A DISPERSIVE NONLOCAL MODEL FOR SHEAR WAVE PROPAGATION IN A PERIODICALLY PERFECT/IMPERFECT CONNECTED MULTI-LAMINATED

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Abstract. *This article present a study of the shear wave propagation based on asymptotic dispersive method for the description of the dynamic processes in a periodically multi-laminate medium with perfect and/or non-perfect contact between the layers. The dispersion relation was derived analytically from the average model. Detailed numerical simulations are provided to illustrate graphically the phase and group velocity. The effects of the unit cell size, wave number and incident angle on the dispersion relation are also examined. The frequency band structure as a function of the wave-vector components is calculated.*

Keywords: *homogenization, periodic multi-laminate medium, imperfect contact, dispersion relation*

1. INTRODUCTION

Periodically multi-laminated elastic composites are of interest in many engineering and geophysical applications. Layered composites have been becoming increasingly important in various branches of modern technology because they offer more attractive and cost effective solutions for a large variety of problems. For example, in structural design, laminated composites find increasing applications because not only they have high strength-to-weight and high stiffness-to-weight ratio but also they are more resistive to environmental effects and require less maintenance.

Various methods have been proposed to describe the dynamic problems of a laminated medium, which take into account structural effects. We can mention the effective modulus theories (Postma, 1955; Behrens, 1967; Aboudi and Weitsman, 1972; Schoenberg and Weitsman, 1975); the effective stiffness theories (Sun *et al.*, 1968; Grot and Achenbach, 1970; Aboudi, 1981); variational methods theories (Hashin and Shtrikman, 1963; Nemat-Nasser *et al.*, 1975; Saurin, 2007; Willis, 2012); continuum mixture theories (Green and Naghdi, 1965; Hegemier. and Nayfeh, 1973; Aboudi, 1973; Ben-Amoz, 1975); continuum theories with microstructure (Mindlin, 1964; Drumheller and Bedford, 1974; Wang and Sun, 2002; Potapenko *et al.*, 2005; Huang and Sun, 2006); and asymptotic homogenization method (Meguid and Kalamkarov, 1994; Chen and Fish, 2001; Parnell and Abrahams, 2006; Andrianov *et al.*, 2008; Smyshlyaev, 2009).

The purpose of this work is to determine the effects of the microstructure, incidence angle, volume fraction and imperfect interface on the dynamic effective properties for multi-laminate medium derived by considering higher order terms in the expansion of the stress and displacement fields. The present investigation is an extension of previous results reported at the literature (Vivar-Pérez *et al.*, 2009; Brito-Santana *et al.*, 2015) for wave propagation in periodically bi-laminated composites.

2. THE PROBLEM FORMULATION

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We consider an anisotropic elastic body of a periodic structure occupying a bounded region $\mathcal{O}^{\varepsilon}$ in $\mathbb{R}^3$ space with Lipschitz boundary $\partial\mathcal{O}^{\varepsilon} = \overline{\partial_1\mathcal{O}^{\varepsilon}} \cup \overline{\partial_2\mathcal{O}^{\varepsilon}}$ such that $\partial_1\mathcal{O}^{\varepsilon} \cap \partial_2\mathcal{O}^{\varepsilon} = \phi$ where $\partial_1\mathcal{O}^{\varepsilon}$ and $\partial_2\mathcal{O}^{\varepsilon}$ are boundary portions. We assume that the region Ω is made up of periodic repetition of the unit cell Y in the form of a parallelepiped with dimensions $\varepsilon y_i\,(i=1,2,3)$ where ε is the ratio of the unit cell size (i.e. period of the structure) related to a typical length in the region. The method is presented for the particular case of anti-plane wave propagation in a periodically multi-laminated composite; see Fig. 1 where the set of interfaces separating of the phases is denoted by Γ^{ε}. The unit cell is presented in the Fig. 2 with Γ denoting the set of interfaces. The medium is assumed to be layered in the x_1 direction, with all material parameters independent of x_2 and x_3. We consider spring structural interface (Benveniste, 1984; Hashin, 1992; Bigoni and Movchan, 2002; Bigoni *et al.*, 2008).

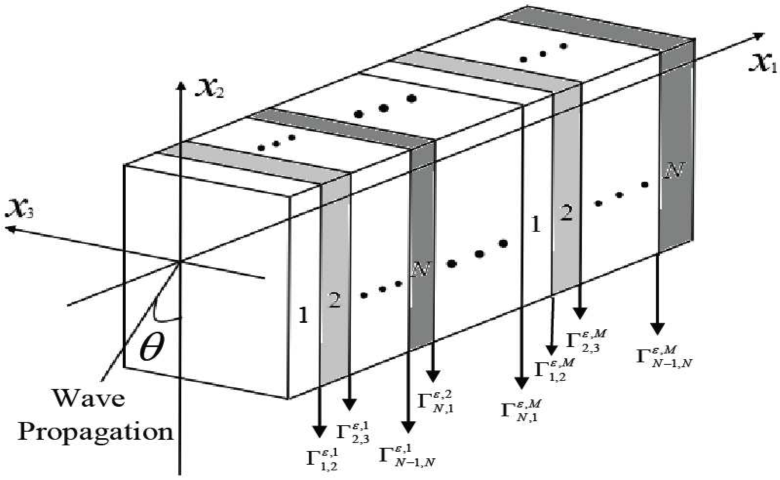

Figure 1. The multi-laminate composite

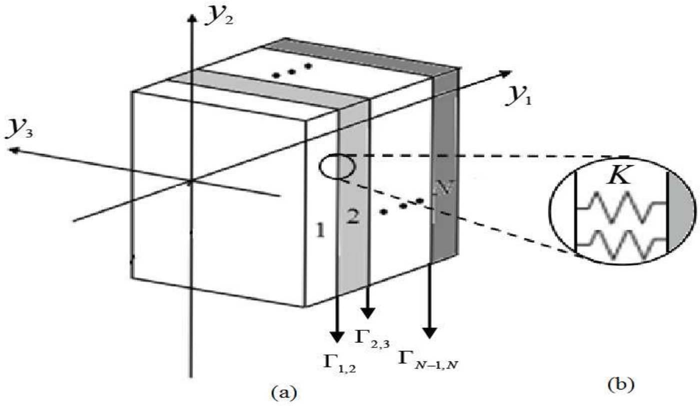

Figure 2. The unit cell and structural interface: (a) unit cell and (b) structural interface

The anti-plane problem is formulated in a bounded subset Ω^{ε} of $\mathbb{R}^2$, i.e. a boundary-value problem within a two-dimensional domain in the x_1x_2-plane. The reference cell with perfect contact interface is denoted by

$$\tilde{Y} = \left\{ \boldsymbol{y} = (y_1, y_2) \in \mathbb{R}^2 : 0 < y_i < l_i,\ i = 1,2 \right\}$$

where l_i are given positive numbers. Note that the subset Ω^{ε} is

$$\Omega^{\varepsilon} = \varepsilon\tilde{Y} = \left\{ \boldsymbol{x} = (x_1, x_2) \in \mathbb{R}^2 : \varepsilon^{-1} x_i \in \tilde{Y},\ i = 1,2 \right\}$$

where $\boldsymbol{y} = \boldsymbol{x}/\varepsilon$.

The anti-plane problem is modeled mathematically in the form

$$\frac{\partial \sigma^{\varepsilon}_{3i}}{\partial x_i} - \rho^{\varepsilon} \frac{\partial^2 u^{\varepsilon}_3}{\partial t^2} = 0 \text{ in } \Omega^{\varepsilon} \times]0,\tau[\tag{1}$$

$$u_3^\varepsilon(\boldsymbol{x},t)\Big|_{t=0} = U(\boldsymbol{x}), \qquad \frac{\partial u_3^\varepsilon(\boldsymbol{x},t)}{\partial t}\Bigg|_{t=0} = g(\boldsymbol{x}) \ \text{ in } \ \Omega^\varepsilon \tag{2}$$

$$u_3^\varepsilon(\boldsymbol{x},t)\Big|_{\partial_1\Omega^\varepsilon} = h(t)\ , \quad \sigma_{3i}^\varepsilon\, n_i\Big|_{\partial_2\Omega^\varepsilon} = q(t) \ \text{ for } t>0 \tag{3}$$

$$\sigma_{3j}^\varepsilon n_j = K\left[\!\left[u_3^\varepsilon\right]\!\right]\ , \ \left[\!\left[\sigma_{3j}^\varepsilon n_j\right]\!\right] = 0 \ \text{ in } \boldsymbol{S}^\varepsilon \tag{4}$$

where $\sigma_{3i}^\varepsilon = C_{3i3j}(x_1/\varepsilon)\partial u_3^\varepsilon/\partial x_j$ and $\partial\Omega^\varepsilon = \partial_1\Omega^\varepsilon \cup \partial_2\Omega^\varepsilon$ such that $\partial_1\Omega^\varepsilon \cap \partial_2\Omega^\varepsilon = \phi$; $[\![\bullet]\!]$ denotes the jump across the interface, i.e. $[\![\bullet]\!] = (\bullet)^{(1)}(\boldsymbol{y}) - (\bullet)^{(2)}(\boldsymbol{y})$ for $\boldsymbol{y}\in\boldsymbol{S}^\varepsilon$ (henceforth, the Latin indices take values 1 and 2); $0 < K(\text{MPa/cm}) < \infty$ denotes the interface stiffness (with $K\to\infty$ corresponding to a perfect interface); $C_{3i3j}(y_1) = C_{3i3j}^{(n)}$ for $y_1 \in T_n$; $n = 1,2,3,\ldots,N$ (T_n is the nth layer of the unit cell); $\boldsymbol{S}^\varepsilon$ is the projection of $\boldsymbol{\Gamma}^\varepsilon$ on x_1x_2-plane; and n_j is the unit vector in the outward normal direction.

2.1 Displacement formulation

Elimination of the stress in (1) leads to the partial differential equation for u_3^ε

$$\frac{\partial}{\partial x_i}\left(C_{3i3j}(y_1)\frac{\partial u_3^\varepsilon}{\partial x_j}\right) - \rho(y_1)\frac{\partial^2 u_3^\varepsilon}{\partial t^2} = 0 \tag{5}$$

where the functions u_3^ε, ρ and C_{3i3j} are l_1- periodic.

We shall assume a time-harmonic solution of (5) in the form

$$u_3^\varepsilon = \text{Re}\left[v(\boldsymbol{x},y_1)e^{\mathrm{i}\omega t}\right] = v(\boldsymbol{x},y_1)\cos\omega t \tag{6}$$

where ω is the circular frequency.

Substituting (6) into (5), we obtain:

$$\frac{\partial}{\partial x_i}\left(C_{3i3j}(y_1)\frac{\partial v(\boldsymbol{x},y_1)}{\partial x_j}\right) + \rho(y_1)\,\omega^2 v(\boldsymbol{x},y_1) = 0 \ \text{ in } \ \Omega^\varepsilon \tag{7}$$

under imperfect contact conditions,

$$C_{3i3j}\frac{\partial v}{\partial x_j}n_i = K[\![v]\!]\ , \ \left[\!\!\left[C_{3i3j}\frac{\partial v}{\partial x_j}n_i\right]\!\!\right] = 0 \ \text{ in } \boldsymbol{S}^\varepsilon \tag{8}$$

By introducing the notation $\sigma_{3i}(\boldsymbol{x},y_1) = C_{3i3j}(y_1)\partial v(\boldsymbol{x},y_1)/\partial x_j$, (7) can be written in the form

$$\frac{\partial\sigma_{3i}}{\partial x_i} + \rho(y_1)\,\omega^2 v(\boldsymbol{x},y_1) = 0 \tag{9}$$

Taking into account a regular asymptotic expansion of the circular frequency ω and following the works reported by Sanchez-Palencia, 1980; Bakhalov and Panasenko, 1989, we begin by writing the series representations

$$\omega = \omega_0 + \varepsilon\cdot\omega_1 + \varepsilon^2\cdot\omega_2 + \cdots = \sum_{\alpha\ge 0}\varepsilon^\alpha\cdot\omega_n \tag{10}$$

$$v(\boldsymbol{x},y_1) = v^{(0)}(\boldsymbol{x}) + \varepsilon\cdot v^{(1)}(\boldsymbol{x},y_1) + \varepsilon^2\cdot v^{(2)}(\boldsymbol{x},y_1) + \cdots = v^{(0)}(\boldsymbol{x}) + \sum_{\alpha\ge 1}\varepsilon^\alpha\cdot v^{(\alpha)}(\boldsymbol{x},y_1) \tag{11}$$

$$\sigma_{3i}(\boldsymbol{x},y_1) = \sigma_{3i}^{(0)}(\boldsymbol{x},y_1) + \varepsilon\cdot\sigma_{3i}^{(1)}(\boldsymbol{x},y_1) + \varepsilon^2\cdot\sigma_{3i}^{(2)}(\boldsymbol{x},y_1) + \cdots = \sum_{\alpha\ge 0}\varepsilon^\alpha\cdot\sigma_{3i}^{(\alpha)}(\boldsymbol{x},y_1) \tag{12}$$

where ω are constants; and $v^{(\alpha)}$ and $\sigma_{3i}^{(\alpha)}$ are periodic with respect to the fast variable y_1. In the case that only functions $v(\boldsymbol{x}, y_1)$ and $\sigma_{3i}(\boldsymbol{x}, y_1)$ are expanded asymptotically, if equating the terms of the order ε^{-2}, ε^{-1}, ε^{0} to zero, we obtain the results from the asymptotic averaging method called as *classical homogenization*.

2.2 Effective medium

In this section the average expression given in Eq. (5), up to $O(\varepsilon^2)$, which is known as the effective equation of motion for a homogeneous dispersive medium, will be presented.

Substituting the above asymptotic expansions (10)-(12) into (9), and rearranging the result by power of ε we can thus write

$$\sum_{\alpha\ge -2} \varepsilon^{\alpha} H_{\alpha}(\boldsymbol{x}, y_1) = 0 \tag{13}$$

The asymptotic sum in (13) vanishes, yielding

$$H_{\alpha}(\boldsymbol{x}, y_1) = 0 \tag{14}$$

which constitutes a recurrent system of partial differential system of equations with unknown functions $\sigma_{3i}^{(\alpha)}(\boldsymbol{x}, y_1)$ in which the solutions $\sigma_{3i}^{(\alpha)}$ and $\sigma_{3i}^{(\alpha+1)}$ for the n-th and (n+1)-th equations are inserted into the next (n+2)-th equation.

Considering

$$\hat{v} = \hat{v}^{(0)} + \varepsilon\, \hat{v}^{(1)} + \varepsilon^2\, \hat{v}^{(2)} \tag{15}$$

we can get $< v > l_1 = \int_0^{l_1} v\, dy_1 \approx \hat{v} l_1$, then it is possible to obtain the averaged model

$$C_{3i3j}^{*} \frac{\partial^2 \hat{v}}{\partial x_i\, \partial x_j} + \omega_0^2\, \hat{\rho}\, \hat{v} = 0 \tag{16}$$

with $C_{3i3j}^{*} = \left\langle C_{3i3j} + C_{3i31} \partial N_j / \partial y_1 \right\rangle$ and $\hat{\rho} = \gamma_k \rho_k$, where N_j are local functions of the first local problem (Brito-Santana *et al.*, 2015), γ_k and ρ_k are the volume fraction and mass density of layer *k*, respectively.

Recall that $u_3 = v \cos \omega t$, therefore $\hat{u}_3 =< u_3 >=< v > \cos \omega t \approx \hat{v} \cos \omega t$. Finally, the effective equation of motion for a homogeneous dispersive medium is obtained as

$$\hat{\rho} \frac{\partial^2 \hat{u}_3}{\partial t^2} = C_{3i3j}^{*} \frac{\partial^2 \hat{u}_3}{\partial x_i\, \partial x_j} + \varepsilon^2\, \hat{\rho}\, \eta\, C_{3i3j}^{*} \frac{\partial^4 \hat{u}_3}{\partial x_i\, \partial x_j\, \partial t^2} \tag{17}$$

where $\eta = \left\langle C_{3i31} \partial N_{jkl} / \partial y_1 + C_{3i3j} N_{kl} - \rho \hat{\rho}^{-1} C_{3i3j}^{*} N_{kl} \right\rangle C_{3l3k}^{*,-1} C_{3j3i}^{*,-1}$ is a microstructure function, N_{kl} and N_{jkl} are local functions of the second and third local problems (Brito-Santana *et al.*, 2015), and $C_{3i3j}^{*,-1}$ are the components of the inverse of the tensor $\boldsymbol{C}^{*}$.

2.3 Dispersion relation, group velocity and frequency band structures

Considering in the dynamic effective medium (17) a harmonic solution of the form

$$\hat{u}_3(\boldsymbol{x}, t) = U_3\, e^{\mathrm{i}\left(k_j x_j - \omega t\right)} \tag{18}$$

where U_3 is the displacement amplitude; $k_1 = \kappa \sin\theta$ and $k_2 = \kappa \cos\theta$ are the propagation vector components with κ being the wave number, we obtain the dimensionless phase velocity

$$\frac{V}{V_0} = \sqrt{\frac{1}{1 + \overline{\eta} \varepsilon^2 (\kappa l_1)^2 (C_{3131}^{*} \sin^2\theta + C_{3232}^{*} \cos^2\theta)}} \tag{19}$$

where κl_1 is the dimensionless wave number and $\overline{\eta} = \eta / l_1^2$. The effective phase velocity obtained from the classical homogenization is

$$V_0 = \sqrt{\frac{C^*_{3131}\sin^2\theta + C^*_{3232}\cos^2\theta}{\hat{\rho}}} \tag{20}$$

The velocity of energy flux (i.e. the group velocity) equals the derivative of frequency with respect to the wave number. Using the relation $\omega = \kappa V$, we can also write the group velocity

$$Vg = \frac{d\omega}{d\kappa} = V + \kappa\frac{dV}{d\kappa} \tag{21}$$

Frequency band structures are the frequencies as functions of the wave-vector components, k_1 and k_2. The pure shear (ω) frequency can be rewritten in the form

$$\omega^2 = \frac{C^*_{3131}k_1^2 + C^*_{3232}k_2^2}{\hat{\rho} + \varepsilon^2\eta\hat{\rho}\,(C^*_{3131}k_1^2 + C^*_{3232}k_2^2)} \tag{22}$$

3. NUMERICAL RESULTS

In order to illustrate the general behavior of different approximations for effective dynamic parameters, some numerical examples for a tri-laminated composite are presented. The material parameters used in the calculations are listed in Tab. 1. For all calculations the length (L_1) of the composite in the x_1 direction is 40 cm, and $\gamma_1 = \gamma_3 = (1-\gamma_2)/2$.

Table 1. Material parameters (Brun *et al.*, 2010)

Materials	μ(MPa)	ρ(kg/m^3)
Layer 1	10	1000
Layer 2 (mesophase)	50	3000
Layer 3	450	5000

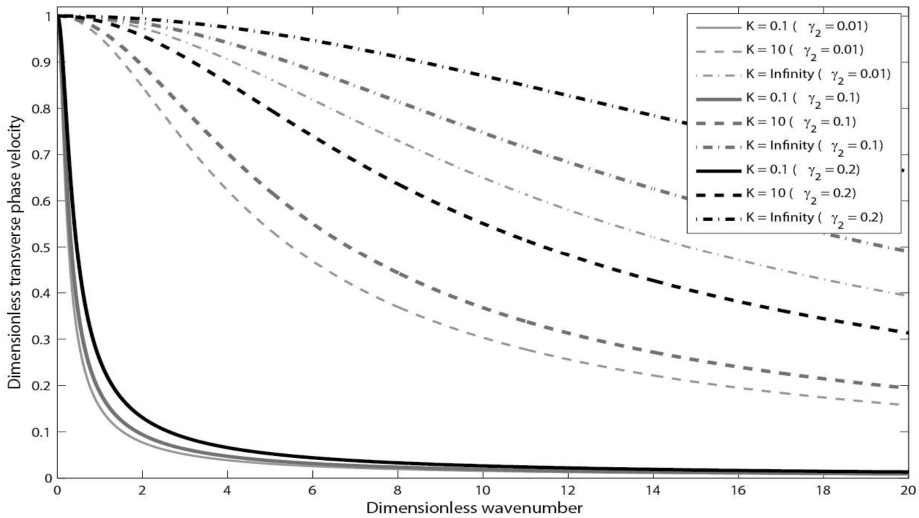

Figure 3. Dispersion relation V/V_0 vs κl_1 for different values of the interface stiffness ($K = 0.1, 10$ and ∞), different values of the mesophase ($\gamma_2 = 0.01, 0.1$ and 0.2) with $\varepsilon = 0.09$ and $\theta = 45^0$

Figure 3 illustrates the dimensionless transverse phase velocity predicted by Eq. (19) versus the wave number for different values of the interface stiffness (*K*), different values of the mesophase (γ_2) with size of the unit cell $\varepsilon = 0.09$ and the incident angle $\theta = 45^0$. This figure shows that the system is more dispersive when the interface stiffness and the mesophase decrease. Note that, the case $K \to \infty$ corresponds to the perfect bonding whereas the case $K \to 0$ to the complete separation of the layers, simulating a completely damaged material.

Figure 4 displays the distribution of the dimensionless transverse phase velocity V/V_0 vs θ (the incident angle). According to this figure, the dimensionless phase velocity increases with the incident angle increasing, and decreases with the interface stiffness decreasing.

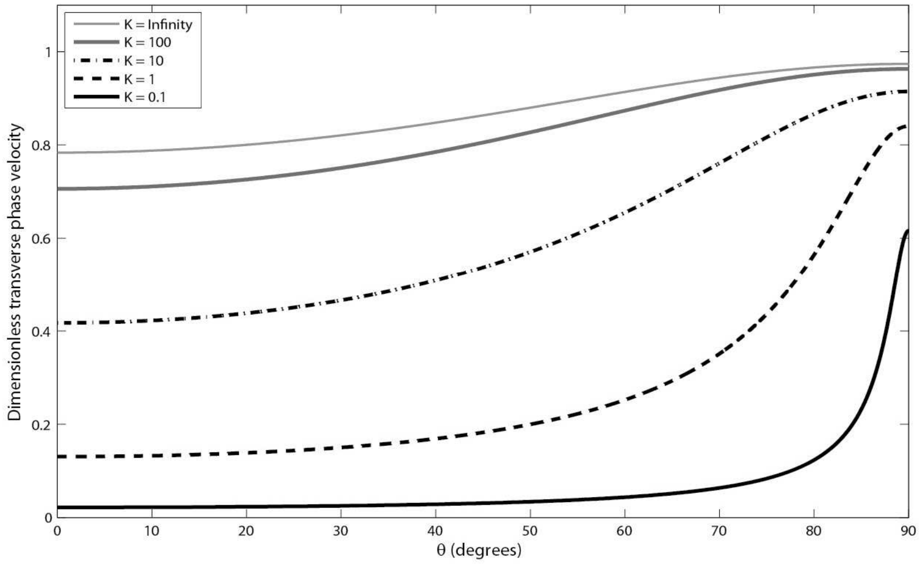

Figure 4. Angular dependence of the dimensionless transverse phase velocity V/V_0, for different values of the interface stiffness ($K = 0.1, 1, 10, 100$ and ∞), with $\gamma_2 = 0.01$, $\varepsilon = 0.09$ and $\kappa l_1 = 5$

Figure 5 show the effects of the interface stiffness and the size of the unit cell on group velocity Vg. In this figure, it can be observed that as the interface stiffness decreases with the fixed size unit cell, Vg decreases. Also, as shown in this figure, the pure shear group velocity decrease with the size of the unit cell increases.

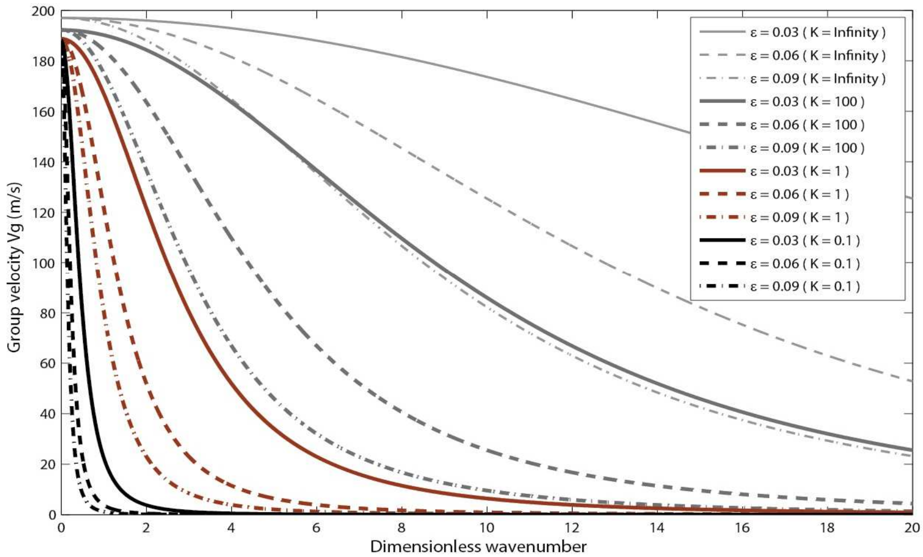

Figure 5. Group velocity Vg vs κl_1 for different values of the interface stiffness ($K = 0.1, 1, 100$ and ∞), different size of the unit cell ($\varepsilon = 0.03, 0.06$ and 0.09) with $\varepsilon = 0.09$ and $\theta = 45^0$

Figure 6 shows the constant-frequency contours together with the direction of energy flow as functions of k_1 and k_2 for the first frequency band of the pure shear wave propagation and different values of the interface stiffness. As it is seen, for suitably small values of k_2 these contours are ellipse. The phase velocity increases with the interface stiffness increasing.

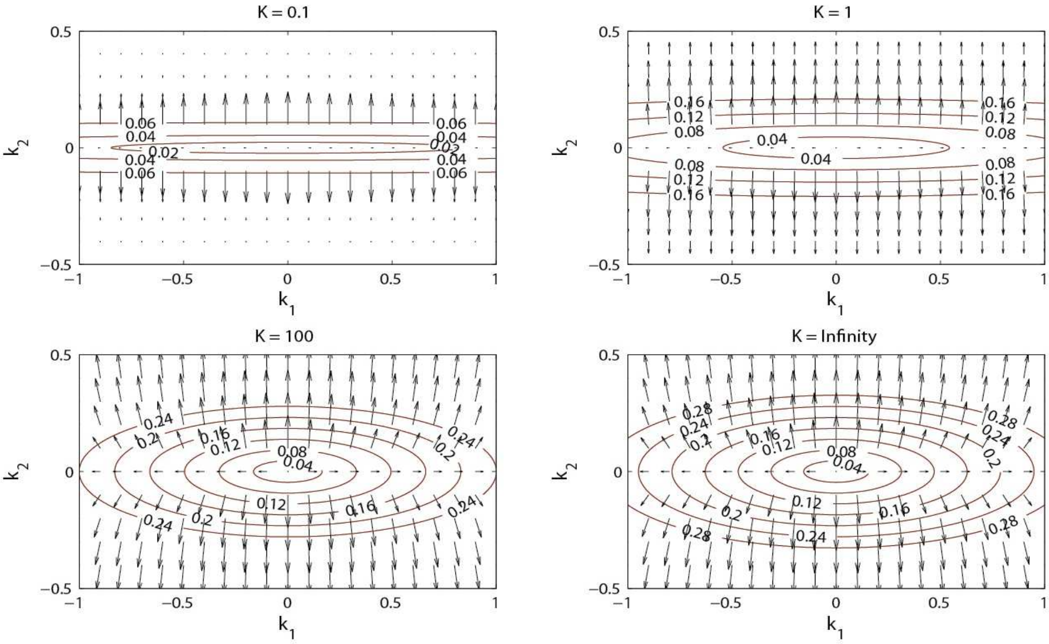

Figure 6. Contours of constant frequency (in MHz) and velocity of energy flux for different values of the interface stiffness ($K = 0.1, 1, 100$ and ∞), with $\varepsilon = 0.09$ and $\gamma_2 = 0.01$

4. CONCLUSION

The paper reports the study of wave propagation in multi-laminate composite materials with perfect and/or non-perfect contact between the layers. The results show that the imperfect contact and the mesophase introduce stronger dispersion in the systems. The case $K \to \infty$ corresponds to the perfect bonding. In addition, as the incident angle increases the dimensionless pure shear phase velocity increases, and decreases when the size of the unit cell increases. The current approach provides the possibility to calculate the frequency band structure as a function of the wave-vector components, and produce contours of constant frequency in the wave-vector components space. The current approach provides possible applications to damage detection in the composites.

5. ACKNOWLEDGEMENTS

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7. RESPONSIBILITY NOTICE

The authors Humberto Brito-Santana, Volnei Tita, Marcelo Leite Ribeiro, Reinaldo Rodríguez-Ramos, Julian Bravo-Castillero and Raúl Guinovart-Díaz are the only responsible for the printed material included in this paper.