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CARTAN'S CONNECTIONS APPLIED TO KINEMATIC AND DYNAMIC CALCULATION IN ROBOTICS: QUATERNIONS APPROACH

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Abstract. *Cartan's connection is an important mathematical object in Differential Geometry that generalizes, to an arbitrary Riemannian space, the concept of angular velocity (and twists) of a non-inertial reference frame. In fact, this is an easy way to calculate the angular velocities (and twists) in system with multiple reference frames, like a robot or a complex mechanism. The concept also can be used in different representations of rotation and twists, like in the Lie groups of unit quaternions and unit dual quaternions. In this work, we apply Cartan's connection to the kinematic and dynamic calculation of a system of multiple reference frames, that could represent a robotic chain or mechanism. Differently from previous works, we represent the transformation between frames as quaternions / unit quaternions, which are known to be better mathematical representations from the numerical point of view. We generalize, to this new quaternion representation, previous results like extended Newton's equation, covariant derivative and Cartan's connection.*

Keywords: *Dual Quaternions, Robotics, Lie Groups, Screw Theory.*

1. INTRODUCTION

Concepts of Differential Geometry and Lie Groups and Lie Algebras have been used in many areas of Engineering for a long time. More recently, a renewed interest can be verified in Mechanism and Robotics (see Yang (1964)), where Quaternions and Dual Quaternios were used, and very complete reference books, like Bottema (1979), appeared. Classical references in Robotics, like Craig (1989); Spong and Vidyasagar (1989), used only vector analysis, but some Lie group's concepts (like orthogonal matrices and $SE(3)$) were also used. A systematic use of Differential Geometry, on the other hand, is far more recent (see Murray *et al.* (1994); Selig (2005)). In Control Theory, Differential Geometry has also been explored (see for example Pait and Colón (2010b); Colón and Pait (2004)).

A *Cartan's connection* is a concept that is fundamental to modern Physics and Differential Geometry, besides the concept of *moving frames*, as presented in Schwarz (1996); Choquet-Bruhat *et al.* (1982); Sharpe (1997). In simple terms, given a manifold M (that could be Riemannian), and a field of reference frames S in M, which associates to each point in $p \in M$ a reference frame (in general, non-inertial) S_p, a Cartan's connection Ξ is a derivative field of S such that Ξ_p contains the information of how the frame S_p changes in each direction. This is the generalized gradient of S, as it also depends on the curvature of M. Computationally, S is a field of matrices, as well as Ξ, but of distinct types. In practical terms, a Cartan's connection is a field of angular velocity in case of pure rotational motion (values in $SO(3)$), and a field twist (differential screw) in general rigid motion (values in $SE(3)$), as can be found in Murray *et al.* (1994). A very interesting application to Mechanics is pointed out, in a limited way, in Sattinger and Weaver (1986), and in Colón (2014). In this paper, this author continues the approach in Sattinger and Weaver (1986) and applies Cartan's connection to obtain some formulas for kinematic chain calculations in Robotics, doing them in the groups $SO(3)$, $SE(3)$ and its Lie algebras. This means that Cartan's connections (velocity fields) assume values in the Lie algebras $\mathfrak{so}(3)$ (of anti-symmetric rotation matrices) and $\mathfrak{se}(3)$ (twists). Another essential concept, associated to Cartan's Connections, is the *covariant derivative*, and by using it, it is possible to rewrite Newton's law and Euler's equation in a form that is *invariant* in any non-inertial frame. This allows the dynamic calculations to be written in an arbitrary non-inertial reference frame. Another advantage of the Cartan's connection is that it can be applied to the description of motion in different representations, as in the quaternions, dual-quaternions and general Clifford algebras (see for example Selig (2005)). In fact, pure rotational motion is described, in terms of unit quaternions, in the Lie group $SU(2)$ (special unitary complex matrices) and the general rigid motion in the Lie group of unit dual quaternions (see preliminary results in Colón (2015)).

In this work, it is shown how kinematics and dynamic calculations can be done in the framework of Cartan's connection, but using quaternion representation, that is, the transformation between frames are matrices in in the Lie group $SU(2)$ (it is assumed that the frames have the same origin), and the Cartan's connection is a field of matrices of $\mathfrak{su}(2)$. This work continues the previous papers Colón (2014) and Colón (2015), presenting new results. In section 2, it is described the kinematics for the motion in $SO(3)$ and $SE(3)$ and dynamics in $SO(3)$ (pure rigid motion) using Cartan's connection, following the point of view of section 14, pages 44-48 of Sattinger and Weaver (1986), which is done in more detail in Colón (2014). The precise notion of Cartan's connections and covariant derivative is presented. In section 3.

, main definitions of quaternions, dual quaternions and Clifford algebras are presented, in preparation for the following sections. It is also presented the concept of principal fiber bundles and associated vector bundles, that are the spaces of poses in quaternion representation. In section 4. it is presented the kinematics calculations for a system of n reference frames in which the transformations are unit quaternions, that is the group $SU(2)$. In section 5. it is presented the dynamic calculation for a system of three reference frames as in section 4. in which the third frame is attached to a rigid body. In section 6. conclusion and suggestions for future work are presented. The point of view of applying Cartan's connection to kinematic and dynamic calculation in Robotics is original of this author, to the best of his knowledge, and the results in section 4.and 5.are new contributions of this paper. The applications of principal and associated fiber bundles (and related concepts) are also contributions of this author.

2. KINEMATICS AND DYNAMICS IN $SO(3)$ AND IN $SE(3)$: CARTAN'S CONNECTION'S POINT OF VIEW

2.1 Kinematics in $SO(3)$ and $SE(3)$

Following the steps of Sattinger and Weaver (1986) and Colón (2014), any point in the Euclidean space $\mathbb{R}^3$ is represented by a 3-vector $\mathbf{r}'$ in a non-inertial frame S', and by a 3-vector $\mathbf{r}$ in the inertial frame S. Suppose that both frames have the same origin. The transformation matrix between the two frames is represented by $R = R(t) \in SO(3)$, whose columns are the 3-vectors of S' expressed in the coordinates of S. Those 3-vectors are related by $\mathbf{r}(t) = R(t)\mathbf{r}'(t)$ and after the application of time derivative, one has $\dot{\mathbf{r}} = R\,(\Omega\mathbf{r}' + \dot{\mathbf{r}}')$, where $\Omega = R^T\dot{R}$ is an anti-symmetric (time variant) matrix called *body angular velocity*, that is a trajectory in the Lie algebra $\mathfrak{so}(3)$. The matrices $R(t)$ and $\Omega(t)$ are operators in $\mathbb{R}^3$ that are representations of the Lie group $SO(3)$ and Lie algebra $\mathfrak{so}(3)$, respectively, in the 3-vector space $\mathbb{R}^3$ (see for example Selig (2005)). The frame S' could also be attached to a rigid body, and $R(t)$ could be viewed as a trajectory in $SO(3)$ describing the rotational motion of the rigid body.

The simplest form of a Cartan's connection is as a field of angular velocity matrices Ω, when the motion of the reference frame (or rigid bidy) is purely rotational. As presented in Sattinger and Weaver (1986), the *covariant derivative* is an operator that acts in 3-vectors of $\mathbb{R}^3$ and is given by $D_t = \partial/\partial t + \Omega$. If one wants to calculate the time derivative of any 3-vector in a non-inertial reference frame S', this operation must be used. In fact, if one applies D_t to $\mathbf{r}'$, it results in $\Omega\mathbf{r}' + \dot{\mathbf{r}}'$, which is the velocity of point $\mathbf{r}'$ as seen in the inertial frame S, but in the coordinates of the non-inertial reference frame S'. Another application of the time derivative results in $\mathbf{a}' = D_t^2\mathbf{r}'$, which is the definition of acceleration in non-inertial reference frames, and given by $\mathbf{a}' = D_t^2\mathbf{r}' = \ddot{\mathbf{r}}' + \dot{\Omega}\mathbf{r}' + 2\Omega\dot{\mathbf{r}}' + \Omega^2\mathbf{r}'$, where $\ddot{\mathbf{r}}'$ is the relative acceleration, $\Omega^2\mathbf{r}'$ is the centrifugal acceleration, and $2\Omega\dot{\mathbf{r}}'$ is the Coriolis acceleration. This is the acceleration of point $\mathbf{r}'$ as seen in the inertial frame S but expressed in the non-inertial frame S'. In order to describe the motion of an arbitrary point of the rigid body, one has to assume that $\dot{\mathbf{r}}' = \ddot{\mathbf{r}}' = 0$.

There is a Lie algebra isomorphism between $(\mathfrak{so}(3), [,])$ and $(\mathbb{R}^3, \times)$, where $[A, B] = AB - BA$ in the Lie bracket and $\times$ is the vector product. Then, to the 3-vectors $\mathbf{r}, \mathbf{r}'$ corresponds matrices $\mathbf{X}, \mathbf{X}' \in \mathfrak{so}(3)$ describing the motion of a point. The operators $R(t)$ and $\Omega(t)$ would act in the space $\mathfrak{so}(3)$, which is known as *adjoint representation*, and the covariant derivative in this new representation is given by $D_t = \frac{\partial}{\partial t} + [\Omega, \cdot]$, where Ω is the Cartan's connection (angular velocity matrix). Applying this operator to a vector $\mathbf{X}' \in \mathfrak{so}(3)$ in the non-inertial reference frame, one has:

$$\begin{aligned}\mathbf{A}' = D_t^2\mathbf{X}' = \left(\frac{\partial}{\partial t} + [\Omega, \cdot]\right)\left(\frac{\partial \mathbf{X}'}{\partial t} + [\Omega, \mathbf{X}']\right) = \ddot{\mathbf{X}}' + \frac{\partial}{\partial t}[\Omega, \mathbf{X}'] + [\Omega, \dot{\mathbf{X}}'] + [\Omega, [\Omega, \mathbf{X}']] = \\ = \ddot{\mathbf{X}}' + [\dot{\Omega}, \mathbf{X}'] + [\Omega, \dot{\mathbf{X}}'] + [\Omega, \dot{\mathbf{X}}'] + [\Omega, [\Omega, \mathbf{X}']]. \quad (1)\end{aligned}$$

One can see that the formula in Eq. (1) is very similar to the formula of acceleration presented above, as both applies to points, but in different representations (see Humphreys (1972)).

For the case of general motion (rotation + translation) of a frame/rigid body, some additional concepts are important: given an arbitrary differentiable manifold M and a Lie group G, a *G-action* of G in M is an application $\phi : G \times M \to M$ such that: 1) $\phi(e, p) = p$, where $p \in M$ is an arbitrary point and $e \in G$ is the identity of the Lie group, and 2) $\phi(g_2, \phi(g_1, p)) = \phi(g_2 g_2, p)$ (group property). The family of invertible applications $\phi_g : M \to M$ indexed by G is the family of transformations of M. It is common to represent $\phi_g(p)$ by simply gp (see for example Choquet-Bruhat *et al.* (1982)). For any point $p \in \mathbb{R}^3$, it is possible to left-multiply it by an element of $SE(3)$, which changes it to another point $q \in \mathbb{R}^3$. If one attaches a non-inertial reference frame S' to a rigid body in $\mathbb{R}^3$, one can represent its position and orientation by another element of the Lie group $SE(3)$. To each point $p \in \mathbb{R}^3$, it is possible to associate a copy of the Lie group $SO(3)$ (let us call it $SO(3)_p$), that is the set of all reference frames at point p. This space is $\mathbb{R}^3 \times SO(3)$, which is a particular type of *principal fiber bundle*. The Lie group $SE(3)$ can be identified with the space $SO(3) \ltimes \mathbb{R}^3$ (semi-direct sum of the two Lie groups), which is topologically equivalent to $\mathbb{R}^3 \times SO(3)$, but is another Lie group (see Levy's decomposition theorem, in Sattinger and Weaver (1986)).

2.2 The concept of principal fiber bundles and general Cartan's connection

A general *principal fiber bundle*, that is represented by (E, B, π, G), is a set of differentiable manifolds E (total space), B (base space), $\pi : E \to B$ is a smooth surjective application known as the projection, and G is a group of transformations. For each point $b \in B$ in the base, the corresponding pre-image $\pi^{-1}(b)$ is called *fiber*, which generates a partition of E in fibers. Each of this fibers are homeomorphic to the group G. In case the we just mentioned, $E = SE(3)$, $B = \mathbb{R}^3$ and $\pi : SE(3) \to \mathbb{R}^3$ is a smooth and surjective application that associates to each reference frame in $SO(3)_p$ its origin, that is, $p \in \mathbb{R}^3$. It is easy to see that the action of $SO(3)$ in $SE(3)$ is such that any element of $SO(3)_p$ remains in $SO(3)_p$ after the action, which means that the action is fiber preserving, and can be interpreted as a rotations without translation. A *section* in this principal fiber bundle is a smooth application that associates to each point $p \in \mathbb{R}^3$ an element in $SO(3)_p$, that is a field of reference frames (see Kobayashi and Nomizu (1969)). A curve $\gamma(t)$ in $\mathbb{R}^3$ represents the translational motion of a rigid body, and a curve $\Gamma(t)$ in $SE(3)$, such that $\gamma(t) = \pi(\Gamma(t))$, is a field of reference frames along $\gamma(t)$ and describes the general motion of a rigid body, that is translational and rotational (see for example Ivey and Landsberg (2003) and Kobayashi and Nomizu (1969)). The Maurer-Cartan form of $SO(3)$ (see Ivey and Landsberg (2003)) is a 1-form given by $\Omega = R^{-1}\,\mathrm{d}R$, where $R \in SO(3)$, and it represents the angular velocities Colón (2014); Sharpe (1997). In the case of the principal fiber bundle $(SE(3), \mathbb{R}^3, \pi, SO(3))$, there is no curvature, and the Maurer-Cartan form is the very Cartan's connection in this situation (as all the spaces in which we will work have no curvature, Cartan's connection is always the Maurer-Cartan form, see comments in Colón (2014) regarding Robotics in curved spaces, like places near massive stars).

Given two arbitrary reference (moving) frames S, S' (neither necessarily inertial), $\mathbf{X}$, $\mathbf{X}^{'}$ its matrices formed by the frame vectors, $\mathbf{X}^{'} = \mathbf{X}A$ the relation of transformation between them, and $\nabla\mathbf{X}^{'} = \mathbf{X}^{'}\Xi^{'}$, $\nabla\mathbf{X} = \mathbf{X}\Xi$, its directional derivatives, the matrix fields Ξ, Ξ' are two Cartan's connections in the reference frames S, S', respectively. The relation between those Cartan's connections are given by $\Xi^{'} = A^{-1}\,\mathrm{d}A + A^{-1}\Xi A$, and is called a frame (gauge) transformation. In fact, this formula can be easily demonstrated:

$$\nabla\mathbf{X}^{'} = \mathbf{X}^{'}\Xi^{'} = \nabla(\mathbf{X}A) = \mathbf{X}\,\mathrm{d}A + \nabla\mathbf{X}A = \mathbf{X}\,\mathrm{d}A + \mathbf{X}\Xi A, \tag{2}$$

which results in $\Xi^{'} = A^{-1}\,\mathrm{d}A + A^{-1}\Xi A$.

If three or more moving frames are involved (see Fig. 1), the following result is important:

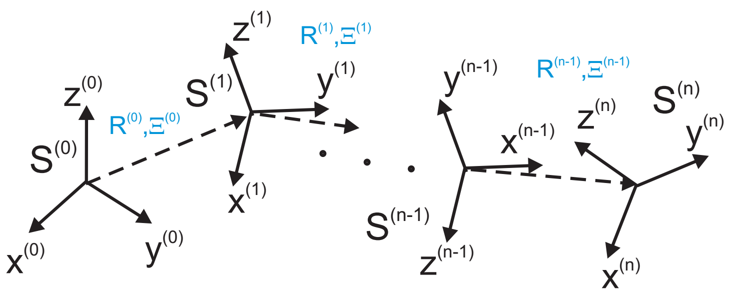

Figure 1. Transformation Between Reference Frames.

Proposition 2.1 (Transformation Formulas) *Given the moving frames $S^{(0)}, S^{(1)}, \cdots, S^{(n-1)}, S^{(n)}$, then the transformation formula for the Cartan's connections are:*

$$\Xi^{(n)} = (A^{(0)}A^{(1)}\cdots A^{(n-1)})^{-1}\,\mathrm{d}(A^{(0)}A^{(1)}\cdots A^{(n-1)}) + (A^{(0)}A^{(1)}\cdots A^{(n-1)})^{-1}\Xi^{(0)}(A^{(0)}A^{(1)}\cdots A^{(n-1)}). \tag{3}$$

that can be found in Colón (2015). In the case of a serial robot with n links, $S^{(0)}$ corresponds to the inertial frame, $S^{(n)}$ is the end-effector frame, and the frames from 1 to $n-1$ are attached to the links. The matrix $A^{(0)}A^{(1)}\cdots A^{(n-1)}$ transform the coordinates from the end-effector to the inertial frame, and could be calculated by using, for example, the Denavit-Hartenberg parametrization (see for example Craig (1989)). Due to the Cartan's theorems (see Colón (2014) or Spivak (2005)), there is a reference frame, in a space without curvature, where the Cartan's connection is null. In the serial robot case, it is $S^{(0)}$, so that $\Xi^{(0)} \equiv 0$. This means that the connection in the end-effector is $\Xi^{(n)} = (A^{(0)}A^{(1)}\cdots A^{(n-1)})^{-1}\,\mathrm{d}(A^{(0)}A^{(1)}\cdots A^{(n-1)})$, which is an essential formula, demonstrated by the author.

2.3 Dynamics of rigid body in $SO(3)$

The dynamic equations for a rigid body was previously presented in Colón (2014) for the case of two frames (one inertial and the other attached to the rigid body). In this case, it is more convenient to work in adjoint representation. Let then the angular momentum, in relation to the non-inertial frame S', be represented by $\mathbf{\Lambda}' \in \mathfrak{so}(3)$, as well as $\mathbf{V}' = D_t\mathbf{X}'$ the velocity (in the reference of S but written in the coordinates of S'). The formula for the angular momentum of a point of mass m is given by $\mathbf{\Lambda}' = m[\mathbf{X}', \mathbf{V}']$ (remember that in this representation, the Lie brackets $[\cdot, \cdot]$ is equivalent to the vector product $\times$). In order to describe the dynamics of a point with mass m, the Newton's law is extended to $mD_t^2\mathbf{r}' = \mathbf{f}'$, that is valid for any non-inertial reference frame, where $\mathbf{f}'$ represents the resulting force in the non-inertial frame S'. This could also be written in terms of the momentum of the particle, calculated as $\mathbf{P}' = mD_t\mathbf{r}'$, and the Newton's law would be reformulated as $D_t\mathbf{P}' = \mathbf{f}'$. It is easy to see that all the inertial forces will appear. This formulas are valid for the standard representation and adjoint representation. In case of rigid motion, the Euler equations can easily be obtained and are shown in Colón (2014).

3. QUATERNIONS, DUAL QUATERNIONS AND CLIFFORD ALGEBRAS

In this section, we present similar results for the case of kinematics and dynamics in unit quaternions and dual quaternions space (results equivalent to those presented in section 2).

3.1 Basic Definitions

In this section, it is presented some basic concepts of quaternions and dual quaternions, and its relations with Clifford algebras and the groups and algebras that will be the main focus of this work, which are $SU(2)$ and $\mathfrak{su}(2)$. More detailed exposition can be found in Colón (2014). Dual quaternions are used in Mechanical Engineering as early as 1964 (see for example Yang (1964)) and the number of related works has been increasing, as can be seen in Bernardes *et al.* (2013, 2014); de Oliveira *et al.* (2014); Radavellli (2014). Quaternions are elements of the form $\overline{q} = a + b\hat{\imath} + c\hat{\jmath} + d\hat{k}$, where a, b, c, d are real numbers and the symbols $\hat{\imath}^2 = \hat{\jmath}^2 = \hat{k}^2 = \hat{\imath}\hat{\jmath}\hat{k} = -1$ define a product in $\mathbb{H}$, which is the vector space of quaternions (which is isomorphic, as a vector space, to $\mathbb{R}^4$). It is also an associative $\mathbb{R}$-algebra, where $1 = (1, 0, 0, 0)$, $\hat{\imath} = (0, 1, 0, 0)$, $\hat{\jmath} = (0, 0, 1, 0)$ e $\hat{k} = (0, 0, 0, 1)$. The conjugate of a quaternion is $\overline{q}^* = a - b\hat{\imath} - c\hat{\jmath} - d\hat{k}$ and $\overline{q}$ can be represented as a pair $(a, \mathbf{v})$, where a is the scalar part, and $\mathbf{v}$ is the 3-vector part. Given two 3-vectors $\mathbf{v}_1, \mathbf{v}_2 \in \mathbb{R}^3$, they can be represented by two quaternions $\overline{q}_1, \overline{q}_2$ with zero scalar parts, and the product $\overline{q}_1\overline{q}_2$ is equal to $(-\mathbf{v}_1 \cdot \mathbf{v}_2, \mathbf{v}_1 \times \mathbf{v}_2)$, where $\cdot$ is the scalar product and $\times$ is the vector product. Every quaternion has an inverse $\overline{q}^{-1}$, which turns $\mathbb{H}$ in a associative division algebra. A pure translation is represented by a pure vector quaternion $\overline{p} = p_x\hat{\imath} + p_y\hat{\jmath} + p_x\hat{k}$, and a rotation must be represented by an unit quaternion $\overline{r} = \cos(\theta/2) + \sin(\theta/2)\mathbf{n}$, where $\mathbf{n}$ is the unit 3-vector in the direction of the rotation (see Selig (2005)).

The vector space $\mathbb{H}$ (which is also a algebra) has norm $\|\overline{q}\|$ which is, by definition, $\sqrt{a^2 + b^2 + c^2 + d^2}$ and a formula to calculate the inverse of $\overline{q}$ is $\overline{q}^{-1} = \overline{q}^*/\|\overline{q}\|$. The unit elements of $\mathbb{H}$ (that is, the elements with norm equals to one) also belong to the space $S^3 \in \mathbb{R}^4$ (that is the thee dimensional spherical surface). In fact, $\hat{\imath}, \hat{\jmath}, \hat{k}$ are versors. The space $\mathbb{H}$ has an algebra homomorphism with the space $M_{2\times 2}(\mathbb{C})$ of 2×2 complex matrices with the form:

$$Q = \begin{bmatrix} a + bi & c - di \\ -c + di & a - bi \end{bmatrix}, \tag{4}$$

with the norm given by the determinant $\|Q\| = \sqrt{\det Q}$ (see Selig (2005)). This means that any unit quaternion can be represented as a 2×2 complex matrix. As the unit elements of an associative algebra forms always a group, as can be seen in Eisenbud (1995), the space of unit quaternions homomorphic to the Lie group $SU(2)$, which is also isomorphic to Lie group S^3. As this last space is simply connected (that is, any closed curve can be continuously deformed to a point), so is $SU(2)$. It is also a fact that $SU(2)$ double covers $SO(3)$ (which is not simply connected), as can be seen in Sattinger and Weaver (1986). This implies that two different elements of $SU(2)$ (in this case, Q and $-Q$) represent the same element $R \in SO(3)$. This fact brings many advantages in the representation of rotation by matrices of $SU(2)$ (and then, by unit quaternions).

Given a real d dimensional vector space V and basis $\{\mathbf{e}_1, \cdots, \mathbf{e}_n\}$ for it, let $g(\mathbf{v}, \mathbf{w})$ be a quadratic (bi-linear) function with matrix $[g_{ab}]$ with signature (n, m, z), (that is, there are n eigenvalues $+1$, m eigenvalues -1 and z zero eigenvalues, and the total size is $d = n + m + z$), a Clifford Algebra $C\ell(n, m, z)$ of V is an associative algebra with product defined by the elements $\mathbf{e}_a\mathbf{e}_b + \mathbf{e}_b\mathbf{e}_a = 2g_{ab}$. The dimension of the algebra is 2^d and its basis is given by $(\mathbf{1}, \mathbf{e}_a, \mathbf{e}_a\mathbf{e}_b, \mathbf{e}_a\mathbf{e}_b\mathbf{e}_c, \cdots, \mathbf{e}_1\mathbf{e}_2\cdots\mathbf{e}_d)$, where $\mathbf{1}$ is the algebra's identity. It is easy to find in the Literature that $C\ell(0, 1, 0) = \mathbb{C}$, which dimension is 2, and $C\ell(0, 2, 0)$ is the algebra of quaternions $\mathbb{H}$. In the case of $d = 2p$ even, it is possible to prove that the Clifford algebra is isomorphic to the space of complex matrices $2^p \times 2^p$ (see Choquet-Bruhat *et al.* (1989)).

Another important Clifford algebra is $C\ell(0, 3, 1)$, that is known as the Dual Quaternions Algebra, that has a degenerate quadratic form. In this case, the identities are $\mathbf{e}_1\mathbf{e}_1 = -1$, $\mathbf{e}_2\mathbf{e}_2 = -1$, $\mathbf{e}_3\mathbf{e}_3 = -1$, $\mathbf{e}_4\mathbf{e}_4 = 0$ and $\mathbf{e}_a\mathbf{e}_b = -\mathbf{e}_b\mathbf{e}_a$ for $a \neq b$. If one defines $\hat{\imath} = \mathbf{e}_1$, $\hat{\jmath} = \mathbf{e}_2$, $\hat{k} = \mathbf{e}_3$ and $\epsilon = \mathbf{e}_4$, the expected dimension is $2^4 = 16$, but it can be proven that ϵ commutes with $\hat{\imath}, \hat{\jmath}, \hat{k}$ and then the product of $\epsilon\hat{\imath}, \epsilon\hat{\jmath}, \epsilon\hat{k}$ are also zero. This fact reduces the number of basis elements to

$(1, \hat{\imath}, \hat{\jmath}, \hat{k}, \epsilon, \epsilon\hat{\imath}, \epsilon\hat{\jmath}, \epsilon\hat{k})$, and the dual quaternions space is isomorphic (in the algebra sense) to the tensor product $\mathbb{H} \otimes \mathbb{H}$, as can be seen in Selig (2005). Any element of this Clifford algebra can be represented by $\underline{q} = \overline{r} + \epsilon(1/2)\overline{pr}$, and if $\overline{p}$ and $\overline{r}$ are translation and rotation quaternions (vector quaternions), it can be proven that this dual quaternion has unitary norm, has inverse, and represents general rigid motion (see Bernardes *et al.* (2013)). Those unit dual quaternions forms a Lie group, and in Selig (2005) it is shown that this group is the semi-direct product $\mathrm{Spin}(3) \ltimes \mathbb{R}^3$, where $\mathrm{Spin}(3)$ is the spin group, and double covers $SO(3) \ltimes \mathbb{R}^3$.

3.2 Principal Fiber Bundles and Associated Vector Bundles

One can also attach to each point in $\mathbb{R}^3$ a vector space V in which a Lie group G has an action, and this new fiber bundle is known as an *associated vector bundle* E to the principal fiber bundle $(P, \mathbb{R}^3, \pi, G)$. A *representation* of a group G in a vector space V is a group homomorphism $\rho : G \to \mathrm{Aut}(V)$ in which each element of G is represented as a linear invertible transformation of V in itself, and $\mathrm{Aut}(V)$ is the set of linear automorphisms in V, that is also a Lie group in the cases we are interested (see Choquet-Bruhat *et al.* (1982)). In fact, a representation of a group is a special type of action of group in a vector space. The derivative application $\rho_g^{'}$ in a point $g \in G$ maps the tangent space T_gG in the tangent space $T_{\rho(g)}\,\mathrm{Aut}(V)$. If one restricts $g = e$ (the identity element in G), the linear application $\rho_e^{'}$ maps the Lie algebra $\mathfrak{g}$ to the Lie algebra of the Lie group $\mathrm{Aut}(V)$, which is the representation of the Lie algebra $\mathfrak{g}$ in V.

As a Cartan's connection Ξ in $(P, \mathbb{R}^3, \pi, G)$ has values in $\mathfrak{g}$, it is possible to calculate $\rho_e^{'}(\Xi)$, which transforms the Cartan's connection in a operator acting in V (or in the associated vector bundle E). The covariant derivative in this situation is the operator in V given by:

$$D_t = \frac{\partial}{\partial t} + \rho_e^{'}(\Xi), \tag{5}$$

where Ξ is the Maurer-Cartan form of G.

As we want to use quaternios in the calculations, from now on we consider the principal fiber bundle $(P, \mathbb{R}^3, \pi, SU(2))$, which associates to each point in $\mathbb{R}^3$ an element of the unit quaternions space (or, more specifically, its matrix representation), which double covers the space $SO(3) \ltimes \mathbb{R}^3$. Consider the adjoint representation of $SU(2)$, in which it acts in its very Lie algebra $\mathfrak{su}(2)$, that is, the associated vector bundle is $(E, \mathbb{R}^3, \pi, \mathfrak{su}(2), SU(2))$. Given the set of Pauli matrices:

$$\sigma_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \quad \sigma_2 = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \quad \sigma_3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \tag{6}$$

any element in $\mathfrak{su}(2)$ can be written as $\mathbf{X} = \mathbf{v} \cdot \sigma = \sum_{i=1}^{3} v^i \sigma_i$, where $\mathbf{v}$ is a 3-vector in $\mathbb{R}^3$ associated to this element (that are also unit quaternions). The rotation (or frame transformation) between two 3-vectors in $\mathbb{R}^3$ is then given by $\mathbf{X}' = e^{-\frac{i\theta}{2}\mathbf{n}\cdot\sigma}\mathbf{X}e^{\frac{i\theta}{2}\mathbf{n}\cdot\sigma}$, where any element of the Lie group $SU(2)$ can be given as a rotation around a unitary 3-vector $\mathbf{n}$, that is $e^{-\frac{i\theta}{2}\mathbf{n}\cdot\sigma}$. The vector product of two 3-vectors $\mathbf{v} \times \mathbf{u}$ is then represented by $[\mathbf{V}, \mathbf{U}]$ in $\mathfrak{su}(2)$, and the Cartan's connection Ξ is a 1-form with values in the Lie algebra $\mathfrak{su}(2)$ and it represents the angular velocity of one frame in relation to another. In a robotic serial kinematic chain, with several reference frames, the Eq. (3) can be used to determine the Cartan's connection (angular velocity) of the n link (end effector) in relation to the inertial reference frame 0.

In order to write down the Euler's equation in this representation, the covariant derivative must be determined to this representation, which is done in the following lema.

Lemma 3.1 *The covariant derivative is the operator that acts in $\mathfrak{su}(2)$ by the formula:*

$$D_t\mathbf{X} = \partial\mathbf{X}/\partial t + [\Xi, \mathbf{X}], \tag{7}$$

where $\mathbf{X} \in \mathfrak{su}(2)$.

Proof: By using the definition presented in Eq. (5), as the representation being used is the adjoint representation, which is given by $\rho(S)\mathbf{X} = S\mathbf{X}S^{-1}$, suppose that a curve $\alpha(t)$ passes through $e \in SU(2)$ such that $\dot{\alpha}(0) = \Xi$ is the element of the Lie algebra. Then $\rho_e^{'}(\Xi)\mathbf{X} = \frac{\mathrm{d}}{\mathrm{d}t}(\alpha(t)\mathbf{X}\alpha(t)^{-1})|_{t=0}$. It is easy to find that:

$$\frac{\mathrm{d}}{\mathrm{d}t}(\alpha(t)\mathbf{X}\alpha(t)^{-1}) = \frac{\mathrm{d}\alpha(t)}{\mathrm{d}t}\mathbf{X}\alpha(t)^{-1} - \alpha(t)\mathbf{X}\frac{1}{\alpha^2(t)}\frac{\mathrm{d}\alpha(t)}{\mathrm{d}t}, \tag{8}$$

which, for $t = 0$, is equal to $\rho_e^{'}(\Xi)\mathbf{X} = \frac{\mathrm{d}\alpha(0)}{\mathrm{d}t}\mathbf{X} - \mathbf{X}\frac{\mathrm{d}\alpha(0)}{\mathrm{d}t} = [\Xi, \mathbf{X}]$, which concludes the proof. ∎

4. KINEMATICS IN $SU(2)$ FOR A SYSTEM OF n REFERENCE FRAMES

It will be considered n reference frames, and applying the formula in proposition 2.1 with $\Xi^{(0)} \equiv 0$ (due to a theorem of Cartan, if the base space has no curvature, there is always a reference frame, the inertial, in which the Cartan's connection is zero, as shown in Spivak (2005)), one has the connection in frame n given by theorem 4.1. In order to demonstrate the theorem, it is necessary the lemma 4.1.

Lemma 4.1 *Given a system of three reference frames $S^{(0)}, S^{(1)}, S^{(2)}$ with transformation matrices $R^{(0)}, R^{(1)}$, the connection in $\Xi^{(2)}$ is given by:*

$$\Xi^{(2)} = [R^{(1)}]^{-1}\Omega^{(0)}R^{(1)} + \Omega^{(1)}, \tag{9}$$

Proof: By applying the formula in Eq. (2.1), we have: $(R^{(0)}R^{(1)})^{-1}\,\mathrm{d}(R^{(0)}R^{(1)}) = [R^{(1)}]^{-1}[R^{(0)}]^{-1}\{(\mathrm{d}R^{(0)})R^{(1)} + R^{(0)}\,\mathrm{d}R^{(1)}] = (R^{(1)})^{-1}[\Omega^{(0)}R^{(1)} + \mathrm{d}R^{(1)}]$, which concludes the proof. ■

Theorem 4.1 *Given a system of $n+1$ reference frames $S^{(0)}$, $S^{(1)}$, $\cdots$, $S^{(n)}$, with matrices of transformation $R^{(0)}$, $R^{(1)}$, $\cdots$, $R^{(n-1)}$ and rotation matrices $\Omega^{(i)} = [R^{(i)}]^{-1}\,\mathrm{d}R^{(i)}$ between $S^{(i)}$ and $S^{(i+1)}$, the Cartan's connection for the frame $S^{(i)}$ can be calculated by the recursive formula:*

$$\Xi^{(i)} = [R^{(i-1)}]^{-1}\Xi^{(i-1)}R^{(i-1)} + \Omega^{(i-1)} \tag{10}$$

for $i = 1, \cdots, n$.

Proof: The proof is by induction. The base of the induction is presented in lemma 4.1. Also, consider that the connection in $S^{(i-1)}$ is given by Eq. (4), that is:

$$\Xi^{(i-1)} = (R^{(0)}R^{(1)}\cdots R^{(i-2)})^{-1}\,\mathrm{d}(R^{(0)}R^{(1)}\cdots R^{(i-2)}) \tag{11}$$

By applying again Eq. (3), we have:

$$\begin{aligned}\Xi^{(i)} &= (\underbrace{R^{(0)}R^{(1)}\cdots R^{(i-2)}}_{Q}R^{(i-1)})^{-1}\,\mathrm{d}(R^{(0)}R^{(1)}\cdots R^{(i-2)}R^{(i-1)}) \\ &= (QR^{(i-1)})^{-1}\,\mathrm{d}(QR^{(i-1)}) = [R^{(i-1)}]^{-1}Q^{-1}\{\,\mathrm{d}QR^{(i-1)} + Q\,\mathrm{d}R^{(i-1)}\} = \\ &= [R^{(i-1)}]^{-1}Q^{-1}\,\mathrm{d}QR^{(i-1)} + [R^{(i-1)}]^{-1}\underbrace{Q^{-1}Q}_{I}\,\mathrm{d}R^{(i-1)} = \\ &= [R^{(i-1)}]^{-1}\Xi^{(i-1)}R^{(i-1)} + [R^{(i-1)}]^{-1}\,\mathrm{d}R^{(i-1)}\end{aligned} \tag{12}$$

and the result follows. ■

Given an arbitrary point $\mathbf{r}^{(i)}$ in the reference frame S^i, in order to calculate its acceleration, one must apply the double operator D_t^2, where D_t is given in Eq. (7), similar to Eq. (1), but the point $\mathbf{r}^{(i)}$ must be represented for a point $\mathbf{X}' \in \mathfrak{su}(2)$, which is possible due to the Lie algebra isomorphism between $\mathbb{R}^3$ and $\mathfrak{su}(2)$ (see Sattinger and Weaver (1986)). The resulting formula will be formally identical to Eq. (1), but if we want to calculate the acceleration of a fixed point (of a rigid body attached to the frame), we must substitute $\ddot{\mathbf{X}}' = \dot{\mathbf{X}}' = 0$, which results in $[\Xi, [\Xi, \mathbf{X}']]$.

5. DYNAMICS IN $SU(2)$ FOR A SYSTEM OF THREE REFERENCE FRAMES

Given a system of n reference frames $S^{(0)}, S^{(1)}, \cdots, S^{(n)}$, in order to obtain the dynamic equation for the motion of the center of mass of the rigid body associated to the frame $S^{(i)}$, we must apply the extended Newton's law $mD_t^2\mathbf{X}^{(i)} = \mathbf{F}^{(i)}$, where m is the mass of i-th body and $\mathbf{F}^{(i)}$ include all the forces acting in the body.

As an example, let us suppose that $n = 2$. The dynamic equation of the center of mass of the rigid body attached to $S^{(2)}$, is $mD_t^2\mathbf{X}^{(2)} = \mathbf{F}^{(2)}$ which has the expression $\mathbf{F}^{(2)} = m(D_t)^2\mathbf{X}^{(2)}$ where $D_t = \frac{\partial}{\partial t} + [[R^{(1)}]^{-1}\Omega^{(0)}R^{(1)} + \Omega^{(1)}, \cdot]$, by using Eq. (4.1). This results in:

$$mD_t^2\mathbf{X}^{(2)} = m\left(\frac{\partial}{\partial t} + [(R^{(1)})^{-1}\Omega^{(0)}R^{(1)} + \Omega^{(1)},\cdot]\right)^2 \mathbf{X}^{(2)} =$$
$$= m\left(\frac{\partial}{\partial t} + [(R^{(1)})^{-1}\Omega^{(0)}R^{(1)} + \Omega^{(1)},\cdot]\right)\left(\dot{\mathbf{X}}^{(2)} + [(R^{(1)})^{-1}\Omega^{(0)}R^{(1)},\mathbf{X}^{(2)}] + [\Omega^{(1)},\mathbf{X}^{(2)}]\right) = F. \quad (13)$$

and after some manipulation, the equation for the motion of the center of mass is obtained.

The dynamic equations are yet under research for the general case. For now, let us suppose the number of reference frames is three (that is, $n = 2$) and all the frames has the same origin. In order to have a more clean notation, let us also suppose that the frames are S, $S^{'}$ and $S^{''}$ The following lemma is important:

Lemma 5.1 *The partial derivative of* $\Xi^{''}$ *is given by:*

$$\frac{\partial}{\partial t}\Xi^{''} = (R')^{-1}\dot{\Omega}R' + \dot{\Omega}' - [\Omega', (R')^{-1}\dot{\Omega}R'] \quad (14)$$

Proof: By applying the time derivative in $(R')^{-1}\Omega R' + \Omega'$, using the fact that $\frac{\partial}{\partial t}(R')^{-1} = -(R')^{-1}\dot{R}'(R')^{-1}$ using the definitions $\Omega' = (R')^{-1}\,\mathrm{d}R'$, $\Omega = R^{-1}\,\mathrm{d}R$, and identifying the term $[\Omega', (R')^{-1}\dot{\Omega}R']$ in the expression, the result follows. ■

Theorem 5.1 *The dynamic equation for the rotational part of the three frame system is given by:*

$$\mathbb{I}_b\{(R')^{-1}\dot{\Omega}R'+\dot{\Omega}'\}+\{(R')^{-1}\Omega R'+\Omega'\}\mathbb{I}_b\{(R')^{-1}\Omega R'+\Omega'\}+\mathbb{I}_b\{(R')^{-1}\Omega R'+\Omega'\}-\mathbb{I}_b[\Omega', (R')^{-1}\dot{\Omega}R'] = T \quad (15)$$

Proof: The velocity of an arbitrary point is $\mathbf{V}'' = D_t\mathbf{X}''$, and for a point in a rigid body one has $\partial\mathbf{X}''/\partial t = 0$, which means that $\mathbf{V}'' = [\Omega, \mathbf{X}'']$, which implies that $\Lambda'' = m[\mathbf{X}'', [\Xi^{(2)}, \mathbf{X}'']] = -m[\mathbf{X}'', [\mathbf{X}'', \Xi^{(2)}]] = \mathbb{I}(\mathbf{X}'')\Xi^{(2)}$. The linear operator $\mathbb{I}(\mathbf{X}'')$ is called the *inertia operator*. Clearly it is non-linear in $\mathbf{X}''$, and in the case of a rigid body, m should be replaced by $\mathrm{d}m$, and an integral in this measure should be carried out, which originates an inertia operator $\mathbb{I}_b$ for the rigid body that is independent of $\mathbf{X}''$. The rotational Newton's law is then $D_t\Lambda'' = \frac{\partial\Lambda''}{\partial t} + [\Xi^{(2)}, \Lambda''] = T$, where Λ'' is the angular momentum expressed in the frame S'' and its expression is given by $\Lambda'' = \mathbb{I}_b\Xi^{(2)}$. After some substitution, one has:

$$D_t\Lambda'' = D_t\mathbb{I}_b\Xi^{(2)} = \mathbb{I}_b\frac{\partial}{\partial t}\Xi^{(2)} + [\Xi^{(2)}, \mathbb{I}_b\Xi^{(2)}] = \mathbb{I}_b\dot{\Xi}^{(2)} + [\Xi^{(2)}, \mathbb{I}_b\Xi^{(2)}] =$$
$$= \mathbb{I}_b\frac{\partial}{\partial t}((R')^{-1}\Omega R' + \Omega') + [((R')^{-1}\Omega R' + \Omega'), \mathbb{I}_b((R')^{-1}\Omega R' + \Omega')] = T. \quad (16)$$

By applying the result of lemma 5.1, and doing additional simple calculations, the statement of the theorem follows. ■

The dynamic equation in theorem 5.1 could additionally be written in separate equations if a basis of eigenvectors of $\mathbb{I}_b$ in $\mathfrak{su}(2)$ was chosen (principal directions). The matrices $(R')^{-1}\Omega R' + \Omega'$ and $(R')^{-1}\dot{\Omega}R' + \dot{\Omega}'$ should then be written in such a basis (and will not be done here due to the lack of space).

6. CONCLUSION AND FUTURE WORK

We presented new results for the kinematics and dynamics modeling for a chain of rigid bodies/reference frames attached to those rigid bodies. Those results uses Cartan's connection with values in the Lie algebra $\mathfrak{su}(2)$ (velocity fields), corresponding to the frame transformations given by the Lie group of unit quaternions $SU(2)$. The kinematic results were obtained for the case of n reference frames with the same origin. The dynamics equations were obtained in the case of three reference frames. The interesting fact from the results obtained is that the kinematic calculations (Cartan's connection, that is the velocity field) are obtained recursively, and in the unit quaternion representation, which is computationally efficient. Future works will include a general formula for kinematics and dynamic calculation for a system with n reference frames with different origins (for example a serial robot with n links) and parallel robots (with kinematic constraints). For those cases, the principal fiber bundle will be $(P, \mathbb{R}^3, \pi,\ \mathrm{Spin}(3) \ltimes \mathbb{R}^3)$, where $\mathrm{Spin}(3) \ltimes \mathbb{R}^3$ is the Lie group of poses (unit dual quaternions). The Cartan's connection Ξ of the group $\mathrm{Spin}(3) \ltimes \mathbb{R}^3$ has values in the Lie algebra $\mathfrak{spin}(3) \ltimes \mathbb{R}^3$. It is expected that new controller designs techniques, based in this theory, could emerge (see Pait and Colón (2010a)).

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