23rd ABCM International Congress of Mechanical Engineering
December 6-11, 2015, Rio de Janeiro, RJ, Brazil

INFLUENCE OF MODAL COUPLING ON THE NONLINEAR VIBRATIONS OF CYLINDRICAL PANELS

Frederico M. A. Silva
Henrique A. R. Sattler
Zenón J. G. N. del Prado
Federal University of Goiás, av. Universitária no. 1488, 74605-220, Goiânia, Goiás.
silvafma@ufg.br, eng.henriquesattler@hotmail.com, zenon@ufg.br

Paulo B. Gonçalves
Pontifical Catholic University of Rio de Janeiro, av. Marquês de São Vicente no. 225, 22451-900, Rio de Janeiro, RJ.
paulo@puc-rio.br

Abstract. *This work analyzes the influence of nonlinear modal coupling on the nonlinear forced vibrations of a simply supported cylindrical panel excited by a time dependent pressure load. The cylindrical panel is modeled by the nonlinear Donnell shallow shell theory. The lateral displacement field is determined by a perturbation procedure and the axial and circumferential displacement are described in terms of the lateral displacement, thus generating a precise low-dimensional model. A general expression for the nonlinear vibration modes is provided in this paper, satisfying all boundary conditions. The discretized equations of motion are obtained by applying the Galerkin method. Various numerical techniques are employed to obtain the cylindrical panel resonance curves, bifurcation scenario and basins of attraction. The results show the influence of geometry and nonlinear modal coupling on the nonlinear response of the cylindrical panel.*

Keywords: *cylindrical panel, modal coupling, nonlinear dynamics*

1. INTRODUCTION

Cylindrical shells and panels are good examples of slender structures, being a three-dimensional solid whose thickness is small compared to the other dimensions. Cylindrical panels are also called open cylindrical shells and are commonly found in several engineering applications. Due to their slenderness and initial curvature, cylindrical panels display a highly nonlinear behavior, usually greater than that of the associated cylindrical shell of same geometry. They display under resonant conditions coexisting solutions, and can move from one steady state to another due to the action of a small perturbation. Also, they may exhibit excessive vibrations capable to leading the structure to collapse. These factors motivated researchers to develop accurate models for the analysis of their behavior to static and dynamic loads.

Amabili and Paidoüssis (2003) and Alijani and Amabili (2014) published an extensive review on the nonlinear free or forced vibrations of cylindrical shells and panels. In a large number of referred works, the importance of the correct modal expansion for the lateral displacement of the shell and panel is highlighted, since different choices may lead to radically different results. Recently, the authors proposed a new solution method for cylindrical shells well adapted to the derivation of efficient reduced order models (Gonçalves *et al.*, 2008; Silva *et al.*, 2011), based on the perturbation technique initially proposed by Gonçalves and Batista (1988). In the present work this methodology is extended to cylindrical panels to obtain a consistent modal expansion for transversal displacement field. Based on the obtained modal expansion, a low order model is proposed to study the nonlinear vibrations of a simply supported cylindrical panel subjected to a lateral harmonic pressure. The consistent modal solution for the displacements field satisfies all boundary conditions for a simply supported panel. The obtained results show the influence of nonlinear modal coupling and quadratic nonlinearities on the resonance curves and basins of attraction of the cylindrical panel.

2. MATHEMATICAL FORMULATION

2.1 Panel equations

Consider a cylindrical panel of radius R, sector angle Θ, chord b, thickness h and length L, made of a linear elastic material with Young's modulus E, Poisson coefficient υ and mass density ρ. The three displacement components u, v and w are related to the cylindrical coordinate system x, θ and z, as shown in Fig 1.

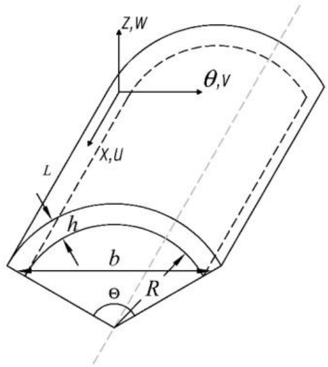


Figure 1. Cylindrical panel geometry and coordinate system

For an isotropic cylindrical panel, the stresses, at an arbitrary point along the shell thickness, are given by:

$$\begin{Bmatrix} \bar{\sigma}_x \\ \bar{\sigma}_\theta \\ \bar{\tau}_{x\theta} \end{Bmatrix} = \frac{E}{(1-\upsilon^2)} \begin{bmatrix} 1 & \upsilon & 0 \\ \upsilon & 1 & 0 \\ 0 & 0 & \frac{(1-\upsilon)}{2} \end{bmatrix} \begin{Bmatrix} \bar{\varepsilon}_x \\ \bar{\varepsilon}_\theta \\ \bar{\gamma}_{x\theta} \end{Bmatrix} \tag{1}$$

The deformations at an arbitrary point $(\bar{\varepsilon}_x, \bar{\varepsilon}_\theta \text{ and } \bar{\gamma}_{x\theta})$ are given in terms of the middle surface strains $(\varepsilon_x, \varepsilon_\theta \text{ and } \gamma_{x\theta})$ and changes of curvature $(\chi_x, \chi_\theta \text{ and } \chi_{x\theta})$ by:

$$\bar{\varepsilon}_x = \varepsilon_x + \chi_x \qquad \bar{\varepsilon}_\theta = \varepsilon_\theta + \chi_\theta \qquad \bar{\gamma}_{x\theta} = \gamma_{x\theta} + 2\chi_{x\theta} \tag{2}$$

These middle surface quantities are given in terms of the displacement components, according to Donnell shallow shell theory, by:

$$\begin{aligned} &\varepsilon_x = u_{,x} + \frac{1}{2} w_{,x}^2 \qquad && \varepsilon_\theta = \frac{1}{R}\left(v_{,\theta} + w\right) + \frac{1}{2R^2} w_{,\theta}^2 \qquad && \gamma_{x\theta} = v_{,x} + \frac{1}{R}\left(u_{,\theta} + w_{,x} w_{,\theta}\right) \\ &\chi_x = -w_{,xx} && \chi_\theta = -\frac{1}{R^2} w_{,\theta\theta} && \chi_{x\theta} = -\frac{1}{R} w_{,x\theta} \end{aligned} \tag{3}$$

The cylindrical panel is subjected to a harmonic lateral pressure of the form:

$$p(t) = P \sin\left(\frac{n\pi\theta}{\Theta}\right) \sin\left(\frac{m\pi x}{L}\right) \cos(\Omega t) \tag{4}$$

where P is the pressure magnitude in N/m²; n and m are, respectively, the number of half-waves in the circumferential and axial direction, Ω is the excitation frequency and t is time.

The nonlinear equations of motion, considering only the transversal inertia and damping forces, are given in terms of the force and moments resultants by:

$$\begin{aligned} &N_{x,x} + \frac{N_{x\theta,\theta}}{R} = 0 \qquad N_{x\theta,x} + \frac{N_{\theta,\theta}}{R} = 0 \\ &\rho h \ddot{w} + 2\eta_1 \rho h \Omega_0 \dot{w} + \eta_2 D \nabla^4 \dot{w} - p(t) - \frac{1}{R^2}\left[R M_{x,xx} + 2M_{x\theta,x\theta} + \frac{M_{\theta,\theta\theta}}{R} + R N_x w_{,xx} \right. \\ &\left. + N_\theta \left(\frac{w_{,\theta\theta}}{R} - 1\right) + 2 N_{x\theta} w_{,x\theta} \right] = 0 \end{aligned} \tag{5}$$

where η_1 and η_2 are, respectively, the linear viscous damping and the viscoelastic material damping coefficients, $D = E\,h^3/12(1-\upsilon^2)$ is the flexural stiffness of the shell, Ω_0 is the lowest natural vibration frequency, $p(t)$ is the lateral pressure and ∇^4 is the bi-harmonic operator defined as: $\nabla^4 = [\partial^2/\partial x^2 + \partial^2/R^2\partial\theta^2]^2$.

The force and moments resultants, in Eq. (5), are obtained by the integration of the stress components along the cylindrical panel thickness as follows:

$$N_x = \int_{-h/2}^{h/2} \bar{\sigma}_x\,dz \qquad N_\theta = \int_{-h/2}^{h/2} \bar{\sigma}_\theta\,dz \qquad N_{x\theta} = \int_{-h/2}^{h/2} \bar{\tau}_{x\theta}\,dz$$
$$M_x = \int_{-h/2}^{h/2} z\,\bar{\sigma}_x\,dz \qquad M_\theta = \int_{-h/2}^{h/2} z\,\bar{\sigma}_\theta\,dz \qquad M_{x\theta} = \int_{-h/2}^{h/2} z\,\bar{\tau}_{x\theta}\,dz \tag{6}$$

For a simply-supported cylindrical panel, the following boundary conditions must be satisfied in the transversal direction:

$$w(0,\theta) = w(L,\theta) = 0 \tag{7}$$
$$w(x,0) = w(x,\Theta) = 0 \tag{8}$$
$$M_x(0,\theta) = M_x(L,\theta) = 0 \tag{9}$$
$$M_\theta(x,0) = M_\theta(x,\Theta) = 0 \tag{10}$$

The displacement field, in this work, is also required to satisfy the following condition:

$$u(L/2,\theta) = 0 \tag{11}$$

The following non-dimensional parameters are now introduced:

$$W = \frac{w}{h} \qquad \xi = \frac{x}{L} \qquad \tau = \Omega_0 t \qquad \bar{\Omega} = \frac{\Omega}{\Omega_0} \tag{12}$$

2.2 2.2 General solution of the shell displacement field by a perturbation procedure and determination of the in-plane displacements *u* and *v*

The numerical model is developed by expanding the transversal displacement component *w* in series in the circumferential and axial variables. From previous investigations on modal solutions for the nonlinear analysis of cylindrical shells (Hunt, et al. 1986; Gonçalves and Batista, 1988; Gonçalves and Del Prado, 2002; Awrejcewicz and Krys'ko, 2003; Gonçalves, et al. 2008; Silva, at al., 2011) it is observed that, in order to obtain a consistent modeling with a limited number of modes, the sum of shape functions for the displacements must express the inherent nonlinear modal couplings. Gonçalves, et al. (2008) and Silva, at al. (2011) have developed a methodology to derive reduced order model for shell vibration analysis trough perturbation techniques, using as variables only the modal amplitudes in the transversal direction. Using these works as a basis, a modal solution for cylindrical panels satisfying the boundary conditions (7)-(10) is here obtained.

The analysis stars by considering as the seed mode or basic initial solution the fundamental vibration mode, that is:

$$W = W_{1,1}(t)\sin(m\pi\xi)\sin\left(\frac{n\pi\theta}{\Theta}\right) \tag{13}$$

By applying a perturbation procedure [13-14] and by imposing the boundary conditions for a simply supported cylindrical panel, Eqs. (7)-(10), the following modal solution that accounts for the essential nonlinear modal coupling is obtained:

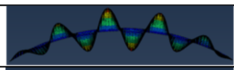
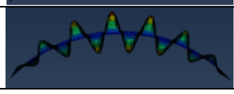
$$\begin{aligned} W = & \sum_{i=1,3,5}^{\infty} \sum_{j=1,3,5}^{\infty} W_{i,j}(\tau) \sin\left(i m \pi \varepsilon\right) \sin\left(\frac{j n \pi \theta}{\Theta}\right) \\ & + \sum_{\alpha=0}^{\infty} \sum_{\beta=0}^{\infty} W_{(2+6\beta),(2+6\alpha)}(\tau) \left\{ -\frac{(3+6\beta)}{(4+12\beta)} \cos(6\beta m \pi \varepsilon) + \cos((2+6\beta) m \pi \varepsilon) \right. \\ & \left. -\frac{(1+6\beta)}{(4+12\beta)} \cos((4+6\beta) m \pi \varepsilon) \right\} \left\{ -\frac{(3+6\alpha)}{(4+12\alpha)} \cos\left(6\alpha \frac{n \pi \theta}{\Theta}\right) + \cos\left((2+6\alpha)\frac{n \pi \theta}{\Theta}\right) \right. \\ & \left. -\frac{(1+6\alpha)}{(4+12\alpha)} \cos\left((4+6\alpha)\frac{n \pi \theta}{\Theta}\right) \right\} \end{aligned} \tag{14}$$

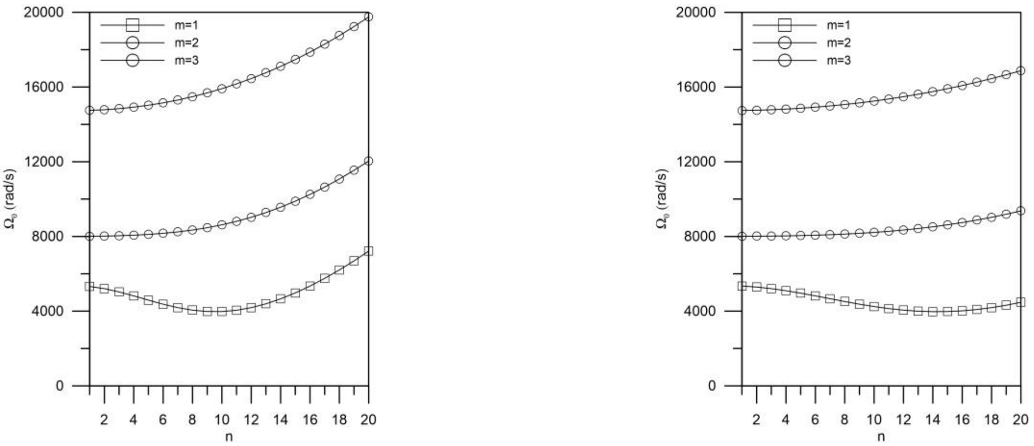
The in-plane displacements u and v are obtained by substituting a selected numbers of relevant modes from Eq. (14) into the in-plane equilibrium equations, and solving the system of linear partial differential equations in u and v and imposing the relevant in-plane boundary and symmetry conditions. Based on this procedure one obtains the necessary number of in-plane modes and write their modal amplitudes in terms of the modal amplitudes $W_{i,j}$ in Eq. (14) [13-14]. Finally, by substituting the adopted expansion for the transversal displacement w together with the obtained expressions for u and v into the equation of motion in the transversal direction, third equation in Eq. (5), and by applying the standard Galerkin method, a consistent discretized system of ordinary differential equations of motion is derived.

3. NUMERICAL RESULTS

Consider a perfect cylindrical panel with the following material properties: E = 2.06 x 10^{11} N/m², υ = 0.3 and ρ = 7850 kg/m³. A cylindrical panel with radius R = 1 m, length L = 0.1 m, thickness h = 0.001 m and two different sector angles Θ are considered. Table 1 shows the lowest natural frequency, associated wave-numbers m and n and the profile of the vibration mode in the circumferential direction. These values will be used throughout the present numerical analysis. As the sector angle Θ increase the circumferential half-wave number n increases and the fundamental frequency decreases. The dense frequency spectrum typical of cylindrical panels is illustrated in Fig. 2.

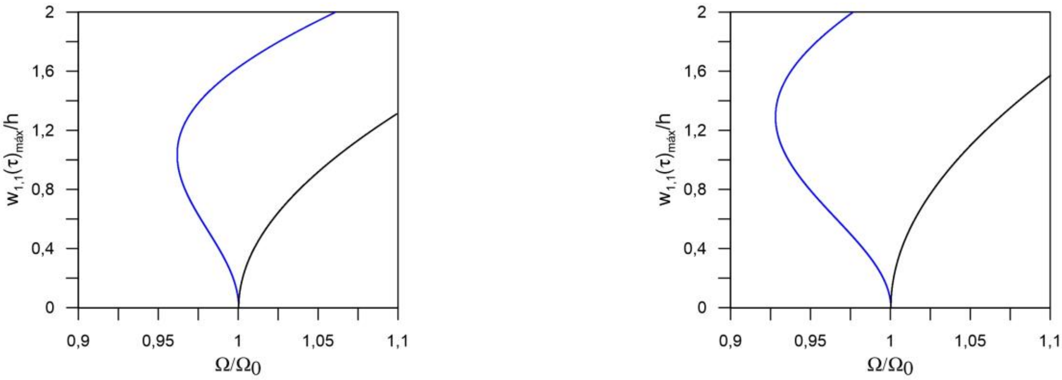
Table 1: Natural frequencies and vibration mode for different cylindrical panel's sector angles.

Sector angle – Θ (rad)	Ω_0 (rad/s)	m	n	Vibration mode
$\pi/3$	3979,52	1	10	
$\pi/2$	3973,39	1	14	



(a) $\Theta=\pi/3$ (b) $\Theta=\pi/2$

Figure 2. Natural frequencies for the cylindrical panel with different sector angles.



(a) $\Theta=\pi/3$ (b) $\Theta=\pi/2$

Figure 3. Frequency-amplitude relation for the cylindrical panel with different sector angles. ($\eta_1 = 0$ and $\eta_2 = 0$). — Model: 1. — Model: 2.

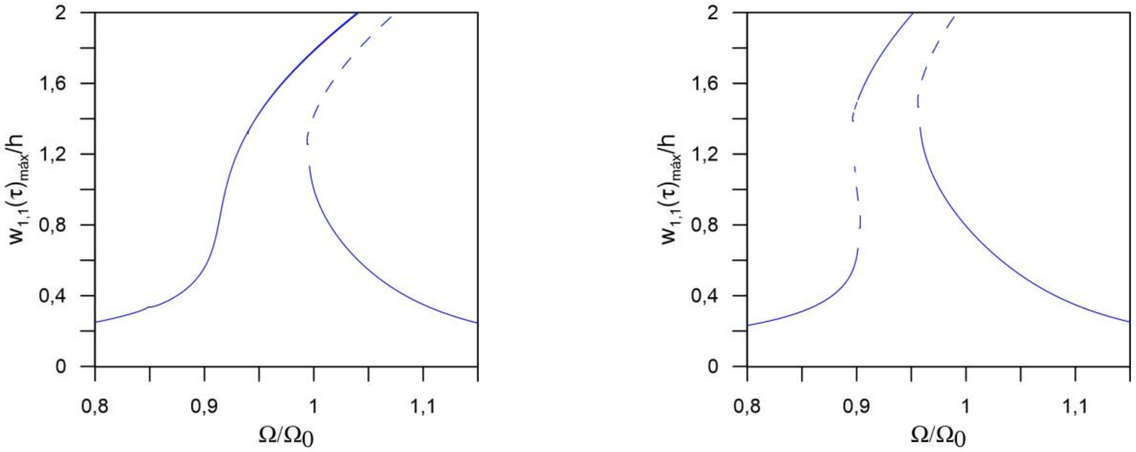
The frequency-amplitude relations of the cylindrical panels are obtained, considering two reduced order models obtained from Eq. (14). They are:

Model 1: $w_{1,1}(\tau) + w_{1,3}(\tau)$.

Model 2: $w_{1,1}(\tau) + \mathbf{w_{2,2}(\tau)}$.

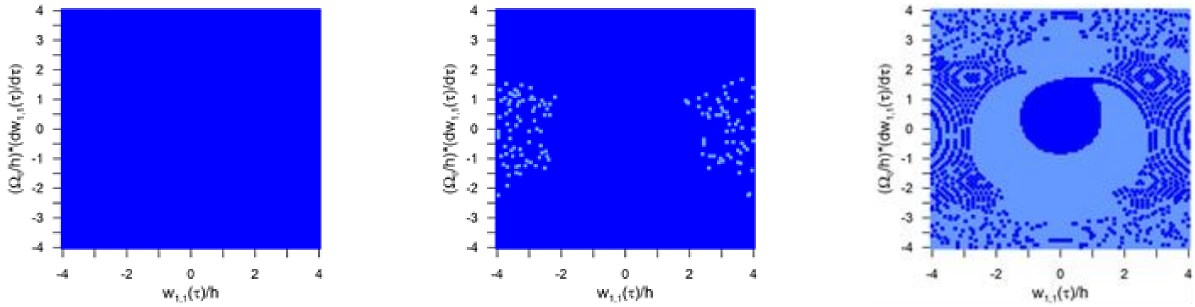
As shown in Fig. 3, where the frequency-amplitude relation is plotted up to vibration amplitudes of the order of two times the shell thickness, the two models lead to two radically different results. While Model 1 leads to a hardening behavior, Model 2, which contains the vibration mode $w_{2,2}(\tau)$, leads to an initially softening behavior. At vibration amplitudes of the order of the shell thickness the curve veers back, displaying at large vibration amplitudes a hardening behavior. This is the expected results for these panel geometries. Mode $w_{2,2}(\tau)$ is associated with quadratic nonlinearities and leads to an asymmetry in the transversal displacement field (Gonçalves and Del Prado, 2002).

Continuation techniques are used to obtain the resonance curves for the harmonically excited panel in the fundamental resonance region. These are shown in Fig. 4, where solid lines represent stable solutions and dashed lines, unstable solutions. The resonance curves, obtained using Model 2, show typical saddle-node bifurcations and coexisting attractors in a large frequency range. As shown in Fig. 4, increasing the sector angle from $\Theta=\pi/3$ to $\Theta=\pi/2$, new saddle-node bifurcations appear along the resonance curves, creating a new unstable path and additional dynamic jumps.



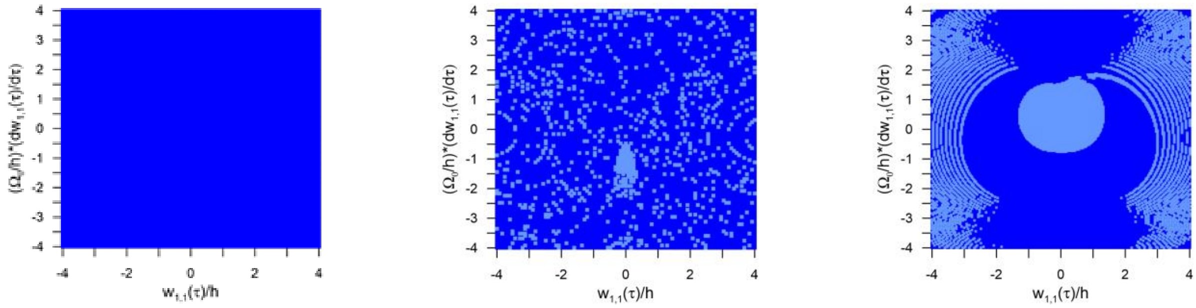
(a) $\Theta=\pi/3$ (b) $\Theta=\pi/2$

Figure 4. Resonance curves, maximum lateral deflection versus excitation frequency. Continuation techniques. Solid line: stable solution. Dashed line: unstable solution. Model: 2. $\eta_1 = 1 \times 10^{-3}$, $\eta_1 = 0$ and P = 10 kN/m².



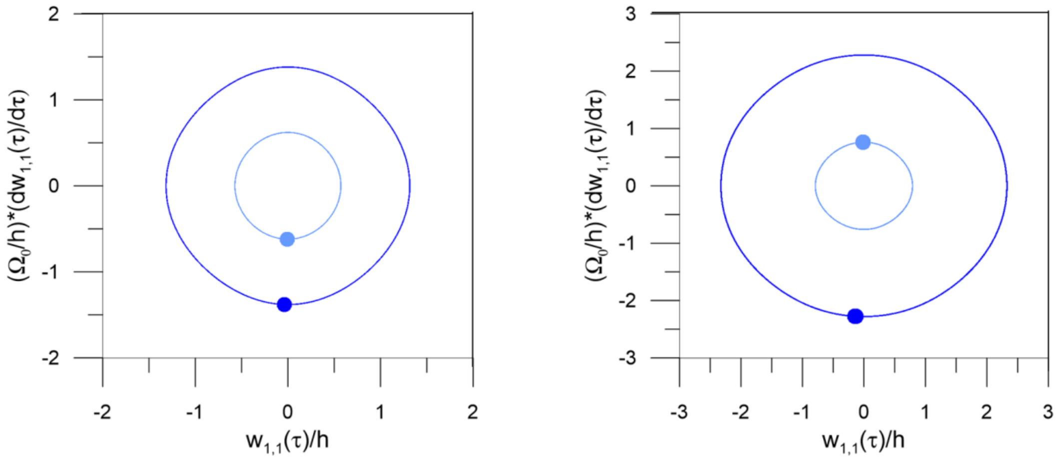
(a) $\Omega = 0.95$ (b) $\Omega = 1.00$ (c) $\Omega = 1.05$

Figure 5. Basin of attraction for a cylindrical panel with sector angle $\Theta=\pi/3$. $\eta_1 = 1 \times 10^{-3}$, $\eta_1 = 0$ and P = 10 kN/m².



(a) ($\Omega = 0.88$ ond $\Omega = 0.92$) (b) $\Omega = 0.90$ (c) $\Omega = 1.00$

Figure 6. Basin of attraction for a cylindrical panel with sector angle $\Theta=\pi/2$. $\eta_1 = 1 \times 10^{-3}$, $\eta_1 = 0$ and P = 10 kN/m².



$\Omega = 0.90$ $\Omega = 1.00$

Figure 7 – Coexisting phase portraits for cylindrical panel with sector angle $\pi/2$. Dark and light blue at basin attraction are coincident. (P = 10 kN).

Figures 5 and 6 show, respectively, selected cross-sections of basins of attraction by the plane $w_{1,1}(\tau)$ x $dw_{1,1}(\tau)/d\tau$ for the cylindrical panels with sector angles $\pi/3$ and $\pi/2$, respectively, considering different values of the forcing frequency in the main resonance region. In Figs. 5(a) and 6(a) only attractor is observed. In the other basin cross-sections two attractors are identified. In these figures the dark blue region corresponds to the large amplitude resonant response, while the light blue region corresponds to initial conditions that converge to the small amplitude non-resonant solution. These coexisting attractors are illustrated in Fig. 7 where projections of the phase portraits onto the plane $w_{1,1}(\tau)$ x $dw_{1,1}(\tau)/d\tau$ for cylindrical panel with sector angle $\pi/2$ are depicted for two values of the forcing frequency. The basins for coexisting attractors show a complex fractal structure that lead to sensitivity to initial conditions. So, future work will study the dynamic integrity of the panel in the main resonance region using local and global integrity factors (Gonçalves et al., 2011)

4. CONCLUDING REMARKS

In the present paper, Donnell shallow shell theory has been applied to model the dynamics of a cylindrical panel under harmonic lateral pressure. Based on a modal solution obtained from perturbation techniques, a general solution for the displacement field is obtained satisfying all boundary and symmetry conditions of a simply-supported cylindrical panel along the four edges. The discretized panel equations are obtained by the standard Galerkin method. The nonlinear resonance curves are obtained considering different models to describe the transversal displacement field and different sector angles. The results corroborate the influence of modal coupling between linear and nonlinear modes on the nonlinear response. Based on this analysis, the nonlinear forced vibrations of the cylindrical panel are investigated. The results show that the modes with cosine functions, derived from perturbation techniques, change the nonlinear behavior of the cylindrical panel from hardening to softening modifying the bifurcation scenario and basins of attraction. This is a work in progress; further work will focus on conducting a detailed bifurcation analysis for different cylindrical panels in order to understand the influence of the several control parameters on its nonlinear vibrations and instabilities.

5. ACKNOWLEDGEMENTS

This work was made possible by the support of the Brazilian Ministry of Education – CAPES, CNPq, FAPERJ-CNE and FAPEG.

6. REFERENCES

Alijani, F. and Amabili, M., 2014. “Non-linear vibrations of shells: a literature review from 2003 to 2013”. *International Journal of Non-Linear Mechanics*, Vol. 58; p. 233-257.

Amabili, M. and Paidoussis, M. P., 2003. “Review of studies on geometrically nonlinear vibrations and dynamics of circular cylindrical shell and panels, with and without fluid-structure interaction”. *Applied Mechanics Reviews*, Vol. 56, p. 349-381.

Awrejcewicz J. and Krys’ko A. V., 2003. “Analysis of complex parametric vibrations of plates and shells using Bubnov-Galerkin approach”. *Archive of Applied Mechanics*, Vol. 73, 495-504.

Gonçalves, P. B. and Batista, R. C., 1988. “Nonlinear vibration analysis of fluid-filled cylindrical shells”. *Journal of Sound and Vibration*, Vol. 127, 133-143.

Gonçalves, P. B. and Prado, Z. J. G. N., 2002. “Nonlinear oscillations and stability of parametrically excited cylindrical shells”. *Meccanica*, Vol. 37, 569-597.

Gonçalves, P. B., Silva, F. M. A. and Prado, Z. J. G. N., 2008. “Low-dimensional models for the nonlinear vibration analysis of cylindrical shells based on a perturbation procedure and proper orthogonal decomposition”. *Journal of Sound and Vibration*, Vol. 315, 641-663.

Hunt, G. W., Williams, K. A. J. and Cowell, R. G., 1986. “Hidden symmetry concepts in the elastic buckling of axially loaded cylinders”. *International Journal of Solid and Structures*, Vol. 22, 1501-1515.

Silva, F. M. A., Gonçalves, P. B. and Prado, Z. J. G. N., 2011. “An alternative procedure for the non-linear vibration analysis of fluid-filled cylindrical shells”. *Nonlinear Dynamics*, Vol. 66, 303-333.

Gonçalves, P.B., Silva, F.M.A., Rega G. and Lenci, S, 2011. “Global dynamics and integrity of a two-dof model of a parametrically excited cylindrical shell”. *Nonlinear Dynamics*, Vol. 63, 61-82.

7. RESPONSIBILITY NOTICE

The authors are the only responsible for the printed material included in this paper.