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OPTIMIZATION OF THE NUMBER OF CONTROL POINTS FOR CURVE FITTING USING B-SPLINES

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Abstract. *In surface reconstruction from a point-cloud, which are usually disorganized, usually cross-sections of the object are used with a points reduction method to reduce the computational cost. After the reduction of the number of points that defines the cross-section of the object, an approximation method is used to obtain the curve, which represents the cross-section shape. When B-Splines curves are used to obtain cross-sections, the control points are calculated from the set of known points, but there is no rule on how many is the number of control points that should be used. In this sense, the objective of this paper is to present an analysis of the number of control points used in curve fitting using B-Spline to obtain a low fitting error and a low time execution of the algorithm. The analysis is performed by the multi-objective optimization method. The objectives evaluated in this method seek to minimize: the least square error, the number of control points and the algorithm resolution time. The objectives are related by a weighted function, whose results are assembled for obtaining the Pareto's curve. The minimum of this curve define the optimum number of control points that allows a good curve fitting, combined with a good performance of the algorithm.*

Keywords: *B-Spline, Curve fitting, Optimization, Multi-Objective, Pareto Curve*

1. INTRODUCTION

The surface reconstruction from a cloud of points is a process with a large computational load, which usually is handled hundreds of millions of points. Reconstructing a surface from all these points is not computationally viable. Therefore, it is necessary to use techniques to simplify the process to rebuild the surface.

The use of cross-sections associated with the method of reduction the number of points is a technique widely employed for reducing the computational load, but if the number of points is drastically reduced the obtained surface does not represent the object shape. Once finished the process of reducing the number of points, it is possible to obtain a curve representing the cross section of the object. One method that has been widely used is the fitting using the B-splines curves (Aquino et al, 2010).

One of the approximation methods using B-spline curves is done applying the least squares method, where it is necessary to define the number of control points. Higher the number of control points, more sensitive becomes the curve with the surface characteristics, decreasing the adjustment error. However, there is also an increase in the processing time due to increase in the number of equations to be solved, featuring a multi-objective problem where it is wanted a good fit allied with a low computational load (Liang, 2002) (Mari 2007), (Pereira, 2014).

Several optimization problems in engineering involve conflicting objectives, so it is necessary to treat them as multi-objective optimization problems, which involve the minimization or maximization of some objective satisfying certain constraints. The result of the optimization problem is not unique, an analysis of all possibilities is therefore necessary to define the best result for the problem. For that is used a weighting function relating the objectives with their respective weights, then drawing the graphs containing curves with the best solutions known as Pareto's curve (Deb, 2001), (Lobato, 2008).

Pereira (2014) used a cut-off factor to define the minimum amount of control points in function of the quantity of points that define the cross section of the object. In this paper is presented a multi-objective problem analysis for the curve fitting of a unitary weighted non-rational B-Spline, where the objective is to obtain the optimum amount of control points that minimize the error using the method of least squares and reducing the time required for the curve fitting.

1.1 Curve fitting and B-Splines

There are two ways for fitting curves: interpolation and approximation, Fig. 1. In an interpolation an exact fit to the data is required i.e. the curve pass through all known points and, for an approximation the curve does not necessarily pass through the points, but close to them. In this paper is used approximation in order to reduce the number of control points to be used for designing B-spline curves.

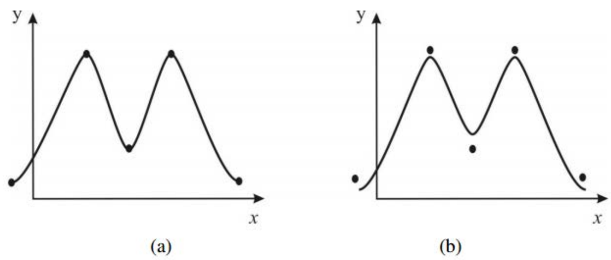


Figure 1. (a) Interpolation, (b) Approach (Ferreira, 2012)

The B-spline is a parametric curve which is usually applied for fitting curves due to ease of use, and changes of its part can be done without changing the entire curve. The equation is also defined in a simple and concise way due to the B-spline basis functions that makes the weighting effect of each control point to draw the curve. B-splines can be classified into two groups, Non-Rational Eq. (1) and Rational Eq. (2).

$$C(u) = \sum_{i=0}^{n} N_{i,p}(u).P_i \quad a \leq u \leq b \tag{1}$$

$$C(u) = \frac{\sum_{i=0}^{n} N_{i,p}(u).P_i.w_i}{\sum_{i=0}^{n} N_{i,p}(u).w_i} \quad a \leq u \leq b \tag{2}$$

Where: $N_{i,p}(u)$ represents the basis functions,
p is the degree of the *B-Spline*
P_i are the control points, $(i = 1, \dots, n)$
w_i are the weights associated to control point,
u are the nodes for which the curve points are calculated;

Rational B-Splines differs from non-rational due to the presence of a weight associated with the control points, where bigger is the weight, more is the curve closer to the control point.

To calculate the B-spline basis functions, one uses the recurrence formula due to deBoor, Cox and Mansfield, also known as Cox-deBoor recurrence formula. This method presents the better computational performance (Piegl; Tiller, 1997).

Given $U = \{u_0, u_1, \dots, u_l\}$ an increasing sequence of real numbers, denominated knots. The *i*-th B-Spline basis function of degree p, $N_{i,p}$, is defined by:

$$N_{i,0}(u) = \begin{cases} 1 & , if\ u_i \leq u < u_{i+1} \\ 0 & , otherwise \end{cases} \tag{3}$$

$$N_{i,p}(u) = \frac{u-u_i}{u_{i+p}-u_i} N_{i,p-1}(u) + \frac{u_{i+p+1}-u}{u_{i+p+1}-u_{i+1}} N_{i+1,p-1}(u) \tag{4}$$

So to represent a B-spline curve is necessary to know the control points, the knots vector and the degree of the curve. Then, the basis function is obtained and the points of the curve are calculated.

1.1.1 Definition of control points by the least squares method

Given a *B-Spline* curve $C(u_k)$, adjusted to a known set of points Q_k $(k = 1, \dots, m)$. The error of fit by the least squares method can be calculated by:

$$f = \sum_{k=1}^{m-1} [C(u_k) - Q_k]^2 \tag{5}$$

$$f = \sum_{k=1}^{m-1} \left[\sum_{i=0}^{n} N_{i,p}(u_k)\, P_i - Q_k\right]^2 \tag{6}$$

That in a matrix form can be written as:

$$f = [NP - Q]^2 \tag{7}$$

Including the diagonal matrix of weights W in Eq. (7) and developing it, one obtains:

$$f = (P^T N^T - Q^T)\mathrm{W}(NP - Q)$$

$$f = P^T N^T WNP - P^T N^T WQ - Q^T WNP + Q^T WQ$$

$$f = P^T N^T WNP - Q^T WNP - Q^T WNP + Q^T WQ$$

$$f = P^T N^T WNP - 2\, Q^T WNP + Q^T WQ \tag{8}$$

Deriving and equaling to zero to find the minimum:

$$\frac{\partial f}{\partial P} = N^T WNP + P^T N^T WN - 2\, Q^T WN$$

$$\frac{\partial f}{\partial P} = N^T WNP + N^T WNP - 2\, N^T WQ$$

$$\frac{\partial f}{\partial P} = 2\, N^T WNP - 2\, N^T WQ = 0 \tag{9}$$

$$N^T WNP = N^T WQ$$

$$P = (N^T WN)^{-1} N^T WQ \tag{10}$$

Where:

$$N = \begin{bmatrix} N_{0,p}(\overline{u_0}) & \cdots & N_{n,p}(\overline{u_0}) \\ \vdots & \ddots & \vdots \\ N_{0,p}(\overline{u_m}) & \cdots & N_{n,p}(\overline{u_m}) \end{bmatrix} \qquad W = \begin{bmatrix} \mathrm{w}_0 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & w_n \end{bmatrix}$$

$$P = \begin{bmatrix} P_0 \\ \vdots \\ P_n \end{bmatrix} \qquad Q = \begin{bmatrix} Q_0 \\ \vdots \\ Q_n \end{bmatrix}$$

Therefore, solving the above system of equations, Eq. (10), it is possible to obtain the control points, which reduces the least square error to a minimum, i.e., the best fit for this condition.

1.1.2 Definition of knots vector

To apply the Eq. (10) are necessary to know the B-Spline basis functions that, in your turn, require the knots vector. According Piegl & Tailor (1997), to define the knots vector (U_k) one can use a procedure using an initial vector ($\overline{u_\mathrm{k}}$) $\in [0,1]$, that can be calculated, for example, by the chord lengths d_c:

$$d_c = \sum_{k=1}^{n} |Q_k - Q_{k-1}|$$

$$\overline{u_0} = 0 \qquad \overline{u_\mathrm{n}} = 1$$

$$\overline{u_\mathrm{k}} = \overline{u_{\mathrm{k}-1}} + \frac{|Q_k - Q_{k-1}|}{d_c},\ k = 1, \dots, n-1 \tag{11}$$

Using the Eq. (11), it is possible to obtain the knots vector in the following way (Piegl & Tailor, 1997):

$$d = \frac{m+1}{n-p+1}$$

$$u_0 = \cdots = u_p = 0 \qquad u_{m-p} = \cdots = u_m = 1$$

$$i = int(j \cdot d) \qquad \alpha = j \cdot d - i$$

$$u_{j+p} = (1-\alpha)\overline{u_{\text{i}-1}} + \alpha\overline{u_{\text{i}}} \qquad j = 1, \dots, n-p \tag{12}$$

2. STUDY OF THE NUMBER OF CONTROL POINTS AS A MULTI-OBJECTIVE PROBLEM

The multi-objective problem proposed in this paper consists of the following objectives:
• Reduce the number of control points;
• Reduction of computational time to solve the method of least squares;
• Reduction the error fitting by least squares methods.

According Vanderplaats (1999) the multi-objective problem-solving is performed using optimization methods for the generation of results to be analyzed, Pareto's curve. However, this work will not use optimization methods because the resolution of Eq. (10) already gives the control points that minimize the error to a minimum, and the other objectives are easily measured.

In this paper the objectives are analyzed according to the number of control points. In the following each response is normalized and then performed the weighted sum of the contribution of each objective, thereby forming the response curve or Pareto's curve.

In order to explain the methodology proposed in this paper, will be used a cubic B-spline curve to adjust a set of points formed by twenty points, represented in Fig. 2.

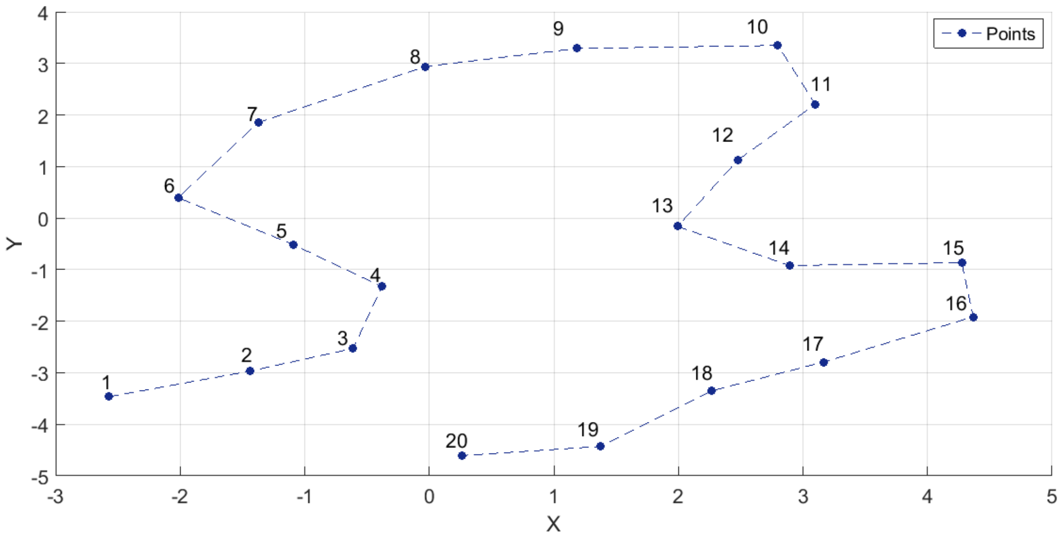


Figure 2. Set of points to be analyzed.

The first step consists in obtaining the least squares error for each number of control points by using Eq. (8), shown in Fig. 3. The analysis is done from the least possible number of control points, given by the degree of B-Spline, up to the number of adjusted points.

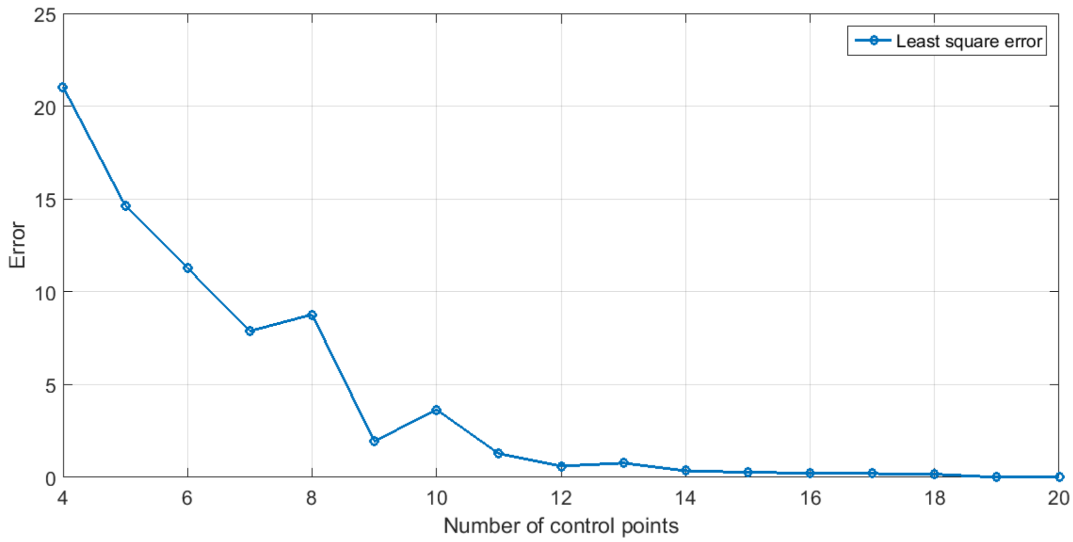


Figure 3. Least square error vs. the number of control points.

Next is estimated the time to solve the method for each number of control points. For this we used the MATLAB 2014 software installed on the operating system LINUX UBUNTU 14.04.2LTS on an HP DM4-2160sf model notebook, Intel i5 2430M 2.4GHz processor, 6GB DDR3 RAM.

As the processing time depends on the operating system, that may have wide range, it is necessary to perform an estimate of these values. For it was held the following statistical procedure:

1. Given a sequence containing *n* values of number of control points to be used to obtain the control points;
2. Calculate the required time to obtain the control points for each *n* value of number of control points;
3. Repeat step 2 for 50,000 times;
4. With this set of time samples, the mean and standard deviation are calculated and generated the curve time vs. the number of control points, as shown in Fig. 4.

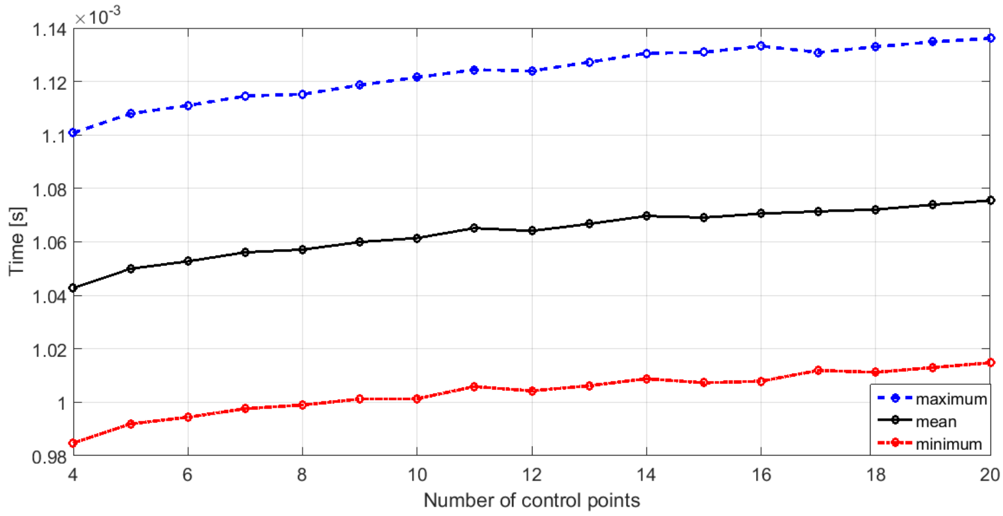


Figure 4. Average time to solve the method vs. the number of control points

As can see in Fig. 4 the time taken to obtain the control points is little sensitive to the increased number of control points. Then, it will not be considered the time for the multi-objective method.

With the results, it is then necessary to use a weighting technique to evaluate the contribution of each objective in order to obtain the solution curve of the problem. As there are only two objectives, the error and the number of the control points, the equation of the objective function to minimize is as follows:

$$F = \frac{w_1 \cdot F_1 + w_2 \cdot F_2}{w1 + w2} \tag{13}$$

Where F_1 and F_2 are the functions that define each objective and w_1 and w_2 their weights respectively.
As the magnitude of the objectives are not comparable, the results are normalized obtaining the following relation:

$$F = \frac{w_1 \cdot \frac{F_1 - |F_1|_{min}}{|F_1|_{max} - |F_1|_{min}} + w_2 \cdot \frac{F_2 - |F_2|_{min}}{|F_2|_{max} - |F_2|_{min}}}{w1 + w2} \tag{14}$$

As the adjusting error is the main objective its weight was adopted as 90% , and for the number of control points a 10% weight.

Finally, the application of Eq. (14) results in the Pareto's curve, which minimizes the expected objectives for the problem, as shown in Fig. 5. Then, the best result corresponds to the lower value of the curve of the objective function F that, for the presented example, corresponds to 60% of the given points. This means that a good cubic B-spline curve for the given points can be obtained using 12 control points, Fig. 6(a).

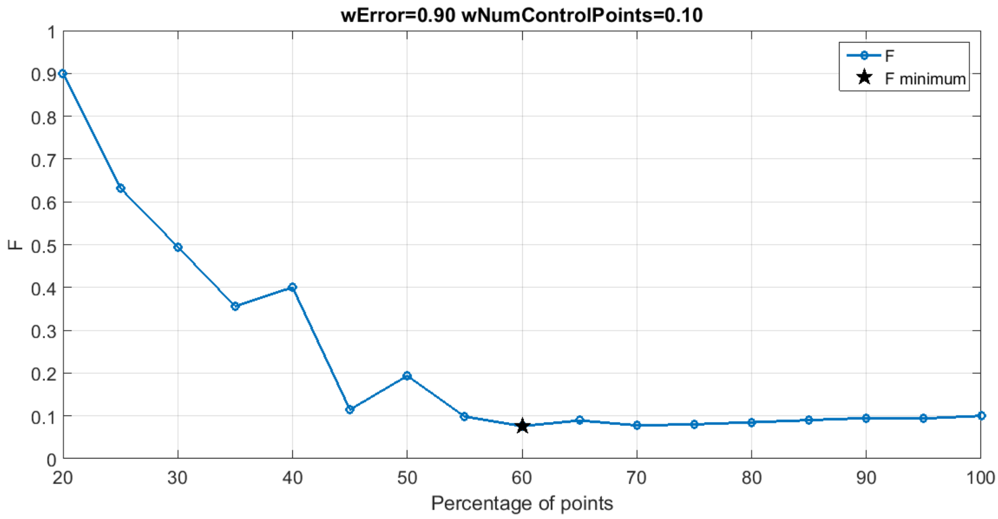


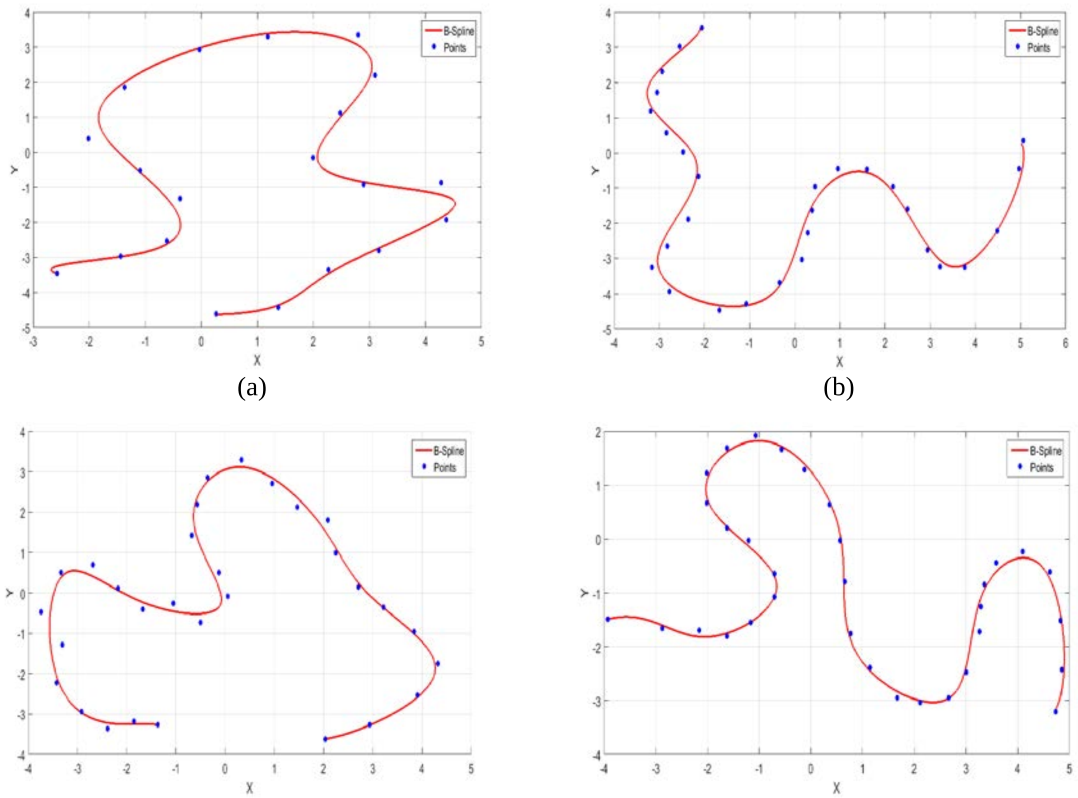
Figure 5. Results of the analysis of the multi-objective problem.

Using three other sets of points as example, there are obtained the results shown in Tab. 1.

Table 1: Results of the application of the multi-objective method for different sets of points.

Curve	Quantity of adjusted points (n)	Minimum F	Optimum number of control points (n_{otm})
Curve A	20	60%	12
Curve B	29	37.93%	11
Curve C	30	46.67%	14
Curve D	33	39.39%	13

Using their optimal number of control points, the curves obtained are shown in Fig. 6.



(a) (b) (c) (d)

Figure 6 - Fitting *B-Spline* cubic curves using the optimal number of control points (a) $n = 20$; $n_{otm} = 12$; (b) $n = 29$; $n_{otm} = 11$; (c) $n = 30$; $n_{otm} = 14$; (d) $n = 33$; $n_{otm} = 13$.

3. CONCLUSION

As observed in Tab. 1, an optimum number of control points can be defined to draw a B-spline curve considering the errors and the number of control points as function of the number of points that defines the curve. Then, one cannot be set a priori a value of the amount of control points in function of the number of points that define the cross section of an object or a curve.

Since the processing time of the method is very small, as can be seen in Fig. 4, the method can be inserted in the program to obtain the curve defined by a known set of points, as for example, for reconstructing a surface from a point-cloud, without compromising the total processing time.

The presented methodology also allows you to vary the weights of the objectives, as function of its importance and / or the user's needs.

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