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OPTIMAL LOW-THRUST TRANSFERS BETWEEN ELLIPTIC COAXIAL NON-COPLANAR ORBITS

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Abstract. *In this paper an analytical first order solution for optimal time-fixed low-thrust limited-power transfers (no rendezvous) between elliptic coaxial non-coplanar orbits in an inverse-square force field is presented. This analytical solution is determined through canonical transformation theory and is expressed in terms of classical orbital elements involving the semi-major axis, the eccentricity and the inclination or the longitude of the ascending node. A simplified solution is determined for transfers between close orbits. An iterative algorithm based on the first order analytical solution is described for solving the two-point boundary value problem of going from an initial orbit to a final orbit.*

Keywords: *Optimal low-thrust limited-power trajectories, transfers between non-coplanar orbits, optimal space trajectories.*

1. INTRODUCTION

The analysis presented in this work has been motivated by the renewed interest in the use of low-thrust propulsion systems in space missions in the last thirty years. Important space missions have made use of low-thrust propulsion systems: NASA-JPL Deep Space 1 and ESA-SMART1. Low-thrust electric propulsion systems are characterized by high specific impulse and low-thrust capability (the ratio between the maximum thrust acceleration and the gravity acceleration on the ground is small, between 10^{-4} and 10^{-2}) and have their greatest benefits for high-energy planetary missions (Marec, 1979; Racca, 2003). Several researchers have obtained numerical and analytical solutions for several maneuvers involving specific initial and final orbits and specific thrust profiles. In the analytical studies, averaging techniques are applied and solutions of the averaged equations are obtained such that only secular behaviour of the optimal solutions is discussed. Few works discuss the inclusion of periodic terms which are, in general, considered only for transfers between close orbits or for transfers between arbitrary elliptic coplanar orbits (Edelbaum, 1965; Marec and Vinh, 1977; Haissig et al, 1992; Kiforenko, 2005; da Silva Fernandes and Carvalho, 2008; Jamison and Coverstone, 2010; Huang, 2012; Quarta and Mengali, 2013).

The main goal of this work is to obtain an approximated analytical solution, which includes short periodic terms, for the problem of optimal time-fixed low-thrust limited-power transfers (no rendez-vous) between elliptic coaxial non-coplanar orbits in a Newtonian central gravity field. The optimization problem is formulated as a Mayer problem of optimal control theory with semi-major axis, eccentricity and inclination or longitude of the ascending node – as state variables. After applying the set of necessary conditions provided by Pontryagin Maximum Principle (Pontryagin et al, 1962) and determining the maximum Hamiltonian which describes the extremal trajectories, the short periodic terms are eliminated through an infinitesimal canonical transformation built through Hori method – a perturbation canonical method based on Lie series (Hori, 1966). The new maximum Hamiltonian resulting from the infinitesimal canonical transformation describes the extremal trajectories for long duration transfers. A set of first integrals are derived for the canonical system of differential equations governed by new Hamiltonian function and a closed-form analytical solution is then obtained through Hamilton-Jacobi theory. For long duration maneuvers the existence of conjugate points is investigated through the Jacobi condition. An iterative algorithm based on the first order analytical solution is described for solving the two-point boundary value problem of going from an initial orbit to a final orbit. For transfers between close orbits a simplified solution is straightforwardly derived by linearizing the new Hamiltonian and the generating function obtained through the Hori method. This simplified solution is given by a linear system of algebraic equations in the initial value of the adjoint variables, such that the two-point boundary value problem can be solved by simple techniques.

2. FORMULATION OF THE OPTIMIZATION PROBLEM

A low-thrust limited-power propulsion system – LP system – is characterized by low-thrust acceleration level and high specific impulse. For such a system, the fuel consumption is described by the variable J defined as (Marec, 1979)

$$J = \frac{1}{2}\int_{t_0}^{t} \gamma^2 dt \tag{1}$$

where γ is the magnitude of the thrust acceleration vector $\boldsymbol{\gamma}$, used as a control variable. The consumption variable J is a monotonic decreasing function of the mass m of the space vehicle,

$$J = P_{\max}\left(\frac{1}{m} - \frac{1}{m_0}\right)$$

where $P_{\max}$ is the maximum power and m_0 is the initial mass. The minimization of the final value of the fuel consumption, J_f, is equivalent to the maximization of m_f.

Consider the motion of a space vehicle M, powered by a limited-power engine in an inverse-square force field. At time t, the state of a space vehicle M is defined by the semi-major axis a, the eccentricity e, the inclination of the orbital plane I or the longitude of the ascending node Ω and the fuel consumption J. The inclination or the longitude of the ascending node is used according to the maneuver considered in the analysis: maneuver with change in the inclination of the orbital plane or maneuver with change in the longitude of the ascending node.

The optimization problem can be formulated as a Mayer problem of optimal control as follows: It is proposed to transfer the space vehicle M from the initial state $(a_0, e_0, I_0, 0)$ at time t_0 to the final state (a_f, e_f, I_f, J_f) at time t_f, such that the final consumption variable J_f is a minimum. The duration of the transfer $t_f - t_0$ is specified. Recall that inclination can be replaced by the longitude of the ascending node.

For transfers between coaxial orbits, the state equations are given by the well-known Gauss equations (Battin, 1987) for semi-major axis, eccentricity and inclination of the orbital plane or longitude of ascending node, according to the maneuver considered. For maneuvers with change in the inclination of the orbital plane, the argument of periapsis is null, i.e. $\omega = 0$ (Marec and Vinh, 1977). So, the state equations are:

$$\frac{da}{dt} = \frac{2}{n\sqrt{1-e^2}}\left[e\sin f\, R + (1+e\cos f)S\right]$$

$$\frac{de}{dt} = \frac{\sqrt{1-e^2}}{na}\left[\sin f\, R + (\cos E + \cos f)S\right]$$

$$\frac{dI}{dt} = \frac{1}{na\sqrt{1-e^2}}\left(\frac{r}{a}\right)\cos f\; W$$

$$\frac{dJ}{dt} = \frac{1}{2}\left(R^2 + S^2 + W^2\right) \tag{2}$$

where R, S and W are, respectively, the radial, the circumferential and the normal components of the thrust acceleration vector, f is the true anomaly, E is the eccentric anomaly and n is the mean motion, $n^2a^3 = \mu$, with μ denoting the gravitational parameter. For maneuvers with change in the longitude of the ascending node, $\omega = 90°$ or $270°$ and $I = 90°$ (Marec and Vinh, 1977). So, the state equations are the same ones previously defined with Ω in the place of I, that is,

$$\frac{d\Omega}{dt} = \frac{1}{na\sqrt{1-e^2}}\left(\frac{r}{a}\right)\cos f\; W\,.$$

Note that the form of differential equation for the longitude of ascending node is consequence of the conditions which define the maneuver; that is, the Gauss equation for the longitude of the ascending node simplifies for $\omega = 90°$ or $270°$

and $I = 90°$. Since the differential equations for I and Ω are exactly the same one, a variable θ is introduced to denote these variables.

Following the Pontryagin Maximum Principle (Pontryagin et al, 1962), the adjoint variables p_a, p_e, p_θ and p_J are introduced and the Hamiltonian function $H\left(a, e, \theta, J, p_a, p_e, p_\theta, p_J, R, S, W\right)$ is formed using the right-hand side of Eq. (2),

$$H = \frac{1}{na\sqrt{1-e^2}}\left\{2a\left[e\sin f\, R + \left(1+e\cos f\right)S\right]p_a + \left(1-e^2\right)\left[\sin f\, R + \left(\cos E + \cos f\right)S\right]p_e \right.$$
$$\left. + \left[\left(\frac{r}{a}\right)\cos f\, W\right]p_\theta\right\} + \frac{p_J}{2}\left(R^2 + S^2 + W^2\right). \tag{3}$$

The control variables R, S and W must be selected from the admissible controls such that the Hamiltonian function reaches its maximum along the optimal trajectory. Thus,

$$R^* = \frac{1}{na\sqrt{1-e^2}}\left[\left(2ae\sin f\right)p_a + \left(1-e^2\right)\sin f\, p_e\right]$$

$$S^* = \frac{1}{na\sqrt{1-e^2}}\left[2a\left(1+e\cos f\right)p_a + \left(1-e^2\right)\left(\cos E + \cos f\right)p_e\right]$$

$$W^* = \frac{1}{na\sqrt{1-e^2}}\left(\frac{r}{a}\right)\cos f\, p_\theta \tag{4}$$

In equations above, it was taken into account that p_J is a first integral and its value is obtained from the transversality condition, $p_J(t_f) = -1$.

Introducing equations above into Eq. (3), one finds the maximum Hamiltonian

$$H^* = \frac{1}{2n^2a^2\left(1-e^2\right)}\left\{4a^2\left(1+2e\cos f + e^2\right)p_a^2 + 4a\left(1-e^2\right)\left(e + \cos f + \cos E\left(1+e\cos f\right)\right)p_a p_e \right.$$
$$\left. + \left(1-e^2\right)^2\left(1 + 2\cos E\cos f + \cos^2 E\right)p_e^2 + \left(\frac{r}{a}\right)^2\cos^2 f\, p_\theta^2\right\}. \tag{5}$$

The problem of determining a first order analytical solution of the system of differential equations governed by the Hamiltonian H^* is solved by means of the theory of canonical transformations as it will be described in the next section.

3. ANALYTICAL FIRST ORDER SOLUTION

In order to eliminate the short periodic terms from the maximum Hamiltonian function H^*, Hori method (Hori, 1966) is applied. It should be noted that H^* contains only terms related to the optimal thrust acceleration. In order to apply Hori method, an additional term which corresponds to the undisturbed Hamiltonian H_0, defined by the dynamics of the two-body problem, is introduced. Thus,

$$\mathsf{H} = H_0 + H^*,$$

where $H_0 = np_M$, with p_M denoting the adjoint variable to the mean anomaly M. It is assumed that H_0 is of zero order and H_γ^* is of the first order in a small parameter defined by the magnitude of the thrust acceleration.

Following the algorithm of Hori method, one finds

$$F_1=\frac{a'}{2\mu}\left\{4a'^2p_a'^2+\frac{5}{2}\left(1-e'^2\right)p_e'^2+\frac{\left(1+4e'^2\right)}{2\left(1-e'^2\right)}p_\theta'^2\right\},\tag{6}$$

$$\begin{aligned}S_1=\frac{1}{2}\sqrt{\frac{a'^5}{\mu^3}}&\left\{8e'\sin E'a'^2p_a'^2+8\left(1-e'^2\right)\sin E'a'p_a'p_e'+\left(1-e'^2\right)\left[-\frac{5}{4}e'\sin E'+\frac{3}{4}\sin 2E'-\frac{1}{12}e'\sin 3E'\right]p_e'^2\right.\\&\left.+\left(1-e'^2\right)^{-1}\left[\left(-\frac{9}{4}e'+e'^3\right)\sin E'+\left(\frac{1}{4}+\frac{1}{2}e'^2\right)\sin 2E'-\frac{1}{12}e'\sin 3E'\right]p_\theta'^2\right\}.\end{aligned}\tag{7}$$

Prime is introduced to denote the new canonical variables defined by the infinitesimal canonical transformation generated by S_1.

Consider the Mathieu transformation defined by the following equations

$$a'=a'' \qquad e'=\sin\varphi \qquad \theta'=\theta''$$

$$p_a'=p_a'' \qquad p_e'=\frac{p_\varphi''}{\cos\varphi} \qquad p_\theta'=p_\theta''.\tag{8}$$

The Hamiltonian function F' is invariant with respect to this transformation,

$$F''=\frac{a''}{2\mu}\left\{4a''^2p_a''^2+\frac{5}{2}p_\varphi''^2+\frac{1}{2}\left(1+5\tan^2\varphi\right)p_\theta''^2\right\}.\tag{9}$$

The averaged canonical system governed by the Hamiltonian function F'' has a set of first integrals which can be derived straightforwardly from the differential equations. The first integrals are given by the following equations:

$$\frac{a''}{2\mu}\left\{4a''^2p_a''^2+\frac{5}{2}p_\varphi''^2+\frac{1}{2}\left(1+5\tan^2\varphi\right)p_\theta''^2\right\}=\mathsf{E},\tag{10}$$

$$p_\varphi''^2+p_\theta''^2\tan^2\varphi=C_1^2,\tag{11}$$

$$p_\theta''=C_2.\tag{12}$$

E, $C_i, i=1,2$ are constants.

Now, consider the canonical transformation,

$$\left(a'',\varphi,\theta'',p_a'',p_\varphi'',p_\theta''\right)\xrightarrow{W}\left(C_1,C_2,\mathsf{E},p_{C_1},p_{C_2},p_{\mathsf{E}}\right),$$

defined by a generating function W such that the constants C_1, C_2 and E become the new generalized coordinates. By applying the separation of variables technique for solving the Hamilton-Jacobi equation (Lanczos, 1971), one gets:

$$W(a'',\varphi,\theta'',C_1,C_2,\mathsf{E})=W_1(a'',C_1,C_2,\mathsf{E})+W_2(\varphi,C_1,C_2,\mathsf{E})+W_3(\theta'',C_1,C_2,\mathsf{E}),$$

with

$$W_1=-\sqrt{\frac{5C^2}{2}}\left\{\left[\frac{4\mu\mathsf{E}}{5C^2a''}-1\right]^{1/2}-\tan^{-1}\left[\frac{4\mu\mathsf{E}}{5C^2a''}-1\right]^{1/2}\right\}, \qquad W_2=\int\sqrt{C_1^2-C_2^2\tan^2\varphi}\,d\varphi, \qquad W_3=C_2\theta''$$

and $5C_1^2+C_2^2=5C^2$.

After some calculations, one finds the solution of the canonical system governed by the Hamiltonian F'' for a given set of initial conditions:

$$a''(t) = \frac{a_0''}{1 + \frac{4a_0''}{\mu}\left(\frac{1}{2}\mathsf{E}t^2 - a_0'' p_{a_0}'' t\right)}, \tag{13}$$

$$a'' \sin^2 k_0 = a_0'' \sin^2\left(\sqrt{2}\psi + k_0\right), \tag{14}$$

$$\psi = \frac{C}{\sqrt{5\left(C_1^2 + C_2^2\right)}}\left(\tau - \tau_0\right), \tag{15}$$

$$\sin\varphi = \sin k_1 \sin\tau , \tag{16}$$

$$\theta'' = k_2 + \tan^{-1}\left(\tan\tau / \sec k_1\right) - \frac{4}{5}\tau\cos k_1 , \tag{17}$$

$$p_a''^2 = \left(\frac{a_0''}{a''}\right)^3 p_{a_0}''^2 + \frac{1}{8} p_{\theta_0}''^2\left(5\sec^2 k_1 - 4\right)\left(\frac{a_0''}{a''^3} - \frac{1}{a''^2}\right), \tag{18}$$

$$p_\varphi''^2 = p_{\theta_0}''^2\left(\sec^2 k_1 - \sec^2\varphi\right), \tag{19}$$

$$p_\theta'' = p_{\theta_0}'' , \tag{20}$$

with the auxiliary constants k_0, k_1 and k_2 defined as functions of the initial value of the adjoint variables by

$$\csc^2 k_0 = \frac{8\left(a_0'' p_{a_0}''\right)^2 + p_{\theta_0}''^2\left(5\sec^2 k_1 - 4\right)}{p_{\theta_0}''^2\left(5\sec^2 k_1 - 4\right)}, \quad \sec^2 k_1 = \frac{p_{\varphi_0}''^2 + p_{\theta_0}''^2 \sec^2\varphi_0}{p_{\theta_0}''^2}, \quad k_2 = \theta_0'' - \tan^{-1}\left(\tan\tau_0 / \sec k_1\right) + \frac{4}{5}\tau_0 \cos k_1 .$$

The constants C, C_1, C_2 and E can also be written as functions of the initial value of the adjoint variables:

$$C^2 = \frac{1}{5} p_{\theta_0}''^2\left(5\sec^2 k_1 - 4\right) \qquad C_2 = p_{\theta_0}''$$

$$C_1^2 = p_{\varphi_0}''^2 + p_{\theta_0}''^2 \tan^2\varphi_0 \qquad 4\mu\mathsf{E} = a_0''\left(8\left(a_0'' p_{a_0}''\right)^2 + p_{\theta_0}''^2\left(5\sec^2 k_1 - 4\right)\right).$$

The initial conditions are $a''(0) = a_0''$, $e''(0) = \sin\varphi_0$ and $\theta''(0) = \theta_0''$, and, τ_0 is obtained from the Eq. (16), that is, $\sin\varphi_0 = \sin k_1 \sin\tau_0$.

Following Hori method (Hori, 1966) and applying the initial conditions, one finds:

$$a(t) = a'(t) + \sqrt{\frac{a'^5}{\mu^3}}\left[8e'\sin E' a'^2 p_a' + 4\left(1 - e'^2\right)\sin E' a' p_e'\right]_{t_0}^{t}, \tag{21}$$

$$e(t) = e'(t) + \sqrt{\frac{a'^5}{\mu^3}}\left[4\left(1 - e'^2\right)\sin E' a' p_a' + \left(1 - e'^2\right)\left[-\frac{5}{4}e'\sin E' + \frac{3}{4}\sin 2E' - \frac{1}{12}e'\sin 3E'\right]p_e'\right]_{t_0}^{t}, \tag{22}$$

$$\theta(t) = \theta'(t) + \sqrt{\frac{a'^5}{\mu^3}}\left[\left(1 - e'^2\right)^{-1}\left[\left(-\frac{9}{4}e' + e'^3\right)\sin E' + \left(\frac{1}{4} + \frac{1}{2}e'^2\right)\sin 2E' - \frac{1}{12}e'\sin 3E'\right]p_\theta'\right]_{t_0}^{t}, \tag{23}$$

with a', e', …, p'_θ previously defined. The eccentric anomaly E' is computed from Kepler´s equation (Battin, 1987) with the mean anomaly given by

$$M'(t) = M'(t_0) + \int_{t_0}^{t} n' dt .$$

4. ANALYSIS OF LONG DURATION TRANSFERS – EXISTENCE OF CONJUGATE POINT

In what follows is presented a discussion about the existence of conjugate points through the analysis of Jacobi condition for long duration transfers, which are described by the general solution of the canonical system governed by the Hamiltonian F''.

Consider Eqns (14) – (17) which define a two-parameter family of extremals in the phase space (a'', φ, θ'') for a given initial phase point $(a''_0, \varphi_0, \theta''_0)$ corresponding to an initial orbit. By eliminating the auxiliary variables τ and ψ, $\alpha = a''/a''_0$ and θ'' can be written as explicit functions of φ, that is, $\alpha = \alpha(\varphi, \varphi_0, k_0, k_1)$ and $\theta'' = \theta''(\varphi, \varphi_0, \theta''_0, k_1)$. The conjugate points to the phase point $(a''_0, \varphi_0, \theta''_0)$ are determined through the roots of the equation (Bliss, 1946)

$$\frac{\partial \alpha}{\partial k_0}\frac{\partial \theta''}{\partial k_1} - \frac{\partial \alpha}{\partial k_1}\frac{\partial \theta''}{\partial k_0} = 0 .$$

Since θ'' does not depend on k_0, $\partial\theta''/\partial k_0 = 0$. On the other hand,

$$\frac{\partial \alpha}{\partial k_0} = -\frac{2\sqrt{\alpha}}{\sin k_0}\left(\alpha - 2\sqrt{\alpha}\cos\left(\sqrt{2}\psi\right) + 1\right)^{1/2} \neq 0 .$$

Thus, the problem of determining the conjugate points reduces to the analysis of the roots of the following equation:

$$\frac{\partial \theta''}{\partial k_1} = 0 .$$

From Eqns (16) and (17), one finds

$$\theta'' = \theta''_0 - \frac{4}{5}\cos k_1 \left\{ \sin^{-1}\left(\frac{\sin\varphi}{\sin k_1}\right) - \sin^{-1}\left(\frac{\sin\varphi_0}{\sin k_1}\right)\right\} + \sin^{-1}\left(\frac{\tan\varphi}{\tan k_1}\right) - \sin^{-1}\left(\frac{\tan\varphi_0}{\tan k_1}\right).$$

The partial derivative $\partial\theta''/\partial k_1$ is then given by

$$\frac{\partial \theta''}{\partial k_1} = -\frac{4}{5}(\tau - \tau_0)\sin k_1 + \frac{4}{5}\cot^2 k_1 \left[-\frac{\sin\varphi_0}{\cos\tau_0} + \frac{\sin\varphi}{\cos\tau}\right] + \frac{\sec^2 k_1}{\tan k_1}\left[\frac{\tan\varphi_0}{\sqrt{\tan^2 k_1 - \tan^2\varphi_0}} - \frac{\tan\varphi}{\sqrt{\tan^2 k_1 - \tan^2\varphi_0}}\right].$$

The auxiliary variable τ is re-introduced to simplify the expression.

Therefore, after some simplifications, one finds that the conjugate points are given by the roots of following equation

$$\left(1 + 4\sin^2 k_1\right)\sin(\tau - \tau_0) + 4(\tau - \tau_0)\sin^2 k_1 \cos\tau\cos\tau_0 = 0 .$$

If a conjugate point τ^* exists, or, equivalently, φ^* (see Eq. (16)), then the extremal does not yield a local minimum.

5. TRANSFERS BETWEEN CLOSE ORBITS

For transfers between close non-coplanar coaxial orbits, a simplified and complete first order solution can be obtained through a linearization of the right-hand side of Eqns (21), (22) and (23) about a reference orbit $\bar{O}$ with semi-major axis $\bar{a}$ and eccentricity $\bar{e}$. This simplified solution can be put in the form

$$\Delta x = Ap_0 , \tag{24}$$

where $\Delta x = \begin{bmatrix} \Delta\alpha & \Delta e & \Delta\theta \end{bmatrix}^T$ denotes the imposed changes on orbital elements (state variables), $\alpha = a/\bar{a}$, $p_\alpha = \bar{a}p_a$, p_0 denotes the 3 × 1 vector of initial values of the adjoint variables, and, A is a 3 × 3 symmetric matrix. In this simplified solution, the adjoint variables are constant and the matrix A is given by:

$$A = \begin{bmatrix} a_{\alpha\alpha} & a_{\alpha e} & a_{\alpha\theta} \\ a_{e\alpha} & a_{ee} & a_{e\theta} \\ a_{\theta\alpha} & a_{\theta e} & a_{\theta\theta} \end{bmatrix}, \tag{25}$$

where:

$$a_{\alpha\alpha} = 4\sqrt{\frac{\bar{a}^5}{\mu^3}}\left[\bar{M} + 2\bar{e}\sin\bar{E}\right]_{t_0}^{t}$$

$$a_{\alpha e} = a_{e\alpha} = 4\sqrt{\frac{\bar{a}^5}{\mu^3}}\left[\left(1-\bar{e}^2\right)\sin\bar{E}\right]_{t_0}^{t}$$

$$a_{\alpha\theta} = a_{\theta\alpha} = 0$$

$$a_{ee} = \sqrt{\frac{\bar{a}^5}{\mu^3}}\left(1-\bar{e}^2\right)\left[\frac{5}{2}\bar{M} - \frac{5}{4}\bar{e}\sin\bar{E} + \frac{3}{4}\sin 2\bar{E} - \frac{1}{12}\bar{e}\sin 3\bar{E}\right]_{t_0}^{t}$$

$$a_{e\theta} = a_{\theta e} = 0$$

$$a_{\theta\theta} = \sqrt{\frac{\bar{a}^5}{\mu^3}}\frac{1}{\left(1-\bar{e}^2\right)}\left[\frac{1+4\bar{e}^2}{2}\bar{M} + \left(-\frac{9}{4}\bar{e} + \bar{e}^3\right)\sin\bar{E} + \left(\frac{1}{4} + \frac{1}{2}\bar{e}^2\right)\sin 2\bar{E} - \frac{1}{12}\bar{e}\sin 3\bar{E}\right]_{t_0}^{t} .$$

$\bar{E}$ is the eccentric anomaly determined from Kepler's equation with the mean anomaly $\bar{M} = \bar{M}_0 + \bar{n}\left(t - t_0\right)$ and t_0 is the initial time. Since the imposed variations on the orbital elements are linear in the adjoint variables, Eq. (24), the solution of the two-point boundary value problem is very simple and can be obtained by standard techniques.

6. SOLUTION OF THE TWO-POINT BOUNDARY VALUE PROBLEM

In this section, an iterative algorithm based on the complete first order analytical solution is briefly described for solving the two-point boundary value problem of going from an initial orbit $O_0 : \left(a_0, e_0, \theta_0\right)$ to a final orbit $O_f : \left(a_f, e_f, \theta_f\right)$ at the prescribed final time t_f.

For a given final time, Eqns (21), (22) and (23) can be represented as follows

$$y_i\left(t_f\right) = g_i\left(t_f, p_{a_0}, p_{e_0}, p_{\theta_0}\right),\ i = 1,2,3,$$

where $y_1 = a$, $y_2 = e$ and $y_3 = \theta$. Note that p_{a_0}, p_{e_0} and p_{θ_0} appear explicitly in the short periodic terms and also implicitly through $a'\left(t\right)$, $e'\left(t\right)$, $\theta'\left(t\right)$ and $E'\left(t\right)$. Thus, functions g_i, $i = 1,2,3$, are nonlinear in these variables.

So, the two-point boundary value problem can be stated as: "find p_{a_0}, p_{e_0} and p_{θ_0} such that $y_1\left(t_f\right) = a_f$, $y_2\left(t_f\right) = e_f$ and $y_3\left(t_f\right) = \theta_f$". This problem can be solved through a Newton-Raphson algorithm (Stoer and Bulirsch, 2002), with the partial derivatives of functions g_i computed numerically by means of a procedure of centered differences.

7. CONCLUSION

An approximated analytical first order solution for optimal time-fixed low-thrust limited power transfers (no rendezvous) between elliptic coaxial non-coplanar orbits in an inverse-square force field is determined through canonical transformation theory. A brief description of the existence of conjugate point for long duration transfers and a simplified solution for transfers between close orbits are also presented.

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