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VIBRATION DISPLACEMENT TRANSMISSIBILITY AND SHOCK DISPLACEMENT RATIO IN SYSTEMS WITH INERTER

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Abstract. *Recently great attention has been given to a mechanical device known as inerter. An inerter is a mechanical system that converts linear into rotational motion. It is the two terminal mechanical analogue of a capacitor when comparing a mechanical single degree of freedom stiffness-mass-damping to an electrical circuit consisting of inductor-resistor-capacitor. The use of inerter has improved handling and grip in Formula 1 racing cars in 2008 and then, and there is an explanation for this in this paper. Displacement transmissibility (DT) analytical equation is obtained for harmonic excitation, and shock displacement ratio (SDR) is obtained by numerical simulation for shock excitation, for systems in which the inerter is included. DT and SDR behavior for each system, under both excitation cases, for underdamped, critically damped and super damped systems, and for different mass ratios are shown, and it can be seen the DT and SDR of each system can be improved in some frequency ranges, and also can be worsen in another frequency ranges. The results of each system are compared with each other to show their differences and to make it possible to discuss their engineering application.*

Keywords: *Inerter, J-Damper, Transmissibility, Shock Excitation, Harmonic Excitation*

1. INTRODUCTION

In 2002, Malcom C. Smith mentioned by the first time the concept of the mechanical inerter (Smith (2002)), which was described as *a mechanical two-node (two-terminal), one-port device with the property that the equal and opposite force applied at the nodes is proportional to the relative acceleration between the nodes*. Even more, acoording to Smith (2002), the capacitor is the mechanical equivalent component of an inerter with both nodes ungrounded. By another way, for an mechanical system without inerter to have an ordinary differential equation to be equivalent of an electrical circuit, the inerter in the mechanical system needs to have one of its terminals grounded, which would be a mass. Recently, after the disclosure of a device used in the suspension of Formula 1 racing cars, called J-Damper, the scientific community had shown interest in the inerter, since it claims to be the J-Damper, and it can improve performance on handling and grip (Chen *et al.* (2009)). Some mechanical components with similar (or equal) behaviour to the inerter have been considered previously. For instance, the inerter was described by Rivin *et al.* (2003) as a vibration isolation system with motion transformation. Rivin *et al.* (2003) also defined inerter's displacement transmissibility transfer function in terms of the inertance (inerter coefficient) and other system parameters. In the literature, there is a device called Lever-Type anti-resonant vibration isolator, described by Yilmaz and Kikuchi (2006), which have a very similar behaviour to the inerter. Apparently, this device was first called DAVI (Dynamic Vibration Anti-resonant Isolator) and had been proposed in 1966 by Flannelly (1966), leading to the first patent of this application in 1969. Other authors have used this device in different applications, for example, Dylejko and MacGillivray (2014) have used it to suppress internal resonance in mechanical systems. There are already many papers and references about this device and in different applications. In vehicles, there is a study with suspensions with inerters in which the author considered multiple performance requirements including, tyre grip, suspension deflection and ride comfort (Hu *et al.* (2014)). Another studies with inerters applied in vehicles were made by Smith and Wang (2004); Wang and Su (2008); Wang and Chan (2010); and Zhang and Hu (2014) . Applications in trains (Jiang *et al.*, 2012) and in civil engineering (Wang *et al.*, 2007) were also studied. In 2011, a hydraulic type inerter was developed by Wang *et al.* (2011), which brings a new possibility to fabricate this device, new possible ways to assemble this device in a suspension system and new ways to control, i.e., adjust it.

2. MATHEMATICAL MODELLING

In this section, the equations for DT for each studied system will be presented as well as the ordinary differential equations for shock excitation which will lead to the SDR results for each system, after numerical simulation.

2.1 System 1: A mass-spring-damper system with a coupled linear inerter

The first system that was investigated is shown in Fig. 1. Its components are: a body with mass m, a spring with linear stiffness k, a damper with linear damping coefficient c and an inerter with linear inertance b. The body's degree of freedom is x and the base movement is ruled by x_0. The degrees of freedom are all time dependant, but the time dependence was suppressed for convenience. The dots indicate the degrees of freedom derivatives in time.

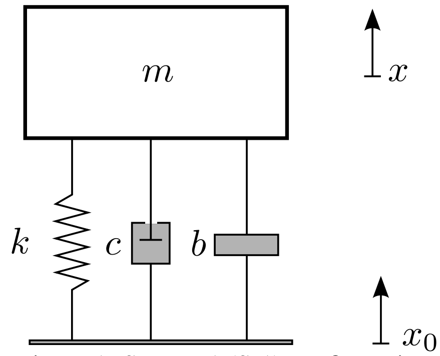


Figure 1: System 1 (S-1) configuration

After applying the second Newton's law, the ODE for this system can be rearranged as Eq. (1).

$$(m+b)\ddot{x} + c\dot{x} + kx = b\ddot{x}_0 + c\dot{x}_0 + kx_0 \tag{1}$$

2.1.1 Harmonic Excitation

For harmonic excitation, Laplace transform was used in Eq. (1) with the objective to pass it from the time domain to the frequency domain. The final displacement transmissibility equation, Eq. (2), was obtained in terms of ξ, which is the damping factor, of μ, which is the mass ratio defined as $\mu = \frac{b}{m}$, and of Ω, which is the frequency ratio between the excitation frequency (ω) and the natural frequency (ω_n) of the system.

$$T_d = \left|\frac{X}{X_0}\right| = \left|\frac{\Omega^2\mu + 2j\Omega\xi + 1}{-\Omega^2(1+\mu) + 2j\Omega\xi + 1}\right| = \sqrt{\frac{(1-\mu\Omega^2)^2 + (2\xi\Omega)^2}{(1-(1+\mu)\Omega^2)^2 + (2\xi\Omega)^2}} \tag{2}$$

This result fits with the one that was first obtained by Rivin *et al.* (2003).

2.1.2 Shock Excitation

For shock excitation, the modelled shock input is the half sine pulse, which starts at time $t = 0$. The new ODE defined by Eq. (3) was obtained in terms of ξ , μ , ω_n, time (t), pulse period (τ) and excitation amplitude (A).

$$(1+\mu)\ddot{x} + 2\omega_n\xi\dot{x} + \omega_n^2 x = -\mu A\frac{\pi^2}{\tau^2}sin\left(\pi\frac{t}{\tau}\right) + 2\omega_n\xi A\frac{\pi}{\tau}cos\left(\pi\frac{t}{\tau}\right) + \omega_n^2 Asin\left(\pi\frac{t}{\tau}\right) \tag{3}$$

Equation (3) was simulated to obtain the SDR behaviour for this system, where the severity factor was defined as $\beta = \frac{\pi}{\omega_n\tau}$. Increasing the severity factor value, for a constant value of natural frequency (which was chosen 1 rad/sec), implies in decreasing the period ratio value, which means the "excitation frequency" is increasing.

2.2 System 2: A mass-spring-damper system with an elastically coupled linear inerter

The second system that was investigated is shown in Fig. 2. The difference between this system and S-1 is that it was added a auxiliary spring with stiffness $N_k k$, where N_k is a real number, in series with the inerter. This new component generated a new degree of freedom x_1, which leads to a second ODE.

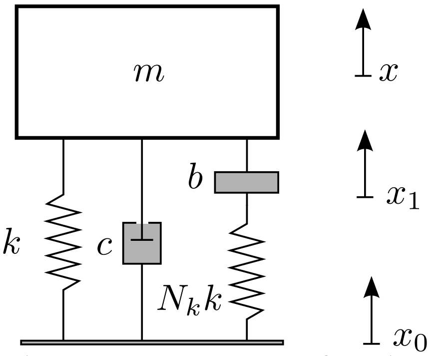


Figure 2: System 2 (S-2) configuration

After applying the second Newton's law, the ODEs for this system can be rearranged as Eq. (4) and Eq. (5).

$$(m+b)\ddot{x} + c\dot{x} + kx = b\ddot{x}_1 + c\dot{x}_0 + kx_0 \tag{4}$$

$$b(\ddot{x} - \ddot{x}_1) = N_k k(x_1 - x_0) \tag{5}$$

Eliminating the degree of freedom x_1 by writing it in terms of x and x_0, the combination of Eq. (4) and Eq. (5), results in Eq. (6).

$$\frac{mb}{N_k k}\ddddot{x} + \frac{cb}{N_k k}\dddot{x} + \left(m + b + \frac{b}{N_k}\right)\ddot{x} + c\dot{x} + kx = \frac{cb}{N_k k}\dddot{x}_0 + \left(b + \frac{b}{N_k}\right)\ddot{x}_0 + c\dot{x}_0 + kx_0 \tag{6}$$

2.2.1 Harmonic Excitation

Considering harmonic excitation, the system in Eq. (6) was written in frequency domain. The final displacement transmissibility equation, Eq. (7), was obtained in terms of ξ, μ, N_k and Ω.

$$T_d = \sqrt{\frac{\left(1 - \Omega^2\mu\left(1 + \frac{1}{N_k}\right)\right)^2 + \left(-\Omega^3\mu\frac{2\xi}{N_k} + \Omega 2\xi\right)^2}{\left(1 + \Omega^4\mu\frac{1}{N_k} - \Omega^2 - \Omega^2\mu\left(1 + \frac{1}{N_k}\right)\right)^2 + \left(-\Omega^3\mu\frac{2\xi}{N_k} + \Omega 2\xi\right)^2}} \tag{7}$$

2.2.2 Shock Excitation

For shock excitation, the modelled shock input is the half sine pulse, which starts at time $t = 0$. The new ODE defined by Eq. (8) was obtained in terms of ξ , μ , ω_n, t, τ and A.

$$\begin{aligned}\frac{\mu}{N_k\omega_n^2}\ddddot{x} + \frac{2\xi\mu}{N_k\omega_n}\dddot{x} + \left(1 + \mu + \frac{\mu}{N_k}\right)\ddot{x} + 2\omega_n\xi\dot{x} + \omega_n^2 x = -\frac{2\xi\mu}{N_k\omega_n}A\frac{\pi^3}{\tau^3}cos\left(\pi\frac{t}{\tau}\right) \\ - \left(\mu + \frac{\mu}{N_k}\right)A\frac{\pi^2}{\tau^2}sin\left(\pi\frac{t}{\tau}\right) + 2\omega_n\xi A\frac{\pi}{\tau}cos\left(\pi\frac{t}{\tau}\right) + \omega_n^2 A sin\left(\pi\frac{t}{\tau}\right)\end{aligned} \tag{8}$$

Equation (8) was simulated to obtain the SDR behaviour for this system.

2.3 System 3: A mass-spring-damper system with a viscously coupled linear inerter

The third system that was investigated is shown in Fig. 3. The difference between this system and S-1 is that it was added an auxiliary damper with damping coefficient $N_c c$, where N_c is a positive real number, in series with the inerter. The addition of this component generated a new degree of freedom x_1, which leads to a second ODE.

After applying the second Newton's law, the ODEs for this system can be rearranged as Eq. (9) and Eq. (10).

$$(m+b)\ddot{x} + c\dot{x} + kx = b\ddot{x}_1 + c\dot{x}_0 + kx_0 \tag{9}$$

$$b(\ddot{x} - \ddot{x}_1) = N_c c(\dot{x}_1 - \dot{x}_0) \tag{10}$$

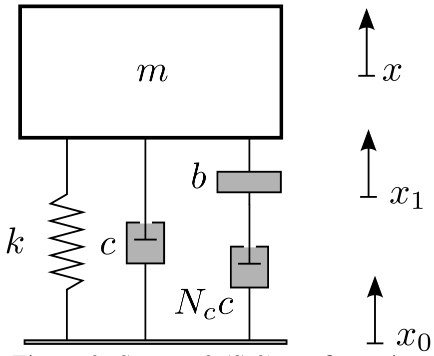


Figure 3: System 3 (S-3) configuration

Eliminating the degree of freedom x_1 by writing it in terms of x and x_0, the combination of Eq. (9) and Eq. (10), results in Eq. (11).

$$\frac{mb}{N_c c}\dddot{x} + \left(m + b + \frac{b}{N_c}\right)\ddot{x} + \left(c + \frac{bk}{N_c c}\right)\dot{x} + kx = \left(b + \frac{b}{N_c}\right)\ddot{x_0} + \left(c + \frac{bk}{N_c c}\right)\dot{x_0} + kx_0 \tag{11}$$

2.3.1 Harmonic Excitation

Considering harmonic excitation, the system in Eq. (11) was written in frequency domain. The final displacement transmissibility equation, Eq. (12), was obtained in terms of ξ, μ, N_c, and of Ω.

$$T_d = \sqrt{\frac{\left(1 - \Omega^2\mu\left(1 + \frac{1}{N_c}\right)\right)^2 + \left(\Omega\left(2\xi + \frac{\mu}{2N_c\xi}\right)\right)^2}{\left(1 - \Omega^2\left(1 + \mu + \frac{\mu}{N_c}\right)\right)^2 + \left(\Omega\left(2\xi + \frac{\mu}{2N_c\xi}\right) - \Omega^3\mu\frac{1}{2N_c\xi}\right)^2}} \tag{12}$$

2.3.2 Shock Excitation

For shock excitation, the modelled shock input is the half sine pulse, which starts at time $t = 0$. The new ODE defined by Eq. (13) was obtained in terms of ξ , μ , ω_n, t, τ and A.

$$\begin{aligned}\frac{\mu}{2N_c\xi\omega_n}\dddot{x} + \left(1 + \mu + \frac{\mu}{N_c}\right)\ddot{x} + \left(2\xi\omega_n + \frac{\mu\omega_n}{2N_c\xi}\right)\dot{x} + \omega_n^2 x = -\left(\mu + \frac{\mu}{N_c}\right)A\frac{\pi^2}{\tau^2}sin\left(\pi\frac{t}{\tau}\right) \\ + \left(2\xi\omega_n + \frac{\mu\omega_n}{2N_c\xi}\right)A\frac{\pi}{\tau}cos\left(\pi\frac{t}{\tau}\right) + \omega_n^2 Asin\left(\pi\frac{t}{\tau}\right)\end{aligned} \tag{13}$$

Equation (13) was simulated to obtain the SDR behaviour for this system.

2.4 System 4: A mass-spring-damper system with a visco-elastically coupled linear inerter

The fourth system that was investigated is shown in Fig. 4. The difference between this system and S-1 is that it was added an auxiliary damper and an auxiliary spring, respectively, with damping coefficient $N_c c$ and stiffness $N_k k$, where N_c and N_k are real numbers, in series with the inerter. The addition of these components generated a new degree of freedom x_1, which leads to a second ODE.

After applying the second Newton's law, the ODEs for this system can be rearranged as Eq. (14) and Eq. (15).

$$(m + b)\ddot{x} + c\dot{x} + kx = b\ddot{x}_1 + c\dot{x}_0 + kx_0 \tag{14}$$

$$b(\ddot{x} - \ddot{x}_1) = N_c c(\dot{x}_1 - \dot{x}_0) + N_k k(x_1 - x_0) \tag{15}$$

Eliminating the degree of freedom x_1 by writting it in terms of x and x_0, the combination of Eq. (14) and Eq. (15), results in Eq. (16).

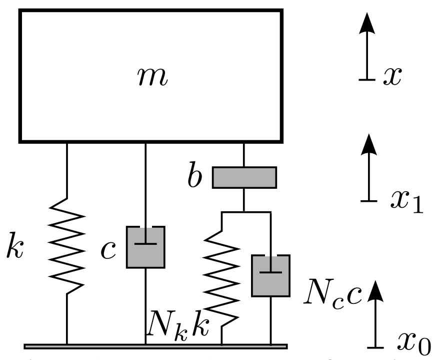


Figure 4: System 4 (S-4) configuration

$$\frac{bm}{N_k k}\dddot{x} + \left(\frac{bc + N_c cm + N_c cb}{N_k k}\right)\ddot{x} + \left(m + b + \frac{bk + N_c c^2}{N_k k}\right)\ddot{x} + \left(c + \frac{N_c c}{N_k}\right)\dot{x} + kx$$
$$= \left(\frac{bc + N_c cb}{N_k k}\right)\dddot{x}_0 + \left(b + \frac{bk + N_c c^2}{N_k k}\right)\ddot{x}_0 + \left(c + \frac{N_c c}{N_k}\right)\dot{x}_0 + kx_0 \quad (16)$$

2.4.1 Harmonic Excitation

Considering harmonic excitation, the system in Eq. (16) was written in frequency domain. The final displacement transmissibility equation, Eq. (17), was obtained in terms of ξ, μ, N_c, N_k and Ω.

$$T_d = \sqrt{\frac{\left(1 - \Omega^2\left(\mu + \frac{\mu}{N_k} + \frac{N_c 4\xi^2}{N_k}\right)\right)^2 + \left(-\frac{2\xi\mu\Omega^3}{N_k}(1 + N_c) + 2\xi\Omega\left(1 + \frac{N_c}{N_k}\right)\right)^2}{\left(1 + \Omega^4\frac{\mu}{N_k} - \Omega^2\left(\mu + \frac{\mu}{N_k} + \frac{N_c 4\xi^2}{N_k}\right)\right)^2 + \left(-\frac{2\xi\Omega^3}{N_k}(N_c + \mu + \mu N_c) + 2\xi\Omega\left(1 + \frac{N_c}{N_k}\right)\right)^2}} \quad (17)$$

2.4.2 Shock Excitation

For shock excitation, the modelled shock input is the half sine pulse, which starts at time $t = 0$. The new ODE defined by Eq. (18) was obtained in terms of ξ , μ , ω_n, t, τ and A.

$$\frac{\mu}{N_k\omega_n^2}\dddot{x} + \left(\frac{2\xi\mu}{N_k\omega_n} + \frac{2N_c\xi}{N_k\omega_n} + \frac{2N_c\xi\mu}{N_k\omega_n}\right)\ddot{x} + \left(1 + \mu + \frac{\mu}{N_k} + \frac{4N_c\xi^2}{N_k}\right)\ddot{x} + 2\xi\omega_n\left(1 + \frac{N_c}{N_k}\right)\dot{x} + \omega_n^2 x =$$
$$= -\left(\frac{2\xi\mu}{N_k\omega_n} + \frac{2N_c\xi\mu}{N_k\omega_n}\right)A\frac{\pi^3}{\tau^3}cos\left(\pi\frac{t}{\tau}\right) - \left(\mu + \frac{\mu}{N_k} + \frac{4N_c\xi^2}{N_k}\right)A\frac{\pi^2}{\tau^2}sin\left(\pi\frac{t}{\tau}\right) + 2\xi\omega_n\left(1 + \frac{N_c}{N_k}\right)A\frac{\pi}{\tau}cos\left(\pi\frac{t}{\tau}\right)$$
$$+\omega_n^2 Asin\left(\pi\frac{t}{\tau}\right) \quad (18)$$

Equation (18) was simulated to obtain the SDR behaviour for this system.

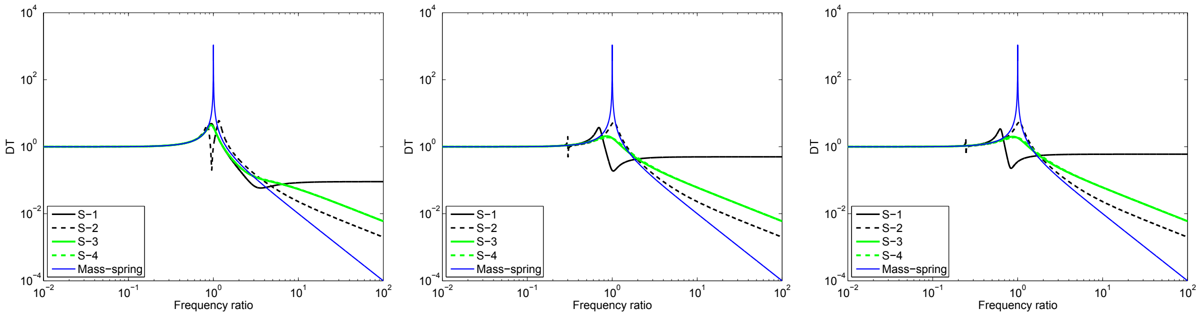
3. COMPARISON OF RESULTS AND DISCUSSION

In this section the results are presented for harmonic (DT) and shock excitation (SDR), comparing the systems. Each figure represent the behaviour for specified values of μ and ξ, which allow a complete investigation about damping influence and mass ratio influence in each system.

3.1 Harmonic Excitation

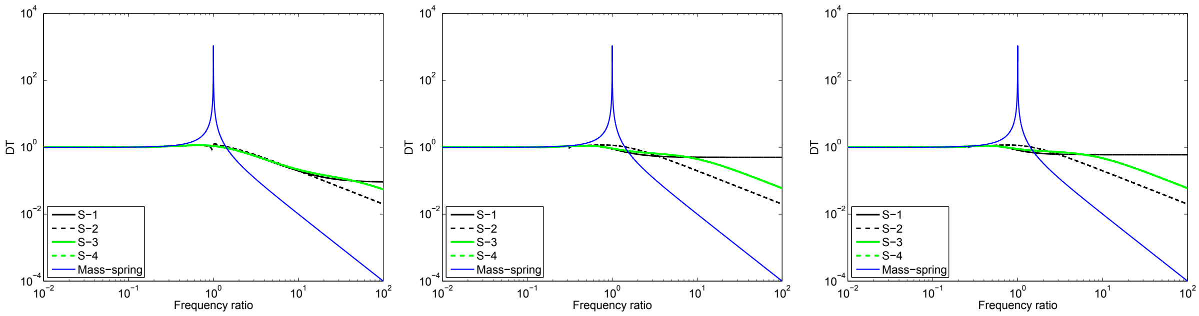
For harmonic excitation, Figs. (5) to (7) show the DT behaviour for systems S-1, S-2, S-3, S-4, and for a Mass-spring system without inerter and damper. The damping factor and mass ratio parameters were changed to show their influence in the DT behaviour.

It can be seen in Figs. (5) to (7) that the use of the inerter to isolate mechanical systems improves the performance in some frequency ranges, specially when the systems are underdamped. When the systems become critically damped



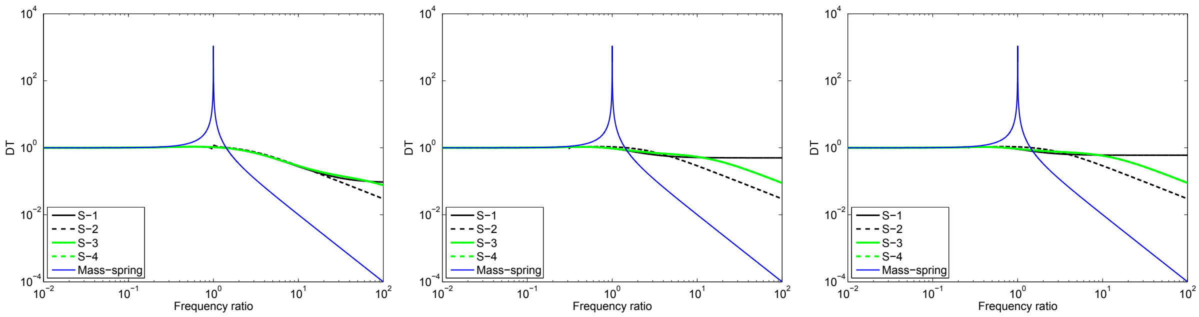
(a) $\mu = 0.1$ (b) $\mu = 1$ (c) $\mu = 1.5$

Figure 5: Compared results for $\xi = 0.1$ - Underdamped systems



(a) $\mu = 0.1$ (b) $\mu = 1$ (c) $\mu = 1.5$

Figure 6: Compared results for $\xi = 1$ - Critically damped systems



(a) $\mu = 0.1$ (b) $\mu = 1$ (c) $\mu = 1.5$

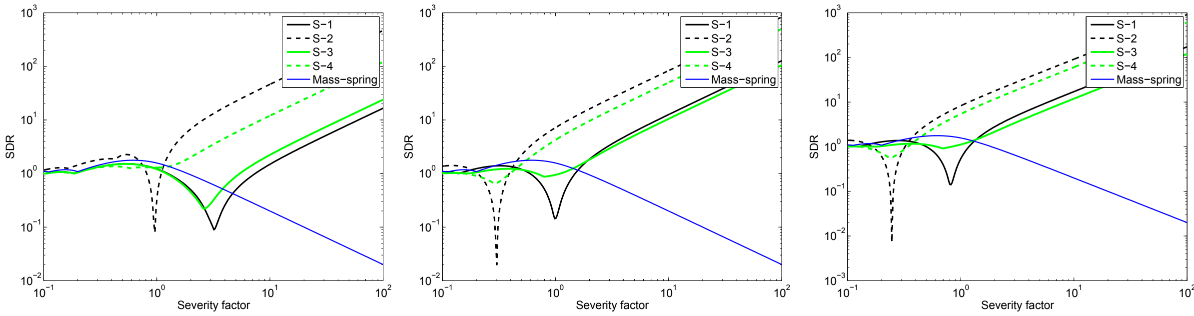
Figure 7: Compared results for $\xi = 1.5$ - Overdamped systems

or overdamped, the influence of the inerter becomes weaker, requiring inertance values greater than the system mass. For some applications, like Formula 1 racing cars, assuming they are underdamped, smaller inertance values implies in lighter suspensions, and can improve the handling and grip by making the wheels more adhered to the race track, since the DT is smaller in some frequency regions, implying that the system mass oscillates less than the wheels. For another applications, inertance values greater than the system mass value can be a good way to isolate systems from external vibration, not necessarily using high damping factors, which would lead to vibration isolator systems similar to tuned mass dampers.

3.2 Shock Excitation

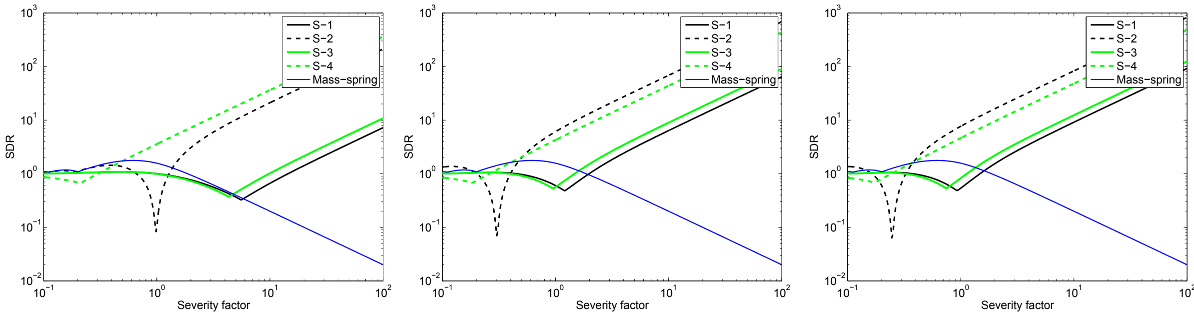
For shock excitation, Figs. (8) to (10) show the SDR behaviour for systems S-1, S-2, S-3, S-4, and for a Mass-spring system without inerter and damper. The damping factor and mass ratio parameters were changed to show their influence in the SDR behaviour.

It can be seen in Figs. (8) to (10) similar behaviours to the DTs. There are severity factor ranges in which the use of inerter improves the SDR, and there are ranges in which not. Differently from DT, the inertance influences more in the systems than the damping factor. For Formula 1 racing cars, the SDR can show better what really occurs in the race track, since chicanes can be approached to shock excitations and not harmonic excitations. By this way, it can be seen the inerter can really improve the car adherence in the race track, improving also the handling and grip, which can lead to the time difference needed to overcome opponents.



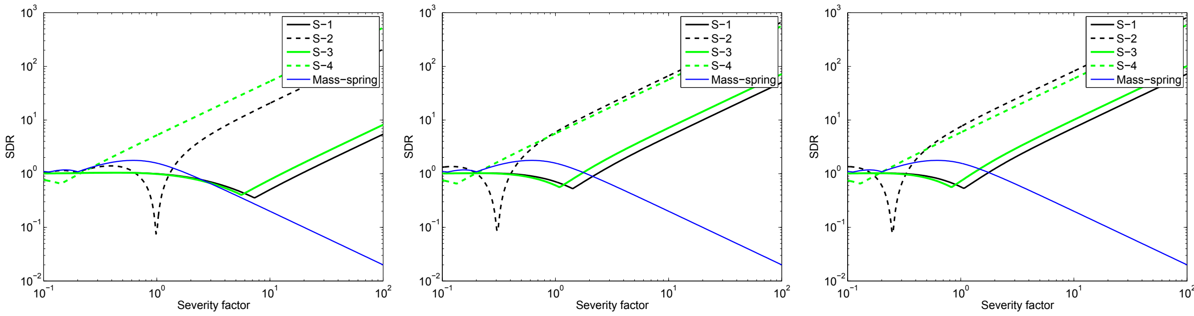
(a) $\mu = 0.1$ (b) $\mu = 1$ (c) $\mu = 1.5$

Figure 8: Compared results for $\xi = 0.1$ - Underdamped systems



(a) $\mu = 0.1$ (b) $\mu = 1$ (c) $\mu = 1.5$

Figure 9: Compared results for $\xi = 1$ - Critically damped systems



(a) $\mu = 0.1$ (b) $\mu = 1$ (c) $\mu = 1.5$

Figure 10: Compared results for $\xi = 1.5$ - Overdamped systems

4. CONCLUSIONS

This paper has presented the influence of the inerter in different mechanical systems in absolute displacement transmissibility (DT) for harmonic excitation and in shock displacement ratio (SDR) for shock excitation. It was concluded the inerter can improve the DT and SDR for some ranges of, respectively, frequency and severity factor values. For Formula 1 racing cars, chicanes can be considered as shock excitations, and it was concluded that the use of inerter, since the car is well designed for this condition, i.e. specific severity factor, can improve the car's adherence in the race track, consequently improving handling and grip, which can shorten a lap time.

5. ACKNOWLEDGEMENTS

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