

ROBUST ROTATIONAL STABILITY ANALYSIS OF A VERTICAL DRILL-STRING

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Abstract. We are interested in predicting rotational instability regions in the operation of a vertical drill-string, where stick-slip is likely to occur. The stick-slip is a severe type of self-excited vibration, which affects mainly polycrystalline diamond compact bits, characterized by large fluctuations in the angular velocity of the bit. Drill-string vibrations is one of the major causes for a decrease in drilling performance. We have modelled the drill-string as a shaft with piecewise constant cross section using Saint-Venant torsion theory with a non-linear boundary condition. Then, we converted the partial differential equation of movement to the discrete domain using the finite elements method. Finally, we represented the dynamical system in the state space and analysed its stability. We will introduce a new definition of stability suitable to predict the likelihood of stick-slip occurrence. Given the difficulty in characterizing some parameters of the model, uncertainties are present, and, to improve the predictability of the analysis, they will be taken into account.

Keywords: Stability, Drill-String, Vibration

1. INTRODUCTION

1.1 MOTIVATION

The drilling of petroleum wells is performed by a drilling rig. In the rotary method (see the figure 1), the drilling is accomplished through the rotational movement of the drill bit and the weight on the bit, causing the fragmentation of the rock. The bit is rotated by a drill-string which transmits rotation from the rotary table (or from the top drive) at the surface to the bit at the bottom of the well (Jansen and van den Steen (1995)). The gravels generated in the interior of the well are continuously removed by the mud¹, which is injected into the drill pipe and returns by the annular gap² (Halliburton (1997); Triggia *et al.* (2001); Arcieri (2013)).

A drill-string in operation is subject to rotational, transversal and axial vibrations (see the figure 2). The rotational vibration occurs mainly with polycrystalline diamond compact bits. It affects the whole drill-string and can easily be detected on the surface by torque fluctuations at the rotary table (or at the top drive). In some extreme cases of rotational vibration, the bit sticks³ while the rotary table (or the top drive) continues to rotate. Thus, the drill-string is twisted until the torque on the bit be equal to the maximum static friction torque between the bit and the rock. And then, when the torque on the bit overcomes the maximum static friction torque, the bit slips⁴, since the dynamic coefficient of friction is often less than or equal to the coefficient of static friction, causing a high angular acceleration of the bit, making the angular velocity of the bit increase up to a few times the imposed angular velocity. This phenomenon is denominated stick-slip. The stick-slip is driven by the rotation of the drill-string, and the whole cycle repeats itself (Mihajlovic *et al.* (2006, 2005); Mihajlovic *et al.*, 2007); Liu *et al.* (2013); Kovalyshen (2013); Germa *et al.* (2009b,a); Richard *et al.* (2004, 2007); Richard and Detournay (2000); Richard *et al.* (2002); Aminfar and Khajepour (2008); López and Castro (2009); López and Suárez (2004); Ritto *et al.* (2010b, 2009, 2010a); Liao *et al.* (2012); Depouhon and Detournay (2014); Dykstra *et al.* (1994); Aadnoy *et al.* (2009); Leine *et al.* (2002); He *et al.* (1995); Kyllingstad and Halsey (1988); Finnie and Bailey (1960); Wu *et al.* (2010, 2012); de Wit *et al.* (2008); Kotsonis and Spanos (1997); Spanos *et al.* (1997, 2003); Yigit and Christoforou (1998, 2000, 1996); Christoforou and Yigit (2003); Franca and Weber (2004)).

Usually, these vibrations reduce the rate of penetration of the bit and the directional control of the drill-string⁵. May

¹Mud: technical jargon for the drilling fluid.

²Annular gap: gap between the drill-string and the wall of the well.

³Stick phase: the bit rotation stops for a finite time interval.

⁴Slip phase: the elastic torsional potential energy is suddenly released.

⁵There are reports of cases where the vibrations of the drill-string increase the efficiency of the drilling process (Dareing (1985); Franca and Weber

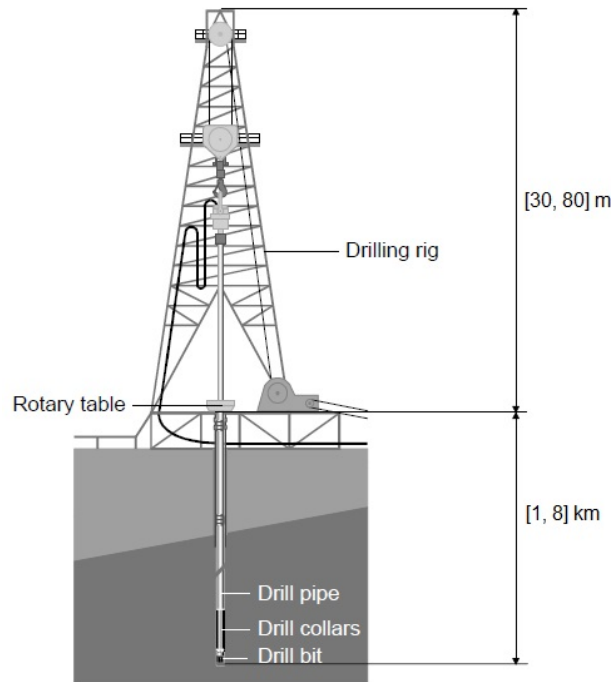


Figure 1. Rotary drilling rig and its main components (Serrarens *et al.* (1998))

cause structural damage to the elements of the drill-string. Increase the chance of failure of its components by fatigue. Interfere in the measurements and may damage electronic sensors. They also increase the wear of the lining pipes of the well. Thus, in most cases, the vibrations of the drill-string are sources of inefficiency in the drilling process; therefore, they increase the cost of drilling the well (Aminfar and Khajepour (2008); Aadnoy *et al.* (2009); Majeed *et al.* (2012); Wu *et al.* (2010, 2012); Ritto *et al.* (2010b, 2009, 2010a); Kyllingstad and Halsey (1988); Spanos *et al.* (2002, 1997, 2003); Yigit and Christoforou (1998, 2000); Christoforou and Yigit (2003, 1997)).

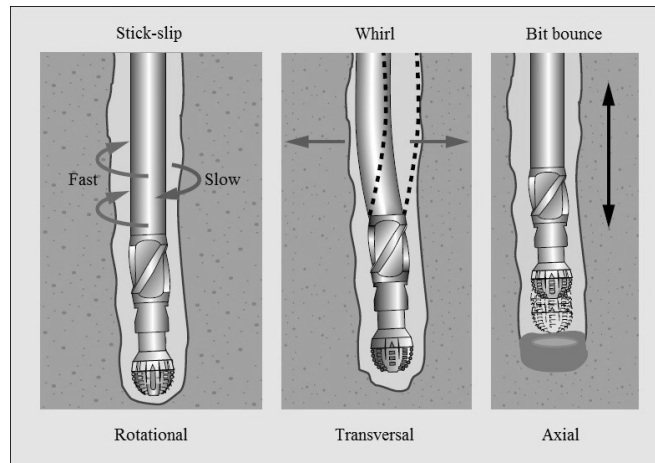


Figure 2. Vibration modes of the drill-string (Schlumberger (2010))

The avoidance of these vibrations is a challenge for technicians and engineers since it can provide important cost savings in drilling operations, in terms of money and exploitation time. For these reasons, an investigation of these vibrations is of interest (Germa *et al.* (2009b,a); Richard *et al.* (2004, 2007); Richard and Detournay (2000); Richard *et al.* (2002); Wu *et al.* (2010, 2012); de Wit *et al.* (2008)).

1.2 OBJECTIVE

The stability map of a dynamical system is a graphical representation of all possible ordered n -uples of n input parameters of the dynamical system with different types of behavior of a given solution of the system (Strogatz (1994)).

(2004); Chin (1994); Kane (1984)).

Each type of behavior of the given solution is represented by a region⁶ in the stability map.

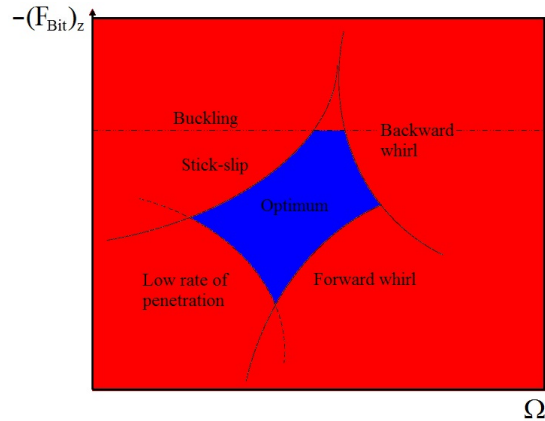


Figure 3. Typical stability map of a drill-string (Wu *et al.* (2010))

In the stability map of the figure 3, the input parameters considered are Ω and $-(F_{Bit})_z$. Where Ω is the imposed angular velocity, $(F_{Bit})_z$ is the weight on the bit. We have put the minus sign in front of $(F_{Bit})_z$ in order to have positive values in the ordinate axis, since $(F_{Bit})_z$ points the opposite direction of the z -axis.

There are some remarkable regions in the figure 3, namely: the stick-slip region⁷, the forward whirl region⁸, the backward whirl region⁹, the buckling region¹⁰, the low rate of penetration region¹¹, the over torque region¹² (not depicted in the figure 3), and the optimum region¹³ (Percy (2014); de Siqueira (2011); Wikipedia (2014)).

The challenge of the driller is to maximize the rate of penetration of the bit while ensuring that the drill-string does not vibrate severely. Thus the driller is found in front of a dilemma: increasing $-(F_{Bit})_z$ may lead to stick-slip; stick-slip usually disappears as Ω surpasses a threshold value, however, increasing Ω may lead to whirl. Keeping both Ω and $-(F_{Bit})_z$ low reduces vibrations, but it negatively affects the rate of penetration of the bit. As a result, the drilling process either suffers from low rate of penetration, or experiences higher rate of penetration but with severe vibrations. The above dilemma is more pronounced when drilling hard or abrasive rocks, because in such cases, the optimum region may even not exist (Wu *et al.* (2010, 2012); Christoforou and Yigit (2003); Richard *et al.* (2004, 2007); Richard and Detournay (2000); Richard *et al.* (2002); Percy (2014); de Siqueira (2011)).

However, these sharp transitions between the regions of the figure 3 are unrealistic, because, even drilling in the same environmental conditions, sometimes the dynamic response of the drill-string is stable, and sometimes is unstable. This can be explained first because we usually do not know with certainty the values of the input parameters. At best, we only know some constraints that these values must satisfy. And second because our mathematical models are simplified representations of the real problem. For example, the damping, which the drill-string is subject to, is extremely difficult to measure. Modeling the bit-rock interaction is not trivial, and simple mathematical models are often used. Thus, uncertainties are present, and, to improve the predictability of the analysis, they should be taken into account (Ritto *et al.* (2010b, 2009, 2010a); Percy (2014)).

The objective of this article is to perform a robust¹⁴ rotational stability analysis of a vertical drill-string of petroleum, in order to identify the instability regions where severe rotational vibrations may occur.

2. MATHEMATICAL MODEL

2.1 STRONG FORM

We have modelled the drill-string as a shaft with piecewise constant cross section using Saint-Venant¹⁵ torsion theory (Meirovitch (1967); Shabana (1997); Krodkiwski (2006)). Thus, the strong form of our problem is:

⁶Region: a graphical representation of a subset of all possible ordered n -uples of the n input parameters considered.

⁷Stick-slip region: the region in which the stick-slip is likely to occur.

⁸Forward whirl region: the region in which the forward whirl is likely to occur.

⁹Backward whirl region: the region in which the backward whirl is likely to occur.

¹⁰Buckling region: the region in which the buckling is likely to occur.

¹¹Low rate of penetration region: the region in which the rate of penetration of the bit is not economically viable.

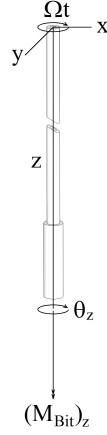
¹²Over torque region: the region in which the torque exceeds the capacity of the drive system.

¹³Optimum region: the complement of the union of the other regions.

¹⁴Robust: considering uncertainties.

¹⁵Adhémar Jean Claude Barré de Saint-Venant (1797 – 1886): was a french engineer and mathematician (Wikipedia (2014)).

Find a function $\theta_z := (z, t) \mapsto \theta_z(z, t)$ with degree of continuity $C^2]0, L[$ in z and $C^2]0, \infty[$ in t where $z \in [0, L]$ and $t \in [0, \infty[$, such that



$$\begin{cases} \frac{\partial}{\partial z} (GJ \frac{\partial \theta_z}{\partial z}) + m_z = \rho J \frac{\partial^2 \theta_z}{\partial t^2}, & \text{if } z \in]0, L[\text{ and } t \in]0, \infty[\\ \theta_z = \Omega t, & \text{if } z = 0 \text{ and } t \in]0, \infty[\\ GJ \frac{\partial \theta_z}{\partial z} = (M_{Bit})_z, & \text{if } z = L \text{ and } t \in]0, \infty[\\ \theta_z = 0, & \text{if } z \in]0, L[\text{ and } t = 0 \\ \frac{\partial \theta_z}{\partial t} = 0, & \text{if } z \in]0, L[\text{ and } t = 0. \end{cases} \quad (1)$$

Where θ_z is the angular displacement of the drill-string cross section around its axis, z is the coordinate along the drill-string axis, t is the time, G is the shear modulus, J is the area moment of inertia of the drill-string cross section around its axis, ρ is the density (see the table 1), m_z is the axial torque per unit length, $(M_{Bit})_z$ is the torque on the bit. $]a$ denotes a real interval open in a , and $[a$ denotes a real interval closed in a . $C^m]a, b[$ denotes the set of functions m times differentiable in $]a, b[$ and with the m -th derivative continuous in $]a, b[$. Note that if a function has degree of continuity $C^m]a, b[$, its n -th derivative has degree of continuity $C^{m-n}]a, b[$. Particularly, a function has degree of continuity $C^0]a, b[$ if it is continuous in $]a, b[$. We usually omit the interval $]a, b[$ when it is the whole real line.

We attend only to the case in which m_z has degree of continuity $C^0]0, L[$ in z and $C^0]0, \infty[$ in t . However, in our problem J has degree of continuity $C^{-1}]0, L[$ in z , so the equations (1) is not a well posed problem, since $\frac{\partial}{\partial z} (GJ \frac{\partial \theta_z}{\partial z})$ does not exist in some $z \in]0, L[$. This motivates us to search for another form (see the subsection 2.3) of our problem with weaker assumptions about the continuity of θ_z . This other form is called weak, and is the form used in finite elements method.

2.2 BIT-ROCK INTERACTION

We have approximated the bit by a solid of revolution (see the figure 4) whose generating line is a function $z := r \mapsto z(r)$, where r, θ, z are the cylindrical coordinates. The infinitesimal force dF_n normal to the surface is given by $dF_n = (f_{Bit})_z (\cos \phi)^2 dS$, where $(f_{Bit})_z$ is the vertical load per unit horizontal area¹⁶, ϕ is the slope angle of the tangent to $z(r)$ at the considered position, dS is the infinitesimal area in which dF_n is acting.

From the geometry we have:

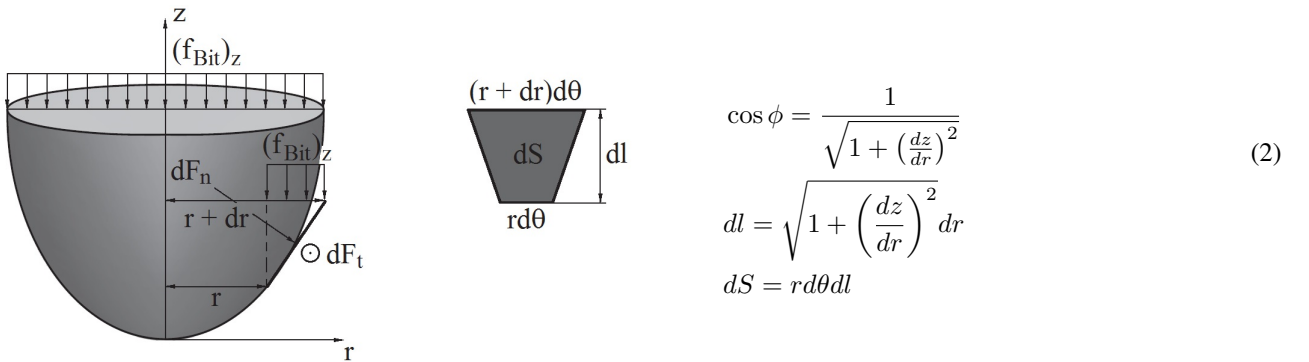


Figure 4. Idealized bit

¹⁶The vertical load per unit horizontal area is $\frac{(F_{Bit})_z}{\pi (R_{Bit})^2}$.

Using the Coulomb¹⁷ law of dry friction, we have $dF_t = \mu dF_n \operatorname{sgn}\left(\frac{\partial \theta_z}{\partial t}\right)$, where dF_t is the infinitesimal force tangential to the surface, μ is the kinetic coefficient of friction, sgn is the sign¹⁸ function.

Substituting the equation for dF_n in the equation of the Coulomb law, using the geometric relations of the figure 4, and integrating over r and over θ , we obtain the torque relative to the z -axis:

$$(M_{Bit})_z = 2\pi \bar{r} \mu (F_{Bit})_z \operatorname{sgn}\left(\frac{\partial \theta_z}{\partial t}\right) \quad (3)$$

Where $\bar{r}$ is the lever-arm characteristic radius, which is defined by:

$$\bar{r} := \frac{1}{\pi (R_{Bit})^2} \int_0^{R_{Bit}} \frac{r^2}{\sqrt{1 + \left(\frac{dz}{dr}\right)^2}} dr \quad (4)$$

Where R_{Bit} is the bit radius.

In order to avoid difficulties when differentiating $(M_{Bit})_z$, we have approximated it by the following C^1 function:

$$(M_{Bit})_z = 2\pi \bar{r} \mu (F_{Bit})_z \left(\tanh\left(\alpha_0 \frac{\partial \theta_z}{\partial t}\right) + \frac{\alpha_1 \frac{\partial \theta_z}{\partial t}}{1 + \alpha_2 \left(\frac{\partial \theta_z}{\partial t}\right)^2} \right) \quad (5)$$

Where $\alpha_0, \alpha_1, \alpha_2$ are the bit-rock interaction constants (see the table 1). The graph of the function (5) is shown in the figure 5.

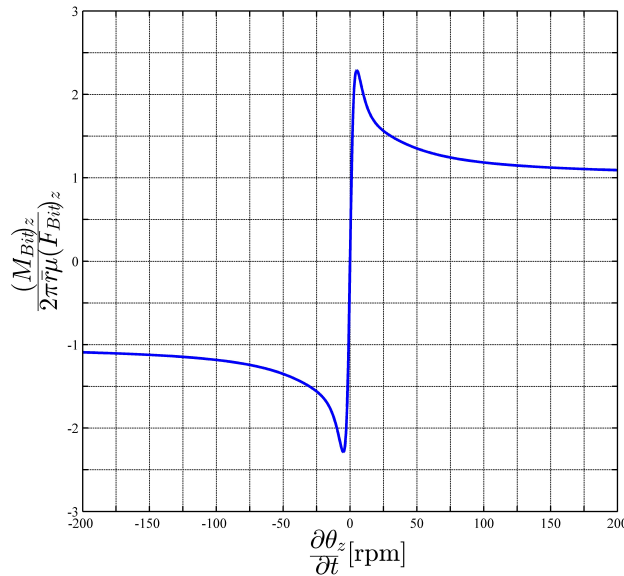


Figure 5. Graph of the torque on the bit

Kovalyshen (2013) proposed four different idealized cutting profiles. We calculated the $\bar{r}$ for each one of them (see the figure 6).

¹⁷Charles-Augustin de Coulomb (1736 – 1806): was a french physicist (Wikipedia (2014)).

¹⁸The sign function extracts the sign of its argument. Whenever $x \neq 0$ we define (Wikipedia (2014)): $\operatorname{sgn}(x) := \frac{x}{|x|}$. Otherwise $\operatorname{sgn}(x) := 0$.

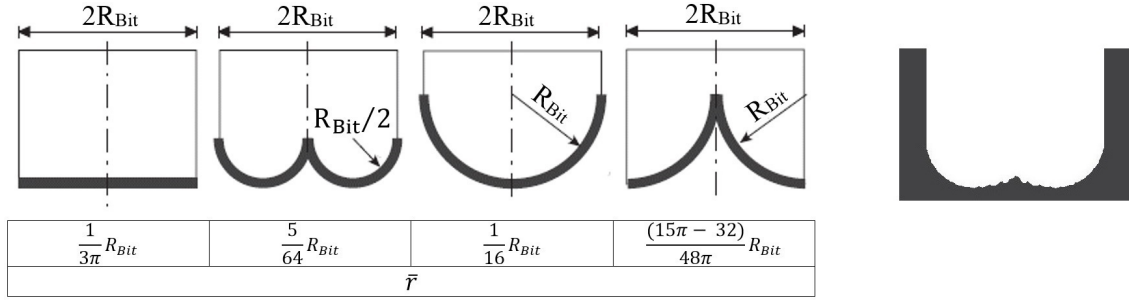


Figure 6. Idealized cut profiles (Kovalyshen (2013)) with their corresponding $\bar{r}$ and a typical cut profile of a PDC bit (Weaver and Clayton (1993))

2.3 WEAK FORM

In order to weaker our assumptions about the continuity of θ_z , we can manipulate the equations (1) using the standard method *multiply by a weight function and integrate by parts* (Roberts (1996)). Thus, we obtain the so called weak form of our problem:

Find a function $\theta_z := (z, t) \mapsto \theta_z(z, t)$ with degree of continuity $C^1]0, L[$ in z and $C^2]0, \infty[$ in t where $z \in [0, L]$ and $t \in [0, \infty[$, such that

$$\left\{ \begin{array}{ll} \left\langle \rho J \frac{\partial^2 \theta_z}{\partial t^2}, W \right\rangle_B + \left\langle G J \frac{\partial \theta_z}{\partial z}, \frac{\partial W}{\partial z} \right\rangle_B = \langle m_z, W \rangle_B + (M_{Bit})_z W(L), & \text{if } t \in]0, \infty[\\ \theta_z = \Omega t, & \text{if } z = 0 \text{ and } t \in]0, \infty[\\ \langle \theta_z, W \rangle_B = 0, & \text{if } t = 0 \\ \left\langle \frac{\partial \theta_z}{\partial t}, W \right\rangle_B = 0, & \text{if } t = 0 \end{array} \right. \quad (6)$$

for all function $W := z \mapsto W(z)$ with degree of continuity $C^0]0, L[$ in z where $z \in [0, L]$ and $W(0) = 0$.

W is called a weight function. B stands for the drill-string body. This form is called weak because its solutions can have lower degree of continuity than the solutions of the strong form. The attempt to solve the weak form instead of the strong form is called method of weighted residuals.

2.4 FEM APPROACH

The system for the finite element B_i is written in shorthand notation:

$$\mathbf{M}_{B_i} \ddot{\mathbf{U}}_{B_i} + \mathbf{K}_{B_i} \mathbf{U}_{B_i} = \mathbf{F}_{B_i} \quad (7)$$

Where $\mathbf{U}_{B_i}$ is called finite element displacement vector, $\mathbf{M}_{B_i}$ is called finite element mass matrix, $\mathbf{K}_{B_i}$ is called finite element stiffness matrix, $\mathbf{F}_{B_i}$ is called finite element force vector.

In our problem, ρ , G , J are piecewise constant functions. If we choose the finite elements such that ρ , G , J do not vary inside them, $\mathbf{M}_{B_i}$ and $\mathbf{K}_{B_i}$ can be easily evaluated (Pilkey (2005)):

$$\mathbf{M}_{B_i} := \frac{\rho J l}{6} \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \quad \mathbf{K}_{B_i} := \frac{G J}{l} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \quad (8)$$

Where l is the finite element length.

In order to get rid of the internal torques (unknown) of the system (7), we can assembly the global finite elements matrices. The first line of the global system of finite elements does not need to be solved, because the first node displacement $\Theta_z(1)$ is known ($\Theta_z(1) = \Omega t$). Thus, it is an algebraic equation for $(M_{RT})_z$, where $(M_{RT})_z$ is the necessary torque expended by the rotary table (or the top drive) to maintain the angular velocity at the top equal to Ω . Thus, we can just remove the degree of freedom of the first node. The first column of the system enters the right hand side as a force.

2.5 DAMPING

Rather than considering a damping term direct in the equation (1), we will insert a damping matrix a posteriori in the global system of finite elements. We should calculate the damping matrix two times. The first one before inserting the boundary conditions, only to obtain the unknown torque $(M_{RT})_z$. The second one after inserting the boundary conditions, to actually solve the system.

Consider the coordinate transformation $\mathbf{U} := \mathbf{Q}\mathbf{u}$, where $\mathbf{U}$ is the finite elements displacement vector, $\mathbf{u}$ is the modal coordinates vector, $\mathbf{Q}$ is the modal matrix¹⁹. In the context of modal analysis, $\mathbf{U}$ is called physical coordinates vector. Observe that we have omitted the B subscript.

Substituting the previous transformation in the global system of finite elements, and pre-multiplying the resulting system by $\mathbf{Q}^\dagger$ in order to preserve the symmetry²⁰ of the coefficients matrices, we obtain:

$$\mathbf{Q}^\dagger \mathbf{M} \mathbf{Q} \ddot{\mathbf{u}} + \mathbf{Q}^\dagger \mathbf{K} \mathbf{Q} \mathbf{u} = \mathbf{Q}^\dagger \mathbf{F} \quad (9)$$

Where the superscript $\dagger$ denotes the transpose of the matrix.

The matrix $\mathbf{Q}$ has the property of simultaneously diagonalize $\mathbf{M}$ and $\mathbf{K}$ (Meirovitch (2001)). We can normalize $\mathbf{Q}$ such that $\mathbf{Q}^\dagger \mathbf{M} \mathbf{Q} := \boldsymbol{\delta}$ and $\mathbf{Q}^\dagger \mathbf{K} \mathbf{Q} := \boldsymbol{\omega}^2$, where $\boldsymbol{\delta}$ is the identity matrix, $\boldsymbol{\omega}^2$ is the eigenvalues matrix²¹, $\boldsymbol{\omega}$ is the natural frequencies matrix²².

The damping matrix can be inserted directly in the uncoupled system assuming that the damping matrix is diagonal:

$$\boldsymbol{\delta} \ddot{\mathbf{u}} + \mathbf{c} \dot{\mathbf{u}} + \boldsymbol{\omega}^2 \mathbf{u} = \mathbf{Q}^\dagger \mathbf{F} \quad (10)$$

The assumption that the damping matrix is diagonal in the uncoupled system is called proportional damping. The matrix $\mathbf{c}$ can be evaluated in fractions of the critical damping $\mathbf{c} := 2\xi\boldsymbol{\omega}$, where ξ is the damping ratio matrix²³. In this article, we have considered the damping ratios of all modes of vibration to have the same value of the damping ratio of the first flexible mode ξ_1 .

The finite elements damping matrix can be obtained transforming $\mathbf{c}$ back to the physical coordinates $\mathbf{C} := \mathbf{Q}^{-\dagger} \mathbf{c} \mathbf{Q}^{-1}$, where the superscript -1 denotes the inverse of the matrix.

3. STABILITY ANALYSIS

3.1 STATE SPACE FORM

The state space form of a n order ordinary differential equation consists in representing the equation by a system of first order ordinary differential equations. This is the form used by MATLAB[®] R2013b ode45 routine to solve differential equations numerically and by us to analyse the stability. We can do that writing our system in an intelligent way:

$$\underbrace{\begin{bmatrix} \dot{\mathbf{U}} \\ \ddot{\mathbf{U}} \end{bmatrix}}_{\mathbf{X}} = \underbrace{\begin{bmatrix} \mathbf{0} & \boldsymbol{\delta} \\ -\mathbf{M}^{-1}\mathbf{K} & -\mathbf{M}^{-1}\mathbf{C} \end{bmatrix}}_{\mathbf{A}} \underbrace{\begin{bmatrix} \mathbf{U} \\ \dot{\mathbf{U}} \end{bmatrix}}_{\mathbf{X}} + \underbrace{\begin{bmatrix} \mathbf{0} \\ \mathbf{M}^{-1}\mathbf{F} \end{bmatrix}}_{\mathbf{B}} \quad (11)$$

Observe that the first line of the equation (11) is just an identity, and the second line is our second order system. Thus, we obtain the so called state space form of our problem:

Find a function $\mathbf{X} := t \mapsto \mathbf{X}(t)$ with degree of continuity $C^1[0, \infty[$ in t where $t \in [0, \infty[$, such that

$$\begin{cases} \dot{\mathbf{X}} = \mathbf{A}\mathbf{X} + \mathbf{B}(\mathbf{X}), & \text{if } t \in]0, \infty[\\ \mathbf{X} = \mathbf{0}, & \text{if } t = 0 \end{cases} \quad (12)$$

¹⁹Modal matrix: matrix of the eigenvectors of $\mathbf{M}^{-1}\mathbf{K}$.

²⁰Symmetric matrix: we say that a matrix $\mathbf{S}$ is symmetric if, and only if $\mathbf{S}^\dagger = \mathbf{S}$.

²¹Eigenvalues matrix: a diagonal matrix whose entries are the eigenvalues of $\mathbf{M}^{-1}\mathbf{K}$. The eigenvalues are the square of the natural frequencies.

²²Natural frequencies matrix: a diagonal matrix whose entries are the natural frequencies.

²³Damping ratio matrix: a diagonal matrix whose entries are the damping ratios of the vibration mode in relation to the critical damping.

Where $\mathbf{X}$ is called state space vector, $\mathbf{A}$ is called state space coefficients matrix, $\mathbf{B}$ is called state space force vector.

3.2 STABILITY DEFINITION

Stability is a controversial topic. There is no consensus on the definition of stability, instead, we can find in the literature several different definitions (Nayfeh and Balachandran (2004); Meiss (2007)). The definition used here is a modification of Lagrange²⁴ stability, which, as will be seen, is suitable to predict the stick-slip occurrence in the long term.

Let Φ be the solution of the initial value problem (12), and let Ψ be a particular²⁵ solution of the system (12) which we want evaluate the stability. We say that Ψ is stable if, and only if Φ converges to Ψ .

In our case, we have evaluated the stability of a particular solution of the form $\Psi := \begin{bmatrix} \Omega t \\ \Omega \end{bmatrix} + \begin{bmatrix} \theta_0 \\ \mathbf{0} \end{bmatrix}$, where Ω is a n dimensional vector whose entries are Ω , θ_0 is a n dimensional constant vector representing the static torsion deformation, which can be obtained forcing Ψ to satisfy the system of differential equations (12).

To draw the stability map, we have considered the initial value problem solution of the linearized system (12). This procedure is called linear stability analysis. It can be done by simple eigenvalues inspection, saving a lot of computational time. And we have:

Ψ is unstable if, and only if at least one eigenvalue of $\mathbf{A} + \nabla \mathbf{B}(\Psi)$ has real part greater than zero.

Ψ is semistable if, and only if the real part of each eigenvalue of $\mathbf{A} + \nabla \mathbf{B}(\Psi)$ is less than or equal to zero, with at least one eigenvalue with zero real part.

Ψ is stable if, and only if the real part of each eigenvalue of $\mathbf{A} + \nabla \mathbf{B}(\Psi)$ is less than zero.

However, the linear stability analysis is limited since it can predict a wrong behavior for the non linear system, specially in the transition regions near the stability line. For instance, in the left hand side of the figure 7 is shown the linear stability map for the drilling of limestone. In the right hand side of the figure 7 is shown the corresponding bit angular velocity of a test point (black square) in the stability map which is supposed to be stable, but it is clearly not stable at all.

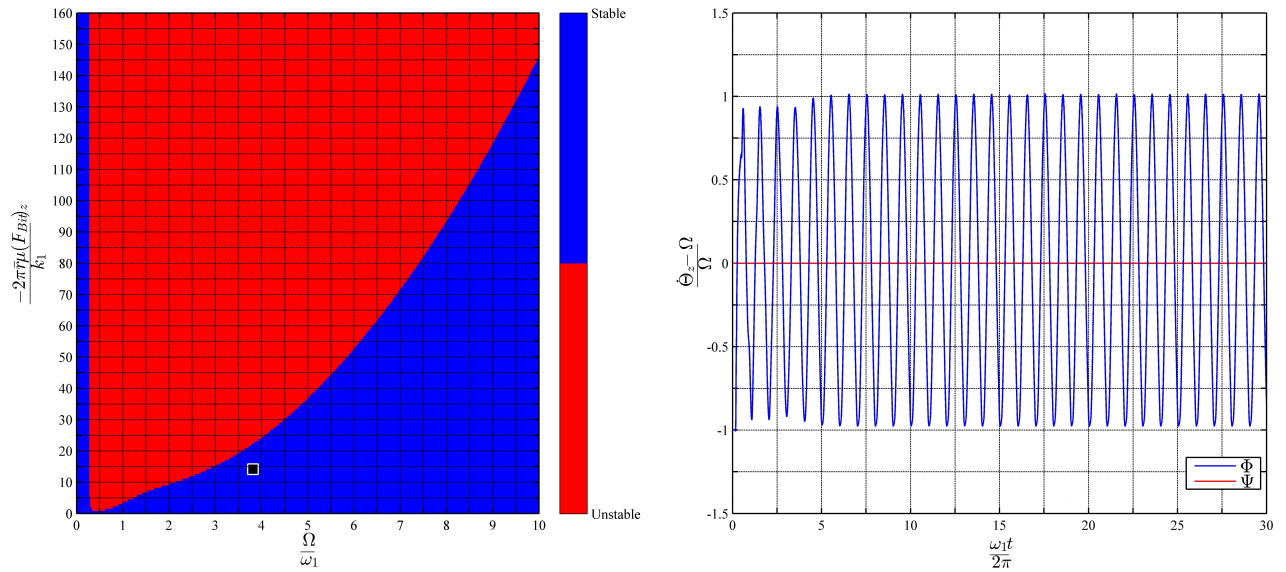


Figure 7. Limitation of the linear stability analysis

Where k_1 is the first eigenvalue of the finite elements stiffness matrix $\mathbf{K}_B$ after inserting the boundary conditions, ω_1

²⁴Joseph-Louis Lagrange (1736 – 1813): was an italian astronomer and mathematician (Wikipedia (2014)).

²⁵Particular solution of a system of ordinary differential equations: a solution of the system which does not necessarily satisfy the initial condition (Boyce and DiPrima (2006)).

is the natural frequency of the first flexible mode of vibration.

4. RESULTS

4.1 STOCHASTIC RESULTS

In this article, we have considered uncertainties in the kinetic coefficient of friction μ and in the damping ratio ξ_1 of the first flexible mode of vibration. Each one were considered separately.

In order to obtain the probability density functions of the stochastic variables, we have used the principle of maximum entropy²⁶, and to obtain the stochastic results, we have run several Monte Carlo²⁷ experiments.

In the case of the kinetic coefficient friction μ the constraint is $a \leq \mu \leq b$. The probability density function of μ which maximizes entropy with this constraint is the uniform density (Kapur and Kesavan (1992)): $f_\mu(x) := \frac{1}{b-a}$. The constants a, b depend on the type of rock which is being drilled (see the table 1).

Table 1. Kinetic coefficient of friction (Kohli and Zoback (2013); Gaffney (1976))

Rock	a	b
Shale	0.404	0.794
Sandstone		
Wet	0.174	0.445
Dry	0.353	0.431
Limestone	0.695	0.754

In the figure 8 are shown the rotation stability maps for different μ .

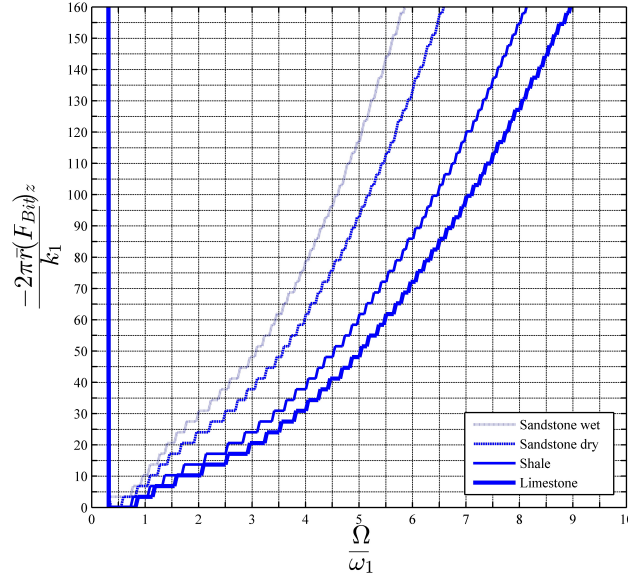


Figure 8. Rotation stability maps for different μ

In the left hand side of the figure 9 is shown the rotation stability map considering uncertainty in the μ for the drilling of limestone. The color represents the probability of being stable. In the right hand side of the figure 9 is shown the corresponding bit angular position of a test point (black square) in the stability map. The color represents the probability density function of the angular position pass by a point of the graph. In the bottom of the figure 9 is shown the corresponding bit angular velocity of the same test point, with the color representing the probability density function of the angular velocity pass by a point of the graph.

²⁶The principle of maximum entropy states that the probability density function to be chosen is the one with largest entropy which respects the variable constraints (Wikipedia (2014)).

²⁷The Monte Carlo method consists in generating samples of the stochastic variables and solving the deterministic problem for each value of the sample. And then make the statistics to find a unknown probability density function (Wikipedia (2014)).

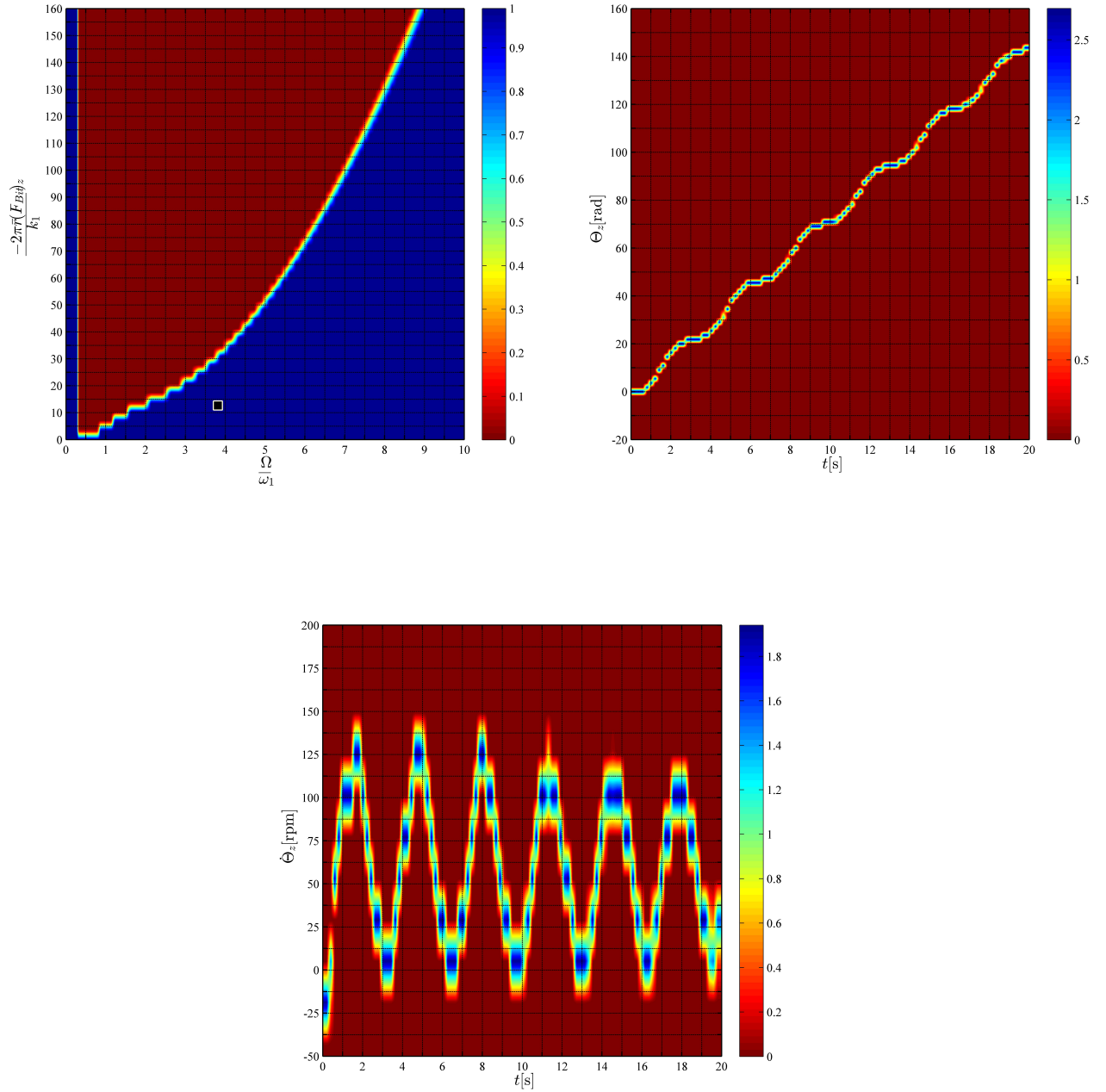


Figure 9. Stochastic results considering uncertainty in the μ for the drilling of limestone.

In the case of the damping ratio of the first flexible mode of vibration ξ_1 the constraint is $0 < \xi_1 \leq 1$. For computational reason, the following two constraints are also imposed $\mu(\ln(\xi_1))$, $\mu(\ln(1 - \xi_1))$. Where μ is the mean²⁸ function. The probability density function of ξ_1 which maximizes entropy with these constraints is the beta density (Kapur and Kesavan (1992)): $f_{\xi_1}(x) := \frac{x^{a-1}(1-x)^{b-1}}{B(a,b)}$. Where B is the beta²⁹ function. The constants a, b are evaluated as follows:

$$\mu(\xi_1) = \frac{a}{a+b} \quad \sigma(\xi_1, \xi_1)^2 = \frac{ab}{(a+b)^2(a+b+1)} \quad (13)$$

Where σ^2 is the variance³⁰ function.

²⁸The mean function measures the central tendency of a stochastic variable. In the case of a continuous stochastic variable y , with probability density function $x \mapsto f_y(x)$, we define (Wikipedia (2014)): $\mu(y) := \int_{\mathbb{R}} x f_y(x) dx$. Not to be confused with the kinetic coefficient of friction.

²⁹The beta function is a special function. Whenever $x, y > 0$ we define (Wikipedia (2014)): $B(x, y) := \int_0^1 t^{x-1}(1-t)^{y-1} dt$.

³⁰The variance function measures how much two stochastic variables change together. In the case of two stochastic variables x, y we define (Wikipedia (2014)): $\sigma(x, y)^2 := \mu((x - \mu(x))(y - \mu(y)))$.

In this article, we have specified the mean $\mu(\xi_1)$ and the coefficient of variation³¹ $cv(\xi_1)$, and then solved the relations (13) to find the constants a , b (see the table 2).

Table 2. Damping ratio of the first flexible mode of vibration

$\mu(\xi_1)$	$cv(\xi_1)$
0.05	1.00
0.10	1.00
0.15	1.00
0.20	1.00
0.25	1.00

In the figure 10 are shown the rotation stability maps for different ξ_1 .

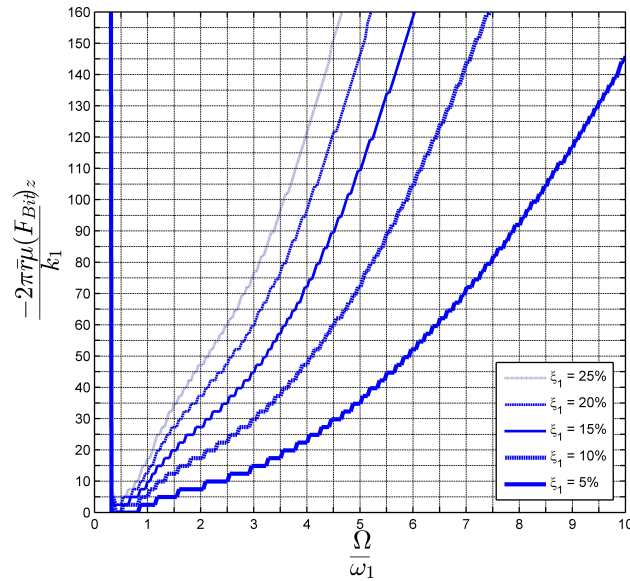


Figure 10. Rotation stability maps for different ξ_1

In the left hand side of the figure 11 is shown the rotation stability map considering uncertainty in the ξ_1 . In the right hand side of the figure 11 is shown the corresponding bit angular position of a test point (black square) in the stability map. In the bottom of the figure 11 is shown the corresponding bit angular velocity of the same test point. The color of the figure 11 has the same meaning of that of the figure 9.

³¹The coefficient of variation function measures the dispersion of a stochastic variable. In the case of a stochastic variable x we define (Wikipedia (2014)): $cv(x) := \frac{\sigma(x,x)}{\mu(x)}$.

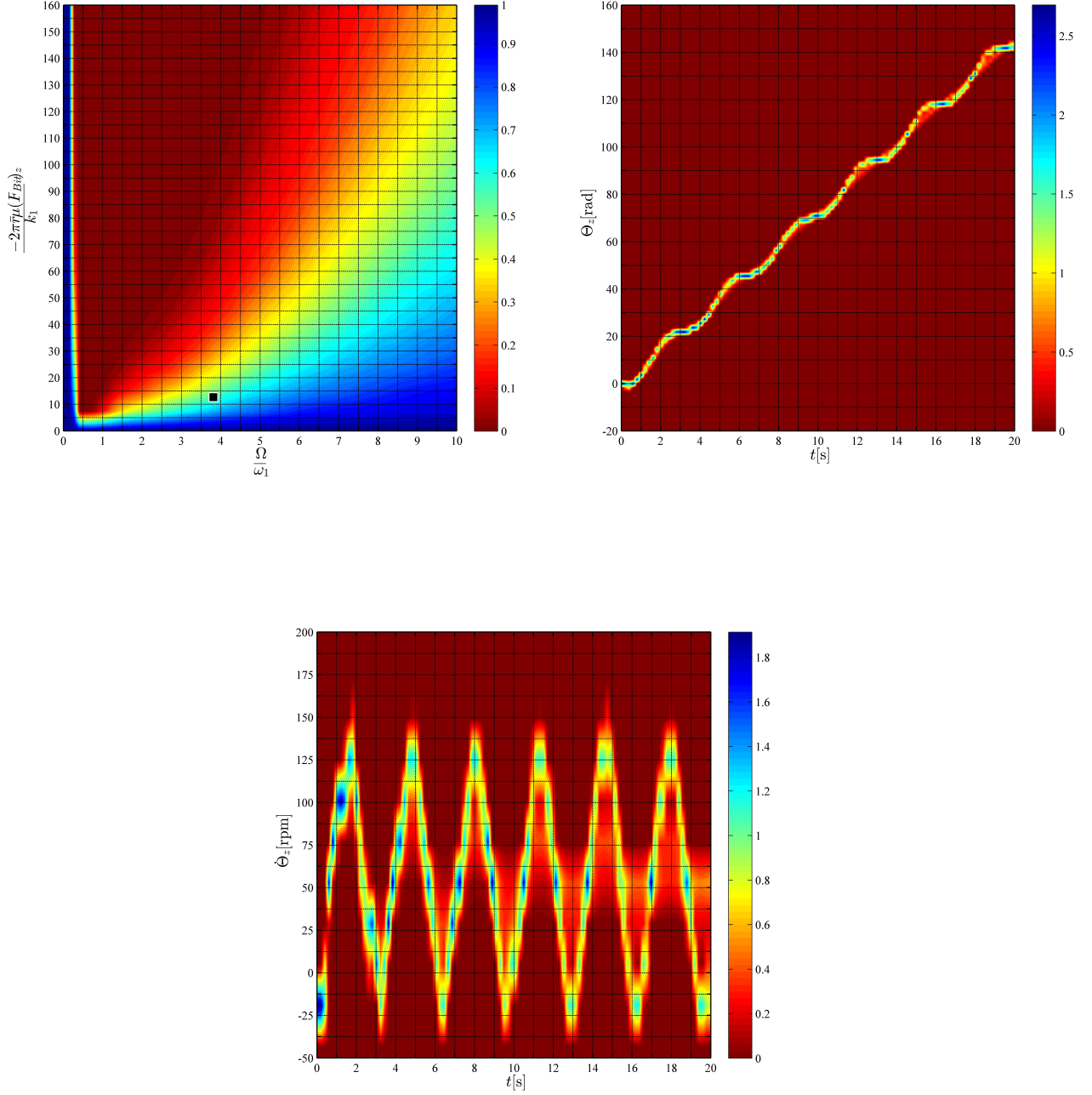


Figure 11. Stochastic results considering uncertainty in the ξ_1 for the drilling of limestone.

5. FINAL REMARKS

5.1 CONCLUSION

We have developed a simple theoretical model of the bit-rock interaction, which easily accommodates different bit shapes. We have used this model to perform a linear stability analysis of the drill-string. The linear stability map is particularly useful for the field engineer when it is not available an experimental stability map, because it qualitatively reproduces the latter. Although the linear stability serves as a guiding principle, it should be used with caution, because it predicts a wrong behavior near the stability curve. This is due to the numerical error introduced by the linearization process. We have considered separately uncertainties in the kinetic coefficient of friction and in the damping ratio. The effect of the uncertainty in the kinetic coefficient of friction is not very interesting, because the stability curve depends linearly on the kinetic coefficient of friction. Thus, the effect of the uncertainty is simply to spread the stability curve. However, the uncertainty in the damping ratio has a much more interesting effect. It significantly changes the stability map, reducing dramatically the stable region area.

5.2 FUTURE WORKS

Our model considers only the degree of freedom of torsion. It would be interesting to improve it including the degrees of freedom of bending and traction. In this way, one could perform the linear stability analysis considering the new degrees of freedom, in order to reproduce the other instability regions of figure 3. Since the linear analysis is unreliable, it should be developed new strategies of stability analysis which avoid solving the dynamical system, because it is costly from the standpoint of the computational time. Particularly, the authors consider promising the strategy of using the linear stability as a preliminary analysis, and then solve the system for a small region offsetted from the stability curve. New models of the bit-rock interaction should be investigated. It would be interesting to consider a stochastic model for bit-rock interaction, and how to perform the stability analysis in this case.

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APPENDIX A. DATA USED IN SIMULATIONS

The values of the drill-string dimensions and material properties are the same used by Percy (2014). The values of the bit-rock interaction constants α_0 , α_1 , α_2 are obtained by least squares fit³² from experimental data of Schlumberger.

Table 3. Values of the deterministic variables

Variable	Value	Unit	Description
L_{DP}	1800	m	Drill pipe length
L_{BHA}	200	m	Bottom hole assembly length
$(R_{DP})_{out}$	0.060	m	Drill pipe outer radius
$(R_{DP})_{in}$	0.0475	m	Drill pipe inner radius
$(R_{BHA})_{out}$	0.075	m	Bottom hole assembly outer radius
$(R_{BHA})_{in}$	0.0475	m	Bottom hole assembly inner radius
$(F_{Bit})_z$	$-12 \cdot 10^3$	N	Weight on the bit
ρ	7850	kg/m ³	Density
E	$200 \cdot 10^9$	Pa	Elastic modulus
G	$70 \cdot 10^9$	Pa	Shear modulus
Ω	70	rpm	Rotary table angular velocity
α_0	0.4775	s	Zeroth constant of the bit-rock interaction
α_1	8.7854	s	First constant of the bit-rock interaction
α_2	4.5595	s ²	Second constant of the bit-rock interaction
μ	0.723	—	Kinetic coefficient friction
ξ	0.05	—	Damping ratio

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³²The least squares fit is a method to approximate experimental data by a function with indeterminate constants. These constants are obtained minimizing the sum of the squares of the errors between each point in the experimental data and the value of the given function at this point.

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