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An Effective Approach for the Control of Uncertain Systems

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Abstract. *This paper develops a new methodology for robust control of uncertain nonlinear systems. Two different controllers are developed and combined to accurately satisfy control requirements in the presence of (a) system uncertainties in our knowledge of the exact physical system, and (b) uncertainties in the given forces acting on the system. The methodology is developed in two steps. First, a controller is designed for the nominal system, which is our best assessment of the actual physical system, and no uncertainties are considered at this step. This controller is realized using a recent finding in the field of analytical dynamics. The control requirements are recast as constraints on the dynamical system, and use of the analytical dynamics result enables us to find the required generalized forces to exactly satisfy the control requirements while simultaneously minimizing a control cost. The discrepancies caused by ignoring the uncertainties are next compensated for by adding a second controller that is obtained by generalizing the concept of sliding mode control. Unlike conventional sliding mode controllers, this additive controller does not suffer from a chattering problem; it also leads to much smaller errors, and faster response. In addition, it allows many different forms of control functions to be used, depending on practical considerations of implementation. A numerical example of a three-degree-of-freedom pendulum is simulated to demonstrate the control strategy developed herein, in which there are uncertainties both in the description of the physical system's parameters and in the description of the gravity field acting on the pendulum. The example illustrates the accuracy, efficiency, and ease of implementation of the new control methodology.*

Keywords: *Constrained motion, Uncertain system, Uncertain given force, Generalized sliding mode control, PID sliding manifold*

1. INTRODUCTION

Nowadays, people are searching for the simplest and most effective way to reach the exact description for dynamical systems that can explain the exact motions of bodies. However, the assessment of parameters for real-life physical systems is known only to within some bounds of uncertainty that may depend on the various levels of their description. The uncertainties that arise in complex mechanical systems stem from two main sources: (i) uncertainties in our knowledge of the physical system, like uncertainties in the stiffness and mass distribution, the nature of damping, etc. and, (ii) uncertainties in our knowledge of the externally applied forces acting on the system.

In the past few years, several researchers have addressed the problem of deriving the equations of motion of systems with uncertainties. In the following we discuss a few of the techniques that have been used in motion control of nonlinear uncertain systems. Model-based controllers proposed in Su and Stepanenko (1994) and Oya, *et al*., 2003 guarantee the asymptotic convergence of motion errors. In Tseng and Chen (2003) and Chang and Chen (2000), a mixed H_2 / H_∞ adaptive tracking control approach and an adaptive fuzzy approach are presented respectively. Su and Stepanenko (1994), Oya, *et al*., 2003 and Tseng and Chen (2003) address the control of configuration variables only related to nonholonomic constraints and ignore the variables subject to holonomic constraints. Wang, *et al*., 2006 consider a tracking control method which deals with both holonomic and nonholonomic Pfaffian constraints. The dynamical model is assumed to be perfectly known with no uncertainties in it. Wang *et al*., 2004 apply robust adaptive techniques to systems with uncertainties and get satisfactory performance under a suitable set of assumptions. A tracking controller based on sliding mode control for systems with nonholonomic Pfaffian constraints with uncertainties is proposed in Song *et al*., 2005. The sliding mode controller uses a specific signum (switching) function which, in general, is known to cause chattering.

The two sources of uncertainty described earlier are simultaneously considered in this paper, and in what follows, all these uncertainties are included in what we call the 'real-life mechanical system,' or the '*actual system*,' whose description is known only imprecisely. While not known precisely, it is assumed, however, that we have estimates of the bounds on the uncertainties involved. Our best assessment of a given actual system will be referred to as the '*nominal system*'. This term naturally also includes the nature of the 'given' forces acting on the system. The constraints on the dynamical system can be holonomic and/or nonholonomic; they are not required to be in Pfaffian form nor functionally independent.

The aim is to develop a closed-form control for a general dynamical system, which when applied to an 'actual system,' causes this system to follow the trajectory that is pre-specified (by the constraints imposed) on the corresponding 'nominal system,' and thereby to satisfy the constraints on the nominal system. First, the nominal system is controlled by using the fundamental equation of mechanics (Udwadia and Kalaba, 1992) that provides a closed form expression for the control without making any linearizations/approximations of the nonlinear system. This control while simultaneously minimizing a suitable control cost ensures that the given constraints are exactly satisfied by the nonlinear nominal system. Next, an additional controller is proposed using a general control scheme based on the idea of a generalized sliding surface; this controller permits the uncertain system to track the nominal system to within user-specified bounds. The additional controller that is developed can be used for a wide class of control laws that can be adapted to the practical limitations of the specific controller being used. With the combined concepts of the fundamental equation in analytical dynamics and a generalization of the notion of a sliding surface, the methodology proposed in this paper provides a general approach to the control of nonlinear uncertain mechanical systems, leading to closed-form nonlinear equations of motion that can guarantee that these systems satisfactorily mimic the motions desired of their 'nominal' counterparts.

To demonstrate the effectiveness of the proposed closed-form equation, we consider an example of a triple pendulum in which the masses and the 'given' force are taken to be imprecisely known. A control force, which needs to be applied to this actual system, so that it follows the given, prescribed constraints assigned to the corresponding nominal system and therefore mimics the behavior of the nominal system, is found. Numerical results are provided showing the simplicity, accuracy, and ease of implementation of the control methodology.

2. ON THE DYNAMICS OF MULTI-BODY SYSTEMS

2.1 Description of the nominal system

We begin by introducing the description of the nominal system, by which we mean our best assessment of the 'actual system,' whose description is known only imprecisely. It is useful to conceptualize the description of such a nominal multi-body system in a three-step procedure (Udwadia, 1996). We do this in the following way: first, we describe the uncontrolled system in which the generalized coordinates are all assumed independent of each other. The equation of motion of this system is given, using Lagrange's equation, by

$$\mathbf{M}(\mathbf{q},t)\ddot{\mathbf{q}} = \mathbf{Q}(\mathbf{q},\dot{\mathbf{q}},t), \tag{1}$$

with the initial conditions

$$\mathbf{q}(t=0) = \mathbf{q_0},\ \dot{\mathbf{q}}(t=0) = \dot{\mathbf{q}}_\mathbf{0}, \tag{2}$$

where $\mathbf{q}$ is the generalized coordinate *n*-vector, $\mathbf{M} > 0$ is the *n* by *n* mass matrix which is a function of $\mathbf{q}$ and *t*, and $\mathbf{Q}$ is an *n*-vector, called the 'given' force, which is a known function of $\mathbf{q}$, $\dot{\mathbf{q}}$, and *t*. From Eq. (1) we find the acceleration of the uncontrolled system given by

$$\mathbf{a} \coloneqq \mathbf{M}^{-1}(\mathbf{q},t)\mathbf{Q}(\mathbf{q},\dot{\mathbf{q}},t). \tag{3}$$

Second, we impose a set of control requirements as constraints on this uncontrolled system. We suppose that the uncontrolled system is now subjected to the *m* sufficiently smooth control requirements given by Udwadia (2002)

$$\varphi_i(\mathbf{q},\dot{\mathbf{q}},t) = 0, \quad i = 1,2,...,m, \tag{4}$$

where $r \leq m$ equations in the equation set (4) are functionally independent. The control constraints described by Eq. (4) include all the usual varieties of holonomic and/or nonholonomic constraints, and then some. The presence of the control requirements does not permit all the components of the *n*-vectors $\mathbf{q_0}$ and $\dot{\mathbf{q}}_\mathbf{0}$ to be independently assigned. We shall assume that the initial conditions Eq. (2) satisfy the *m* control requirements. (If not, the control constraints can be expressed in an alternative form so that they are asymptotically satisfied (Udwadia, 2003), see Section 3). Differentiating the control requirements Eq. (4) with respect to time *t* we obtain the relation (Udwadia, 2008)

$$\mathbf{A}(\mathbf{q},\dot{\mathbf{q}},t)\ddot{\mathbf{q}} = \mathbf{b}(\mathbf{q},\dot{\mathbf{q}},t), \tag{5}$$

where $\mathbf{A}$ is an *m* by *n* matrix whose rank is *r,* and $\mathbf{b}$ is an *m*-vector. We note that each row of $\mathbf{A}$ arises by appropriately differentiating one of the *m* control requirements in the set given in Eq. (4).

In the third and final step, the equation of motion of the '*controlled nominal system*,' or the '*nominal system*' for short, is given by

$$\mathbf{M}(\mathbf{q},t)\ddot{\mathbf{q}} = \mathbf{Q}(\mathbf{q},\dot{\mathbf{q}},t) + \mathbf{Q^c}(\mathbf{q},\dot{\mathbf{q}},t), \tag{6}$$

where $\mathbf{Q^c}$ is the control force *n*-vector that arises to ensure that the control requirements Eq. (5) are satisfied. It can be explicitly expressed as (Udwadia and Kalaba, 2007 and Udwadia, 2003)

$$\mathbf{Q^c}(t) := \mathbf{Q^c}(\mathbf{q}(t),\dot{\mathbf{q}}(t),t) = \mathbf{A^T} \;\; \mathbf{AM^{-1}A^T} \;^{+} \;\; \mathbf{b} - \mathbf{Aa} \;\; , \tag{7}$$

wherein the various quantities have been defined in the previous two steps and the superscript "+" denotes the Moore-Penrose (MP) inverse of a matrix. In the above equation, and in what follows, we shall suppress the arguments of the various quantities unless required for clarity. The control force given in Eq. (7) is optimal in the sense that it minimizes the control cost $\mathbf{Q^{cT}M^{-1}Q^c}$ at *each* instant of time (Udwadia, 2003 and Udwadia, 2008). Thus using Eq. (7) in Eq. (6) and pre-multiplying both sides of the equation with $\mathbf{M^{-1}}$, the acceleration of the nominal system that satisfies the constraint Eq. (4) can be expressed as

$$\ddot{\mathbf{q}} = \mathbf{a} + \mathbf{M^{-1}A^T} \;\; \mathbf{AM^{-1}A^T} \;^{+} \;\; \mathbf{b} - \mathbf{Aa} \;\; := \mathbf{a} + \mathbf{M^{-1}Q^c}(t). \tag{8}$$

2.2 Description of the actual system

We assume that the mass matrix of the uncertain real-life system, which we do not know exactly, is $\mathbf{M_a} := \mathbf{M} + \delta\mathbf{M} > 0$, where $\mathbf{M} > 0$ is the *n* by *n* nominal mass matrix—our best estimate of the mass matrix of the actual system—, and $\delta\mathbf{M}$ is the *n* by *n* matrix that characterizes our uncertainty in the mass matrix of the actual system. From here on the subscript 'a' denotes the *actual*, real-life system whose knowledge is uncertain. Similarly, the 'given' force *n*-vector acting on the real-life system is taken to be $\mathbf{Q_a} := \mathbf{Q} + \delta\mathbf{Q}$, where the *n*-vector $\mathbf{Q}$ denotes the 'given' force on the nominal system, and $\delta\mathbf{Q}$ denotes the *n*-vector of uncertainty in $\mathbf{Q}$.

We note that the control force $\mathbf{Q^c}(t)$ obtained from Eq. (7) is based on the nominal description of the system. By applying this control force to the actual system so that

$$\mathbf{M_a}\ddot{\tilde{\mathbf{q}}} := \mathbf{Q_a}(\tilde{\mathbf{q}},\dot{\tilde{\mathbf{q}}},t) + \mathbf{Q^c}(t), \tag{9}$$

one obtains a different state ($\tilde{\mathbf{q}},\dot{\tilde{\mathbf{q}}}$) from that obtained for the (controlled) nominal system ($\mathbf{q},\dot{\mathbf{q}}$). This causes the trajectories of the actual system and the nominal system to differ with a corresponding error in satisfaction of our desired trajectory requirements given in Eq. (4).

2.3 Description of the controlled actual system

To compensate for the above-mentioned uncertainties, the control force given by the second member on the right-hand side of Eq. (9), $\mathbf{Q^c}(t)$, needs to be modified since it was calculated on the basis of the nominal system and is now instead being applied to the actual unknown system. We do this by adding an additional control force $\mathbf{Q^u}$, resulting in a new state $(\mathbf{q_c},\dot{\mathbf{q}}_\mathbf{c})$ (see Eq. (10)). We define the difference between $\mathbf{q_c}(t)$ and $\mathbf{q}(t)$ as the tracking error $\mathbf{e}(t)$ and introduce the additive controller based on a generalization of the notion of a sliding surface (see Utkin (1977) and Udwadia and Wanichanon (2014)). Thus, the equation of motion of the controlled actual system becomes

$$\mathbf{M_a}(\mathbf{q_c},t)\ddot{\mathbf{q}}_\mathbf{c} = \mathbf{Q_a}(\mathbf{q_c},\dot{\mathbf{q}}_\mathbf{c},t) + \mathbf{Q}^\mathbf{c}(t) + \mathbf{Q}^\mathbf{u} \tag{10}$$

where $\mathbf{q_c}$ is the generalized coordinate *n*-vector of the controlled actual system, $\mathbf{Q}^\mathbf{c}(t)$ is the control force which is obtained from the corresponding nominal system and which causes the nominal system to exactly satisfy the constraint Eq. (5), and $\mathbf{Q}^\mathbf{u}$ is the additional control force *n*-vector which we shall develop below in closed form. We now refer to Eq. (10) as the description of the '*controlled actual system*,' or the '*controlled system*,' for short. Pre-multiplying both sides of Eq. (10) by $\mathbf{M_a^{-1}}$, the acceleration of this controlled system can then be expressed as

$$\ddot{\mathbf{q}}_\mathbf{c} = \mathbf{a_a} + \mathbf{M_a^{-1}}\,\mathbf{Q}^\mathbf{c}(t) + \mathbf{M_a^{-1}}\,\mathbf{M}\mathbf{u}. \tag{11}$$

Here $\mathbf{a_a} := \mathbf{M_a^{-1}}\mathbf{Q_a}$ and $\mathbf{Q}^\mathbf{u} := \mathbf{Mu}$, where $\mathbf{u}$ is the additional generalized acceleration provided by the additional control forces $\mathbf{Q}^\mathbf{u}$ to compensate for uncertainties in our knowledge of the actual system.

Defining the tracking error as

$$\mathbf{e}(t) = \mathbf{q_c}(t) - \mathbf{q}(t) \tag{12}$$

and differentiating Eq. (12) twice with respect to time, we get

$$\ddot{\mathbf{e}} = \ddot{\mathbf{q}}_\mathbf{c} - \ddot{\mathbf{q}}, \tag{13}$$

which upon use of Eq. (8) and Eq. (11) yields

$$\begin{aligned}\ddot{\mathbf{e}} &= \mathbf{a_a}(\mathbf{q_c},\dot{\mathbf{q}}_\mathbf{c},t) - \mathbf{a}(\mathbf{q},\dot{\mathbf{q}},t) + \left[\mathbf{M_a}^{-1}(\mathbf{q_c},t) - \mathbf{M}^{-1}(\mathbf{q},t)\right]\mathbf{Q}^\mathbf{c}(t) + \mathbf{M_a^{-1}}(\mathbf{q_c},t)\mathbf{M}(\mathbf{q},t)\mathbf{u} \\ &:= \delta\ddot{\mathbf{q}} + \mathbf{M_a^{-1}}\mathbf{Mu} = \delta\ddot{\mathbf{q}} + [\mathbf{I} - (\mathbf{I} - \mathbf{M_a^{-1}}\mathbf{M})]\mathbf{u} := \delta\ddot{\mathbf{q}} + \mathbf{u} - \bar{\mathbf{M}}\mathbf{u}.\end{aligned} \tag{14}$$

where we have defined

$$\delta\ddot{\mathbf{q}}(\mathbf{q},\dot{\mathbf{q}},\mathbf{q_c},\dot{\mathbf{q}}_\mathbf{c},t) := \left[\mathbf{a_a}(\mathbf{q_c},\dot{\mathbf{q}}_\mathbf{c},t) - \mathbf{a}(\mathbf{q},\dot{\mathbf{q}},t)\right] + \left[\mathbf{M_a}^{-1}(\mathbf{q_c},t) - \mathbf{M}^{-1}(\mathbf{q},t)\right]\mathbf{Q}^\mathbf{c}(t), \tag{15}$$

and

$$\begin{aligned}\bar{\mathbf{M}} &= \mathbf{I} - \mathbf{M_a^{-1}}(\mathbf{q_c},t)\,\mathbf{M}(\mathbf{q},t) = \mathbf{I} - \ \mathbf{M}(\mathbf{q_c},t) + \delta\mathbf{M_{p,q}}(\mathbf{q_c},t)\ ^{-1}\,\mathbf{M}(\mathbf{q},t) \\ &= \mathbf{I} - \ \mathbf{M}(\mathbf{q},t) + \delta\mathbf{M_q}(\mathbf{q},\mathbf{q_c},t) + \delta\mathbf{M_{p,q}}(\mathbf{q},\mathbf{q_c},t)\ ^{-1}\,\mathbf{M}(\mathbf{q},t) = \mathbf{I} - \ \mathbf{I} + \mathbf{M}^{-1}(\mathbf{q},t)\ [\delta\mathbf{M_q}(\mathbf{q},\mathbf{q_c},t) + \delta\mathbf{M_{p,q}}(\mathbf{q},\mathbf{q_c},t)]\ ^{-1} \\ &\approx \mathbf{I} - \ \mathbf{I} + \mathbf{M}^{-1}(\mathbf{q},t)\ [\delta\mathbf{M_q}(\mathbf{q},\mathbf{q},t) + \delta\mathbf{M_{p,q}}(\mathbf{q},\mathbf{q},t)]\ ^{-1},\end{aligned} \tag{16}$$

with $\mathbf{a_a} := \mathbf{M_a^{-1}}\mathbf{Q_a}$, $\mathbf{M_a} := \mathbf{M}(\mathbf{q_c},t) + \delta\mathbf{M}(\mathbf{q_c},t)$, and $\mathbf{Q_a} := \mathbf{Q}(\mathbf{q_c},\dot{\mathbf{q}}_\mathbf{c},t) + \delta\mathbf{Q}(\mathbf{q_c},\dot{\mathbf{q}}_\mathbf{c},t)$.

Equation (16) points out that uncertainties in the actual system's description cause the dynamical mass matrix of the actual system $\mathbf{M_a}(\mathbf{q_c},t)$ to differ from the nominal system's mass matrix $\mathbf{M}(\mathbf{q},t)$ because of two reasons: (1) the parameters (the masses in this case) describing the actual system are different from those of the nominal system, and (2) the controlled response of the actual system, $\mathbf{q_c}(t)$, is different from that of the nominal system, $\mathbf{q}(t)$. The first uncertainty leads to the perturbation $\delta\mathbf{M_{p,q}}(\mathbf{q},\mathbf{q_c},t)$, and the second to $\delta\mathbf{M_q}(\mathbf{q},\mathbf{q_c},t)$. A similar situation applies to the uncertain given (generalized) force $\mathbf{Q_a}(\mathbf{q_c},\dot{\mathbf{q}}_\mathbf{c},t)$. The last approximation in Eq. (16) assumes that the control accomplishes the objective and makes $\mathbf{q_c}(t) \approx \mathbf{q}(t)$.

Since $\mathbf{M_a}$ and $\mathbf{Q_a}$ are uncertain, the value of $\delta\ddot{\mathbf{q}}$ is also uncertain. However, here it is assumed that one has an estimate of a bound on it so that

$$\left\|\delta\ddot{\mathbf{q}}\right\|_\infty \leq \Gamma, \tag{17}$$

where Γ is a positive constant and $\left\|\cdot\right\|_\infty$ denotes an infinity-norm of a vector.

We now define the generalized sliding surface $\mathbf{s}$ as

$$\mathbf{s} \equiv \dot{\mathbf{e}} + b\mathbf{e} + k\int \mathbf{e}dt \quad b > 0,\ k \geq 0 \ , \tag{18}$$

where *b* and *k* are constants, and $\mathbf{s} = \begin{bmatrix} s_1 & s_2 & \cdots & s_n \end{bmatrix}^\mathbf{T}$. From Eq. (18), we have

$$s_i = \dot{e}_i + be_i + k\int e_i dt \quad i = 1,2,\cdots,n \ . \tag{19}$$

It is not difficult to show that if $s_i = 0$, then e_i converges to zero asymptotically as $t \to \infty$.

Now, let us define the Lyapunov function *V* by:

$$V = \frac{1}{2}\mathbf{s}^\mathbf{T}\mathbf{s} = \frac{1}{2}\sum_{i=1}^{n} s_i^2. \tag{20}$$

Its derivative is given by

$$\dot{V} = \mathbf{s}^\mathbf{T}\dot{\mathbf{s}} = \sum_{i=1}^{n} s_i\dot{s}_i = \sum_{i=1}^{n} s_i\left(\ddot{e}_i + b\dot{e}_i + ke_i\right) = \sum_{i=1}^{n} s_i\left(\delta\ddot{q}_i + u_i - \bar{M}u_i + b\dot{e}_i + ke_i\right). \tag{21}$$

Then, the aim is to find an additional control force u_i so that $\dot{V}$ is negative. The conventional sliding mode controller utilizes the following control force u_i:

$$u_i = -b\dot{e}_i - ke_i - \Gamma\,\mathrm{sgn}\left(s_i\right), \tag{22}$$

by which Eq. (21) becomes

$$\begin{aligned} \dot{V} &= \sum_{i=1}^{n} s_i\left(\delta\ddot{q}_i + u_i + b\dot{e}_i + ke_i\right) = \sum_{i=1}^{n} s_i\left[\delta\ddot{q}_i - \Gamma\,\mathrm{sgn}\left(s_i\right) - \bar{M}u_i\right], \\ &= \sum_{i=1}^{n} s_i\left[\delta\ddot{q}_i - \Gamma\,\mathrm{sgn}\left(s_i\right) - \bar{M}\left(-b\dot{e} - ke - \Gamma\,\mathrm{sgn}\left(s\right)\right)_i\right] \approx \sum_{i=1}^{n} s_i\left[\delta\ddot{q}_i - \Gamma\,\mathrm{sgn}\left(s_i\right)\right]. \end{aligned} \tag{23}$$

We note that the last term above obtained by the approximation that $\bar{M}\dot{e}_i, \bar{M}e_i << 1$ and $\bar{\mathbf{M}} << \mathbf{I}$. Thus $\dot{V} < 0$ always holds. However, the discontinuity of the $\mathrm{sgn}(\cdot)$ function generally results in undesirable chattering. In order to avert chattering, let us first consider a region where $\left|s_i\right| > \varepsilon$ holds ($\varepsilon > 0$ is a small positive number), then the following inequality is satisfied:

$$s_i\delta\ddot{q}_i < \frac{\Gamma}{\varepsilon}s_i^2. \tag{24}$$

<u>Proof</u>: If s_i and $\delta\ddot{q}_i$ have opposite signs, then Eq. (24) always holds because the right hand side is always positive. Hence, let us consider the following two cases in which s_i and $\delta\ddot{q}_i$ have the same sign.

Case 1. When $s_i > 0$ and $0 \le \delta\ddot{q}_i \le \Gamma$, since $s_i > 0$, we have

$$\frac{s_i}{\varepsilon} > 1 \tag{25}$$

from the assumption of $\left|s_i\right| > \varepsilon$. Multiplying both sides by $\Gamma s_i \; > 0$ yields

$$\Gamma s_i < \frac{\Gamma}{\varepsilon} s_i^2. \tag{26}$$

From the assumption that $s_i > 0$ and $\delta\ddot{q}_i \le \Gamma$, we have

$$s_i \delta\ddot{q}_i \le \Gamma s_i, \tag{27}$$

and from Eq. (26) it follows that $s_i \delta\ddot{q}_i < \frac{\Gamma}{\varepsilon} s_i^2$.

Case 2. When $s_i < 0$ and $-\Gamma \le \delta\ddot{q}_i \le 0$, since $s_i < 0$, $\left|s_i\right| = -s_i$ and again from $\left|s_i\right| > \varepsilon$, we have

$$-\frac{s_i}{\varepsilon} > 1. \tag{28}$$

Multiplying both sides by $-\Gamma s_i \; > 0$ yields

$$-\Gamma s_i < \frac{\Gamma}{\varepsilon} s_i^2. \tag{29}$$

From the assumption that $s_i < 0$ and $-\Gamma \le \delta\ddot{q}_i$, we have

$$s_i \delta\ddot{q}_i \le -\Gamma s_i, \tag{30}$$

and it follows from Eq. (29) that $s_i \delta\ddot{q} < \frac{\Gamma}{\varepsilon} s_i^2$. ❑

Then, using relation (24), Eq. (21) becomes

$$\dot{V} = \sum_{i=1}^{n} s_i \;\; \delta\ddot{q}_i + u_i - \bar{M}u_i + b\dot{e}_i + ke_i \;\; < \sum_{i=1}^{n} \left[\frac{\Gamma}{\varepsilon} s_i^2 + s_i \;\; u_i - \bar{M}u_i + b\dot{e}_i + ke_i \;\; \right]. \tag{31}$$

Let us choose the additional acceleration u_i as

$$u_i = -b\dot{e}_i - ke_i - \frac{\Gamma}{\varepsilon} s_i - f \;\; s_i \;\; , \tag{32}$$

where $f \;\; s_i$ is a function to be determined. Then, Eq. (31) becomes

$$\begin{aligned} \dot{V} &< \sum_{i=1}^{n} \left[\frac{\Gamma}{\varepsilon} s_i^2 + s_i \;\; u_i - \bar{M}u_i + b\dot{e}_i + ke_i \;\; \right] = \sum_{i=1}^{n} \left[s_i \;\; -f \;\; s_i \;\; - \bar{M}u_i \;\; \right] \\ &= \sum_{i=1}^{n} \left[-s_i \left(f \;\; s_i \;\; + \bar{M} \left(-b\dot{e} - ke - \frac{\Gamma}{\varepsilon} s - f \;\; s \;\; \right)_i \right) \right] \approx \sum_{i=1}^{n} \left[-s_i f \;\; s_i \;\; \right]. \end{aligned} \tag{33}$$

Again, the last term above obtained from the approximation that $\bar{M}\dot{e}_i, \bar{M}e_i, \bar{M}\int e_i dt << 1$ and $\bar{\mathbf{M}} << \mathbf{I}$. In brief, in the region $|s_i| > \varepsilon$, $\dot{V} < 0$ is guaranteed when $f(s_i) = 0$ or, when s_i and $f(s_i)$ have the same sign. Conventionally the discontinuous signum function is used for $f(s_i)$ as noted in Eq. (22), resulting in the chattering problem, but now one can find a number of the continuous functions $f(s_i)$ that effectively avert such chattering.

Selection of the function $f(s_i)$ will depend on the practical control cost to be minimized or on the control performance associated with the control requirements. In this paper special attention is paid to the case where

$$f(s_i) = \eta s_i, \quad \eta \geq 0 \tag{34}$$

so that s_i and $f(s_i)$ have the same sign and $\dot{V} < 0$ is guaranteed. Then, using Eqs. (18) and (32) the control law $u_i(t)$ is given by

$$\begin{aligned} u_i(t) &= -b\dot{e}_i - ke_i - (\frac{\Gamma}{\varepsilon} + \eta)s_i \\ &= -\left(k + \frac{\Gamma b}{\varepsilon} + \eta b\right)e_i - k\left(\frac{\Gamma}{\varepsilon} + \eta\right)\int e_i dt - \left(b + \frac{\Gamma}{\varepsilon} + \eta\right)\dot{e}_i \\ &= G_P e_i + G_I \int e_i dt + G_D \dot{e}_i, \end{aligned} \tag{35}$$

where $k \geq 0,\ b > 0,\ \varepsilon > 0,\ \Gamma > 0,$ and $\eta \geq 0$. Eq. (35) shows that the control $u_i(t)$ is nothing but conventional PID control with the control gains given by $G_p = -\left(k + \frac{\Gamma b}{\varepsilon} + \eta b\right)$, $G_I = -k\left(\frac{\Gamma}{\varepsilon} + \eta\right)$, and $G_D = -\left(b + \frac{\Gamma}{\varepsilon} + \eta\right)$.

Up to now, it has been shown that in the region $|s_i| > \varepsilon$, $\dot{V} < 0$ is guaranteed if the control law, Eq. (32), is used where $f(s_i) = 0$ or s_i and $f(s_i)$ have the same sign. On the contrary, if $|s_i| \leq \varepsilon$, $\dot{V} < 0$ is not guaranteed and the errors may not converge to zero (although they are bounded). This is because the control $u_i(t)$ forces the states into the region bounded by $|s_i| > \varepsilon$ instead of onto the sliding surface $s_i = 0$. However, if ε is small enough, the errors will also be small so that the original nonlinear nominal system can be *linearized*. Then, by linear control theory, the PID controller, Eq. (35), can force the errors to go to much smaller values with proper gain tuning. In brief, the use of the PID controller given by Eq. (35) forces the states into the region bounded by $|s_i| = \varepsilon$, and once the states enter this region we know from linear control theory that the errors will converge to very small values. Thus by taking advantage of both linear and nonlinear control theory improved performance is obtained with no chattering and with faster response, in the sense that gain-tuning is simplified and robustness is guaranteed.

3. NUMERICAL RESULTS AND SIMULATION

To demonstrate the applicability of the proposed methodology, we introduce an example of a simple multi-body system. It is straightforward to extend this example to more general situations.

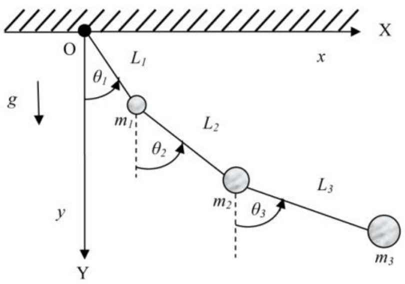

Figure 1. Triple pendulum with the datum at origin O

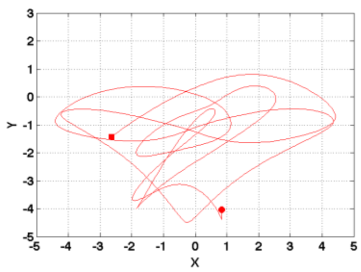

Figure 2. Motion of mass m_3, starting at the circle

Numerical results and simulations of the nominal system: Consider a planar pendulum consisting of three masses m_1, m_2, and m_3 suspended from massless rods of lengths L_1, L_2, and L_3 moving in the XY-plane (see Fig. 1). An inertial frame of reference is fixed at the point of suspension, O, of the triple pendulum and potential energy is measured using the line OX as the datum. Though simple, the system can exhibit complex dynamics.

The masses are constrained to move so that the total energy, $E(t)$, of the system is required to equal the sum of the energies (kinetic and potential) of only the two masses m_2 and m_3, i.e., $E(t) = E_2(t) + E_3(t)$, where we have denoted $E_i(t)$ as the total energy of mass m_i.

Using the generalized coordinate 3-vector $\mathbf{q}(t) = [\theta_1(t), \theta_2(t), \theta_3(t)]^T$ whose components, in the absence of the above-mentioned energy constraint, are independent of one another, Lagrange's equation for the system yields

$$\mathbf{M}(\mathbf{q};\ m_1, m_2, m_3, g)\ddot{\mathbf{q}} = \mathbf{Q}(\mathbf{q}, \dot{\mathbf{q}};\ m_1, m_2, m_3, g) \tag{36}$$

where the elements of the 3 by 3 symmetric matrix $\mathbf{M}$ are given by

$$\begin{aligned} &M_{11} = (m_1 + m_2 + m_3)L_1^2;\ M_{12} = (m_2 + m_3)L_1L_2\cos(\theta_{12});\ M_{13} = m_3L_1L_3\cos(\theta_{13}) \\ &M_{22} = (m_2 + m_3)L_2^2;\ M_{23} = m_3L_2L_3\cos(\theta_{23});\ M_{33} = m_3L_3^2, \end{aligned} \tag{37}$$

and the elements of the 3-vector $\mathbf{Q}$ are given by (see Eq. (1))

$$\begin{aligned} Q_1 &= -(m_2 + m_3)L_1L_2\dot{\theta}_2^2\sin(\theta_{12}) - m_3L_1L_3\dot{\theta}_3^2\sin(\theta_{13}) - (m_1 + m_2 + m_3)gL_1\sin\theta_1 \\ Q_2 &= (m_2 + m_3)L_1L_2\dot{\theta}_1^2\sin(\theta_{12}) - m_3L_2L_3\dot{\theta}_3^2\sin(\theta_{23}) - (m_2 + m_3)gL_2\sin\theta_2 \\ Q_3 &= m_3L_1L_3\dot{\theta}_1^2\sin(\theta_{13}) + m_3L_2L_3\dot{\theta}_2^2\sin(\theta_{23}) - m_3gL_3\sin\theta_3. \end{aligned} \tag{38}$$

In the above, we have denoted $\theta_{ij}(t) = \theta_i(t) - \theta_j(t)$, and we explicitly show in Eq. (36) the physical parameters m_1, m_2, and m_3, and the gravitational acceleration parameter *g*, which we will later on consider to be known only imprecisely.

In the second step we describe the energy constraint $E(t) = E_2(t) + E_3(t)$, which is equivalent to the relation $E_1(t) = 0$, where the energy E_1 of mass m_1 is given by

$$E_1 = \frac{1}{2}m_1L_1^2\dot{\theta}_1^2 - m_1gL_1\cos\theta_1. \tag{39}$$

Since the system may not initially (at time *t* = 0) satisfy this constraint, we modify the constraint using the trajectory stabilization relation (Udwadia, 2003),

$$\dot{E}_1 + \alpha E_1 = 0, \tag{40}$$

where $\alpha > 0$ is a positive function. By Eq. (39) and Eq. (40) we obtain the constraint equation

$$\mathbf{A}\,\ddot{\mathbf{q}} := \begin{bmatrix} L_1^2\dot{\theta}_1 & 0 & 0 \end{bmatrix}\ddot{\mathbf{q}} = -\,gL_1\sin\theta_1\dot{\theta}_1 - \alpha(\frac{1}{2}L_1^2\dot{\theta}_1^2 - gL_1\cos\theta_1) := \mathbf{b}. \tag{41}$$

For the final step, to obtain the equations of motion of the (controlled) nominal system we use the information from Eqs. (36)-(38) and (41) in Eq. (8), and get

$$\ddot{\mathbf{q}} = \mathbf{a} + \mathbf{M}^{-1}\mathbf{A}^{\mathbf{T}}(\mathbf{A}\mathbf{M}^{-1}\mathbf{A}^{\mathbf{T}})^{+}(\mathbf{b} - \mathbf{A}\mathbf{a}). \tag{42}$$

In what follows, we shall assume that the real-life triple pendulum described above has masses whose values are imprecisely known, and that our best assessment of their (nominal) values is: $m_1 = 1$ kg, $m_2 = 2$ kg, and $m_3 = 3$ kg. The lengths of the massless rods are $L_1 = 1$ m, $L_2 = 1.5$ m, and $L_3 = 2$ m. At $t = 0$, the masses are located with the angles of $\theta_1(0) = 1$ rad, $\theta_2(0) = 0$ rad, and $\theta_3(0) = 0$ rad (see Fig. 1). The initial velocities are taken to be $\dot{\theta}_1(0) = 0.01$ rad/s, $\dot{\theta}_2(0) = 0$ rad/s, and $\dot{\theta}_3(0) = 0$ rad/s . Since these initial conditions do not satisfy the constraint, $E_1 = 0$, the parameter α in Eq. (40) is chosen to be $0.1\|\mathbf{A}\|_2^2$ where $\|\mathbf{A}\|_2$ is the L^2 norm of the matrix $\mathbf{A}$ in Eq. (41). The acceleration due to gravity is downwards, and its nominal value is taken to be $g = 9.81$ m/s^2. Numerical integration throughout this paper is done in the *Matlab* environment, using a variable time step integrator with a relative error tolerance of 10^{-8} and an absolute error tolerance of 10^{-12}.

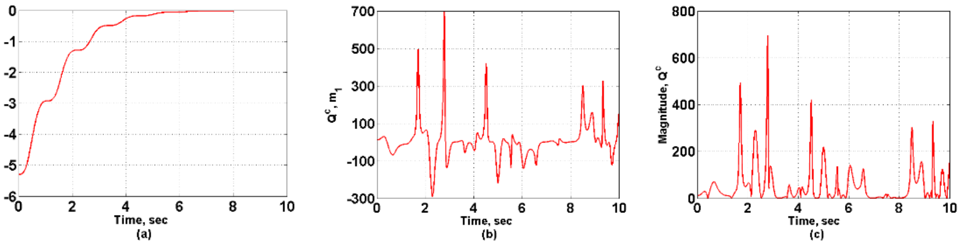

Figure 3. (a) Energy E_1 in N-m (b) Control torque (N-m) on mass m_1 of the nominal system to satisfy $E=E_2+E_3$ (*c*) L^2 norm of Control Torque

Figure 3(a) shows the energy of mass m_1 versus time *t*. Figure 3(b) shows the control force $\mathbf{Q^c}$ so the nominal system follows the desired constraint $E_1 = 0$. No control torques are applied to m_2 and m_3. Figure 3(c) shows the L^2 norm of $\mathbf{Q^c}$.

Numerical results and simulations of the actual system: We now consider the actual system and suppose that there is an uncertainty of up to $\pm 10\%$ in each of the nominal values that were used of the three masses m_1, m_2, and m_3. We also assume that the acceleration, *g*, due to gravity is uncertain with maximum amplitude of $\pm 5\%$ of its nominal value of 9.81 m/s^2. With imperfect knowledge of the parameters and forces in the system, in order to control the actual system's motion so that it tracks the motion of the controlled nominal system, and thereby satisfies the constraints imposed on the nominal system, we thus use the controller $\mathbf{u}$ given by Eq. (35). The equation of motion of the controlled actual system, Eq. (11), then becomes

$$\ddot{\mathbf{q}}_\mathbf{c} = \mathbf{a_a} + \mathbf{M_a^{-1}}\,\mathbf{Q^c}(t) - \mathbf{M_a^{-1}M}\left[\left(k + \frac{\Gamma b}{\varepsilon} + \eta b\right)\mathbf{e} + k\left(\frac{\Gamma}{\varepsilon} + \eta\right)\int \mathbf{e}\,dt + \left(b + \frac{\Gamma}{\varepsilon} + \eta\right)\dot{\mathbf{e}}\right] \tag{43}$$

that will cause the actual system to track the trajectory of the nominal system, thereby compensating for the uncertainty in our knowledge of the actual system.

In order to illustrate the efficacy of our control force in compensating for our lack of exact knowledge of the actual system, we pick the set $\delta m_1 = 0.1$ kg, $\delta m_2 = -0.2$ kg, $\delta m_3 = 0.3$ kg and $\delta g = 0.49$ m/s^2 which is assumed to represent our actual system. To check the performance of our controller, we perform a simulation using Eq. (43) by choosing $\Gamma = 40$, and the parameters $k = 10$, $b = k$, $\eta = 50$, and $\varepsilon = 10^{-2}$, to specify our controller. The chosen set of deviations from the nominal values ($m_1 + \delta m_1 = 1.1$ kg, $m_2 + \delta m_2 = 1.8$ kg, $m_3 + \delta m_3 = 3.3$ kg, and $g + \delta g = 10.3$ m/s^2) represents simply one possibility for the parameters of the actual system, which we use here solely for purposes of demonstration.

The trajectories of the three masses of the actual system follow near-exactly those of the nominal system (Fig. 2). The maximum generalized displacement error $(q - q_c)$ between the nominal system Eq. (8) and the controlled actual system Eq. (43) is computationally found to be of the order of $O(10^{-4})$ as shown in Fig 4.

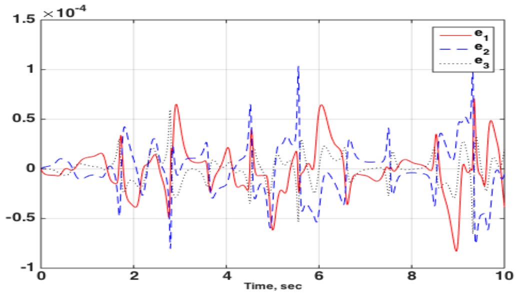

Figure 4. Tracking errors between the controlled nominal system and the controlled actual system (q-q_c)

This illustrates the performance of the closed-form Eq. (43); the controlled actual system tracks the trajectories pre-specified by the nominal system in the presence of the $\pm 10\%$ uncertainties in masses and the $\pm 5\%$ uncertainty in gravity field, along with the constraint imposed on it given by Eq. (40). Also, the continuous linear function $f_i(s)$ given in Eq. (34) eliminates chattering. Pre-multiplying Eq. (43) by $\mathbf{M}_\mathbf{a}$, we obtain

$$\mathbf{M}_\mathbf{a}\ddot{\mathbf{q}}_\mathbf{c} = \mathbf{Q}_\mathbf{a} + \mathbf{Q}^\mathbf{c} - \mathbf{M}\left(\left(k + \frac{\Gamma b}{\varepsilon} + \eta b\right)\mathbf{e} + k\left(\frac{\Gamma}{\varepsilon} + \eta\right)\int \mathbf{e}dt + \left(b + \frac{\Gamma}{\varepsilon} + \eta\right)\dot{\mathbf{e}}\right) := \mathbf{Q}_\mathbf{a} + \mathbf{Q}^\mathbf{c} + \mathbf{Q}^\mathbf{u}. \tag{44}$$

The total control torque applied to the actual system is given by $\mathbf{Q}^\mathbf{T} = \mathbf{Q}^\mathbf{c} + \mathbf{Q}^\mathbf{u}$. Here $\mathbf{Q}^\mathbf{c}(t)$ is the control torque obtained from the nominal system, and $\mathbf{Q}^\mathbf{u}$ is the torque applied by the additional compensating controller to compensate for our inexact knowledge of the actual system. The control torques $\mathbf{Q}^\mathbf{T}$ and $\mathbf{Q}^\mathbf{u}$ on the masses m_1, m_2, and m_3 of the actual pendulum are shown in Fig. 5.

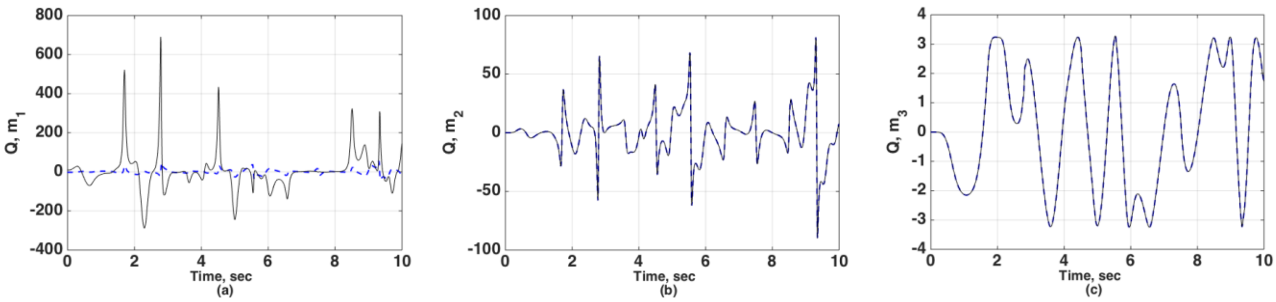

Figure 5. Control torques (N-m) on the controlled actual system; left, mass m_1, center m_2, right m_3. The solid line shows the total control torque, $\mathbf{Q}^\mathbf{T}$, and the dashed line shows the additional torque $\mathbf{Q}^\mathbf{u}$ needed to compensate for uncertainties in describing the actual system

4. CONCLUSION

The general closed-form equation of motion for the control of uncertain nonlinear multi-body systems has been developed. The novelty in the approach developed here is that we first use the *fundamental equation of mechanics* to obtain an exact control force for the nominal, nonlinear, non-autonomous, mechanical system. This control force, $\mathbf{Q}^\mathbf{c}$, ensures that the trajectory constraints are exactly satisfied by the nominal system and that it optimizes the control cost given by $\mathbf{Q}^{\mathbf{c}^T}\mathbf{M}^{-1}\mathbf{Q}^\mathbf{c}$ at *each* instant of time. Control of the *actual system*, in which both the mass matrix and the 'given' force vector may be only imprecisely known, is then carried out using the generalized concept of sliding surfaces. This yields an additional compensating controller that turns out to have PID-form. Instead of using the standardly-used discontinuous signum functions or saturation functions (Utkin, 1977), the generalized sliding surface approach uses continuously functions. This results in trajectories approaching the sliding surface without chattering. In general, the parameters and functions that define the compensating control force can be chosen depending on practical considerations, and adjusted so that the desired tracking error is achieved when the uncertain system is required to track the nominal system. Thus greater flexibility is afforded when dealing with large, complex multi-body systems. Using the simple example of an uncertainly known triple pendulum under an uncertain gravity force, the simplicity and effectiveness of the approach is illustrated. The approach to obtaining closed-form control developed herein appears general enough to be useful for complex dynamical systems in which uncertainties may arise in both the system's description and the given (generalized) forces acting on them.

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