0^\infty xp(x)dx = 1$. The corresponding cumulative distribution is $P(x) = \int_0^x p(y)dy$.

3. Averaged behavior

Let $x_k, k = 1, ..., N$ be the ordered sequence of failure threshold l_k: $x_1 \leq x_2 \leq \cdots \leq x_N$. Under the conditions of democratic redistribution, the external elongation on the bundle, z_k, when the kth bond is about to break can be expressed as

$$z_k = \frac{\lambda+1}{\lambda}[(1+\frac{1-k}{N})\frac{1}{\Lambda}+1]x_k \tag{5}$$

where

$$\Lambda = \frac{\lambda_f(\lambda+1)}{\lambda}. \tag{6}$$

The average displacement $\bar{z}(x)$ at elongation x is easy to assess for large N by noting that $P(x) \simeq k/(N+1)$. Thus, Eq. (5) is approximated by

$$\bar{z}(x) = \frac{\lambda+1}{\lambda}([1-P(x)]\frac{1}{\Lambda}+1)x \tag{7}$$

The average tension can be written in the form

$$\bar{\sigma}(x) = N[1-P(x)]x \tag{8}$$

By using Eq. (7) and Eq. (8) we can use strain x as a parameter to obtain the average stress-strain relation $\bar{\sigma} = \bar{\sigma}(\bar{z})$.

The introduction of parameter λ is responsible for the fact that loading and unloading paths do not coincide. Along the reverse path, the rebinding or healing of the bonds, is described by the equation

$$\bar{z}_r(x) = \left[(1-P(x))\frac{1}{\Lambda}+1\right]x. \tag{9}$$

The corresponding stress can be found from

$$\bar{\sigma}_r(x) = N\frac{\lambda}{\lambda+1}[1-P(x)]x. \tag{10}$$

As in the case of loading we can also obtain the average stress-strain relation $\bar{\sigma}_r = \bar{\sigma}_r(\bar{z}_r)$ in the case of unloading. Finally we mention that along the loading path the maximum value z_c corresponds to x_c which is a solution of the equation

$$[1-P(x_c)] - p(x_c)x_c + \Lambda = 0 \tag{11}$$

Similar equation can be obtained for the unloading path. To illustrate these relations we assume that the thresholds obey Weibull distribution which is broadly used in reliability engineering (Weibull, 1951; de S. Jayatilaka, 1979). In this paper we use the one-parameter Weibull distribution

$$p(x) = \rho x^{\rho-1}e^{-x^\rho}. \tag{12}$$

The corresponding cumulative distribution is then

$$P(x) = 1 - e^{-x^\rho}. \tag{13}$$

Here ρ is the Weibull slope and a new non-dimensional parameter of the problem which is related to the variance of disorder through

$$\text{Var}(X) = \Gamma\left(1+\frac{2}{\rho}\right) - \left(\Gamma\left(1+\frac{1}{\rho}\right)\right)^2 \tag{14}$$

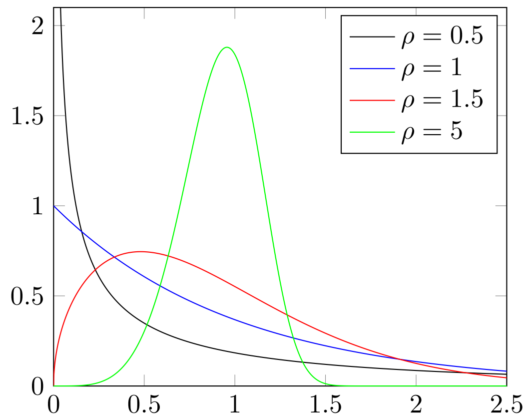


Figure 2. The Weibull probability density function for several values of shape parameter ρ

where $\Gamma(x)$ is the Gamma function and it is clear that the distribution becomes progressively more and more narrow as the parameter ρ increases, see Fig. (2) . In case of Weibull distribution of thresholds we can solve Eq. (11) explicitly to obtain along the loading path that

$$x_c = \left(\frac{1}{\rho} - W\left(-\Lambda\frac{e^{1/\rho}}{\rho}\right)\right)^{1/\rho}, \tag{15}$$

where $W(x)$ is the Lambert function defined by the equation $z = W(z)e^{W(z)}$. Since this function is not injective, the relation $W(x)$ is multivalued (except at 0). This function makes sense for $z \geq -1/e$, which gives the inequality

$$\Lambda \leq e^{-\frac{1}{\rho}-1}\rho. \tag{16}$$

This relation defines in the averaged model the boundary between the brittle and ductile regimes, see Fig. (3) (a). Figure (3)

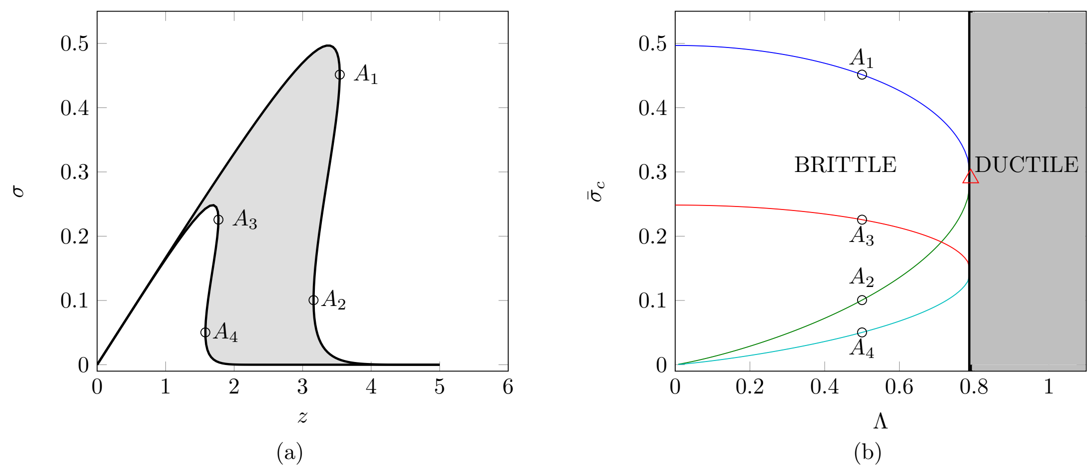


Figure 3. (a) Typical stress-strain behavior in the brittle regime ($\Lambda = 0.5$). In the hard device loading A_1, A_2 are the spinodal points for the forward loading path, while A_3, A_4 are the spinodal points for the unloading path. (b) Brittle to ductile transition at $\rho = 3$.

(b) shows the values of the critical tensions, σ_c, obtained at x_c, for fixed disorder ρ and varying elasticity parameter Λ. The point Λ_c where the two curves meet marks the transition between brittle to ductile behaviour. In a brittle regime when the control parameter z reaches the point A_1 the system undergoes a finite jump. In the ductile regimes such jump does not exist and the evolution is continuous. If we interpret the spinodal stress as the order parameter, it is clear from Fig. (3) (b) that both direct and reverse transitions are of the second order (super-critical pitchfork bifurcation).

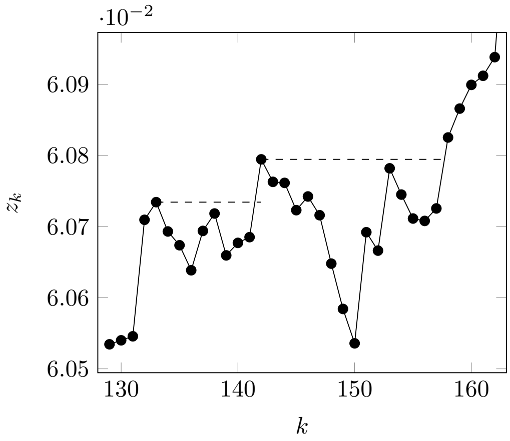


Figure 4. Fluctuations of the birst thresholds z_k for a bundle containing $N = 1000$ in the window with $129 \leq k \leq 162$. The first dashed black line ilustrates a burst of size $\Delta = 8$ and the second line marks a burst of size $\Delta = 15$.

4. Avalanche distribution

The averaged behavior described in the previous section completely neglects fluctuations. Indeed, it is easy to see that the sequence of external imposed displacement $\{z_k\}$ do not form a monotonically increasing or decreasing sequence, but oscillates. From Eq. (5) we see that z_k is the product of a monotonically increasing fluctuating factor x_i and a monotonically decreasing factor $(N + 1 - i)$ plus a constant. Now suppose that an applied displacement z endanger the dissociation of $k - 1$ bonds. This is equivalent to say that $z > x_k > x_j$, for all $j < k$. Assume that the the subsequent values satisfy $x_{k+j} \leq x_k$, with $j = 1, 2, \ldots, \Delta - 1$, but $x_{k+\Delta} > x_k$, this means that all the bonds $x_k, x_{k+1}, \ldots, x_{k+\Delta-1}$ will break without changing the external load, until a new stable position is reached, and the bundle is in equilibrium. This simultaneous failure of bonds due to the increasing of stress because of redistribution of the load forms a burst or an avalanche of size Δ and in view of the acoustic emission data it is of interest to study the expected number of bursts $D(\Delta)$ of size Δ. This counting of burst is illustrated in the Fig. (4).

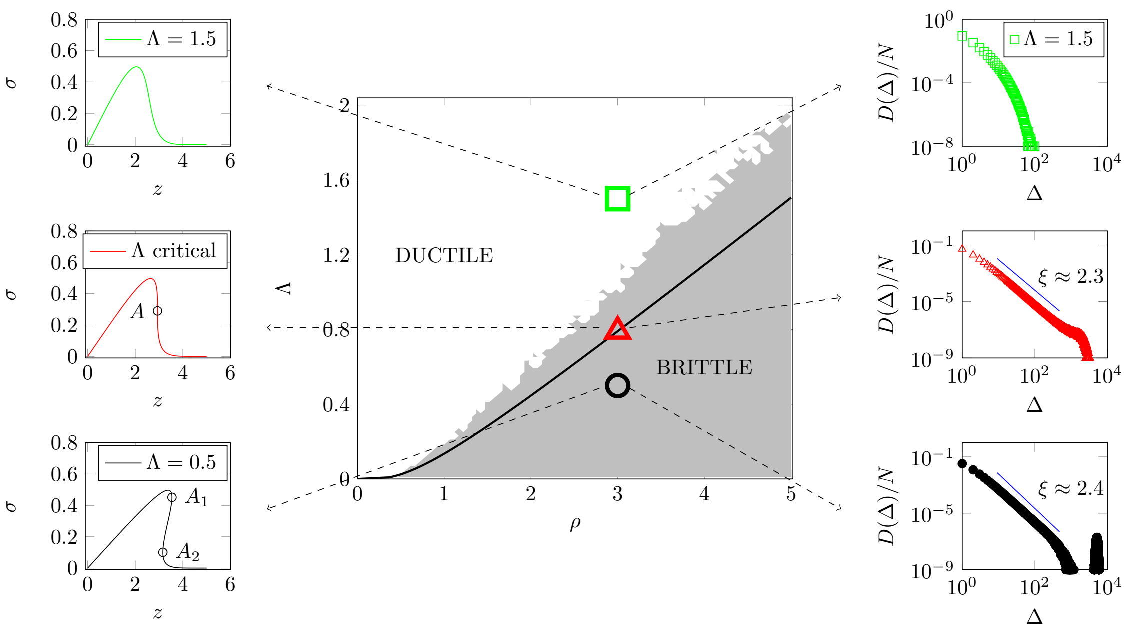


Figure 5. Brittle-to-ductile transition (critical) line in the (ρ, Λ) phase diagram.

For a classical democratic FBM loaded in a soft device it can be shown analytically that the frequency of bursts in which Δ fibers break simultaneously asymptotically follows an universal power law distribution with the distribution $D(\Delta)/N \propto \Delta^{-\xi}$, where $\xi = 5/2$ (Hemmer and Hansen, 1992; Pradhan *et al.*, 2010). Closely following this work we can derive at large N a new formula for the case of hard device accounting for the additional elasticity parameter Λ:

$$\frac{D(\Delta)}{N} = \int_0^{x_c} \frac{\Delta^{\Delta-1}}{\Delta!} \left(\frac{1}{1+\Lambda}\right)^{\Delta-1} \left[\frac{p(x)\,x}{1-P(x)}\right]^{\Delta-1} \exp\left[-\Delta \frac{p(x)\,x}{1-P(x)}\right] \left[1 - \frac{p(x)\,x}{1-P(x)+\Lambda}\right] p(x)\,dx. \quad (17)$$

Here x_c is the value of x which maximizes the equation (7). From this formula we can derive an asymptotic relation near $x = x_c$. First, we may rewrite equation (17) as

$$\frac{D(\Delta)}{N} = \frac{\Delta^{\Delta-1}}{\Delta!} \left(\frac{1}{1+\Lambda}\right)^{\Delta-1} \int_0^{x_C} f(x) e^{[-g(x)+\ln g(x)]\Delta} dx \quad (18)$$

with $f(x) = \left[1 - \frac{p(x)\,x}{1-P(x)+\Lambda}\right] \frac{p(x)}{g(x)}$ and $g(x) = \frac{p(x)\,x}{1-P(x)}$. To compute this integral one may employ a saddle-point like approximation, we observe that for large Δ the exponential term reach its maximum for $x = x_0$ such that $\frac{p(x_0)\,x_0}{1-P(x_0)} = 1$. We expand the function $f(x)$ around x_0 to first order and the function $h(x) = [-g(x) + \ln g(x)]\Delta$ to second order,

$$f(x) \approx f(x_0) + f'(x_0)(x - x_0) \quad (19)$$

$$h(x) \approx -\Delta[g(x_0) + \frac{g'^2(x_0)}{2}(x - x_0)^2] \quad (20)$$

using these expansions and the Stirling approximation $\Delta! \approx \Delta^\Delta e^{-\Delta}\sqrt{2\pi\Delta}$ we obtain the asymptotic result

$$\frac{D(\Delta)}{N} \approx \frac{\Delta^{-3/2}}{\sqrt{2\pi}} \left(\frac{1}{1+\Lambda}\right)^{\Delta-1} \left[C_1\Delta^{-1/2} + C_2\Delta^{-1}\right] \quad (21)$$

with $C_1 = \frac{\sqrt{2\pi}}{g'(x_0)} f(x_0)$ and $C_2 = -\frac{f'(x_0)}{[g'(x_0)]^2}$.

The known disorder insensitive, mean-field scaling relation can be now obtained in the limit of small Λ when the loading is effectively soft. However, now we can also consider the separate roles of parameters Λ (series elasticity) and ρ (disorder).

For small values of Λ and fixed ρ our system is similar to the soft device case, therefore it is expected that the burst distribution in this case will follow the asymptotic power-law behaviour, see Fig. (5). As Λ increases at fixed ρ the system exhibits a transition from brittle to ductile behavior, see Fig. 5. The grey region in the Fig. 5 corresponds to the parameter values (ρ and Λ) where the power law behavior can be claimed confidently. To construct the boundary we implemented an algorithm generating for every pair (ρ, Λ) the corresponding burst distribution and then applying a Kolmogorov-Smirnov test to assert the validity of the hypothesis of power law distribution. The fluctuations around the critical line are due to the tolerance error in the identification of the power law. Notice that the ductile behavior is obtained when the parameter Λ is greater than the value suggested by the averaged theory, see Eq. (16). As we advance deeper into the ductile regime we observe progressive loss of power law tails in the distribution of burst distribution.

The existence of a line of critical points in ρ, Λ plane separating brittle and ductile regimes indicates that during brittle to ductile transition the correlation length diverges. The corresponding regimes are scale free and acoustic emission test should detect the crackling noise. Below this line we observe super-critical regimes (SNAP) while the regimes above this line are sub-critical (POP). Similar phase diagram can be constructed for the unloading path in the system that is able to rebind. In this case, which is of significant importance for biological systems where adhesive bonds can usually heal, one would see some hysteresis, however the statistical properties of avalanches in direct and reverse processes remain the same.

5. Conclusions

We used democratic FBM to study the brittle-ductile transition in a prototypical elastic system with mean field interactions. This transition can be approached by tuning of either the variance of the disorder or the appropriate combination of internal stiffness parameters. Despite the fact that temperature in this athermal model is equal to zero, our study provides compelling evidence that the brittle-to-ductile transition is remarkably similar to the thermodynamic phase transition of the second order with diverging correlation length and power law distributed fluctuations. We associate ductile fracture with subcritical regimes characterized by poping noise and brittle fracture with supercritical regimes characterized by system size snap events. The exponent of the power law distributed avalanches predicted by our augmented FBM model in the supercritical (brittle) and in the critical (transition) regimes is (almost) the same as in the standard FBM model where only snap regimes have been observed previously and ductility was automatically excluded.

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