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An Investigation of the Boundary-Layer Equations of a Dilute and Monodispersed Emulsion of Very Viscous Drops

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Abstract. *This work deals with the rheology of the boundary-layer of a dilute and monodispersed emulsion of very viscous drops. Assuming a regime of small deformations for the drops, which occurs in high-drop viscosity ratios, a constitutive model to describe the macroscopic behavior of the emulsion is presented. This model is based on two equations: the first, describes how the geometry of the drop varies along the flow; the second, calculates the additional stress that arises from the development of the droplet shape on the microscopic scale. Using this model and invoking the boundary-layer hypothesis, we present a system of two partial differential equations that describes the boundary-layer behavior of an emulsion. The numerical solution is performed by a Galerkin Finite Element Method with an additional stabilization technique. The results for the emulsion boundary-layer show the existence of a similar velocity profile and nonlinear effects when compared to a Newtonian fluid. Based on the numerical results for some terms of the boundary-layer model, we also propose some simplifications on the original equation, obtaining a formulation similar to that of Newtonian fluids.*

Keywords: *Emulsion, Boundary-Layer, Finite Element Method.*

1. INTRODUCTION

An emulsion is a biphasic mixture of two immiscible viscous liquids in which one is dispersed in the other in the form of drops. Emulsions are produced by a process of generation and breakup of drops in a couple of fluids which sets out two phases, one dispersed and another continuous. Although both phases are composed by simple materials, such as water and oil, for example, emulsions are typically classified as non-Newtonian fluids. As matter of fact, the flow that occurs in the scale of the drops is characterized by complex physical phenomena, making the equivalent continuous fluid present nonlinear behavior in many cases. In this way, the rheology of emulsions is difficult to predict or control due to the complex connection between the microscale of the drops and the macroscopic flow. The study of the mechanical behavior of emulsions is supported by the wide variety and range of their applications. In fact, emulsions can be found in many sectors of materials processing, food processing and pharmaceutical and petroleum industries (Larson, 1999).

Investigations about the mechanical behavior of emulsions begin with the work of Taylor (1932), which extended the results of Einstein (1906) to obtain the effective viscosity of dilute emulsions of spherical drops of high surface tension. Deformation of droplets in simple shear and extensional flows were also studied by Taylor (1934). In this paper, the author defined a scalar measure of deformation of drops with ellipsoidal shape. Subsequently, many theoretical papers have investigated the deformation of drops in different types of flows and its effects on the rheology of a dilute emulsion (Frankel and Acrivos, 1970; Barthès-Biesel and Acrivos, 1973). Numerical strategies for the study of emulsions are available in a great number of articles found in the area. Most of these studies performs the flow in the small scale, investigating the conditions for drop deformation and breakup (Pozrikidis, 1993; Bazhlekov *et al.*, 2004) and the interaction between two or more drops (Loewenberg and Hinch, 1997; Zinchenko *et al.*, 1999). More recently, Oliveira and Cunha (2011) explored a different limit, where the drop deformation is limited by a high-drop viscosity ratio, i.e., the viscosity ratio between dispersed and continuous phases is $\lambda \gg 1$. In this case, the droplets do not break under shear and as the particles are pulled out by straining, they are also spun by vorticity. In this paper, the authors also presented a new constitutive model to describe dilute and monodisperse emulsions of very viscous drops seen as a homogeneous equivalent media.

Within this context, and motivated by various applications, this paper aims to investigate a boundary-layer model for high-drop viscosity emulsions. For this, we invoke the boundary-layer hypothesis and use the constitutive equations developed by Oliveira and Cunha (2011) to obtain a system of two partial differential equations that describes the boundary-layer behavior of a dilute emulsion over a flat plate. These equations are solved numerically by the Galerkin Finite Element Method (GFEM) with an additional stabilization technique to avoid numerical instabilities. The results are used to investigate the existence of a similar velocity profile and nonlinear effects inside the boundary-layer region. Using the numerical results and imposing some simplifications, we also propose a pioneering model to describe the boundary-layer of an emulsion on a flat plate. The work is organized in an introduction, a description of the mathematical formulation, including the constitutive model and the boundary-layer equations for the emulsion, a section describing the main aspects of the numerical method, a section devoted to the results and discussion, a brief conclusion and references.

2. MATHEMATICAL FORMULATION

2.1 Microhydrodynamics analysis

We consider an emulsion consisting of neutrally buoyant identical viscous drops with viscosity $\mu' = \lambda\mu$ dispersed in an incompressible Newtonian liquid of viscosity μ. Whereas the drop is convected by the flow, a characteristic scale of velocity experienced by the particle, U_a, can be given by $U_a = \dot{\gamma}_c a$, where $\dot{\gamma}_c$ is a typical shear rate of the flow and a is the droplet radius before deformation. Assuming a typical droplet with diameter between 1 μm and 100 μm and considering the thermophysical properties of fluids in usual emulsions and common speeds in flows of practical interest, we see that the Reynolds number of the flow outside the drop is very small – $Re_o = \rho a^2 \dot{\gamma}_c/\mu \ll 1$. Furthermore, for drops of oil-water emulsion, in which the viscosity ratio is $\lambda \sim 100$, the Reynolds number of the flow inside the drop is also very small – $Re_i = Re_o/\lambda \ll 1$. Thus, both flows can be considered free from inertia effects and are governed by the Stokes incompressible equations, namely (Kim and Karilla, 1991)

$$\nabla \cdot \boldsymbol{u} = 0, \tag{1}$$

$$\nabla \cdot \boldsymbol{\sigma} = 0, \tag{2}$$

where $\boldsymbol{u}$ is the Eulerian velocity field and $\boldsymbol{\sigma}$ is the local stress tensor. It is important to remember that both the dispersed phase and the continuous phase are Newtonian fluids. Thus, except for the difference between the viscosities, it follows that

$$\boldsymbol{\sigma} = -P\boldsymbol{I} + \mu[(\nabla\boldsymbol{u}) + (\nabla\boldsymbol{u})^T]. \tag{3}$$

Here, $P = p - \rho\boldsymbol{g}\cdot\boldsymbol{x}$ is the modified pressure, $\boldsymbol{I}$ is the unit tensor, $\boldsymbol{g}$ is the gravitational acceleration, $\boldsymbol{x}$ is the position vector and $p = -(\boldsymbol{\sigma} : \boldsymbol{I})/3$ is the mechanical pressure.

2.2 Constitutive model for the emulsion

Even if any suspension may be treated as a biphasic media, in most cases, the small size of the particles and its spatial distribution allow the mixture to be treated as a homogeneous equivalent fluid whose mechanical average behavior is intrinsically linked to the dynamics in the microscale. In this way, the average stress tensor *for* a suspension can be modeled by the usual Newtonian stress tensor for the continuous phase modified by a term associated to the existence of the dispersed particles. So, the constitutive equation for a suspension can be written as $\boldsymbol{\sigma} = -P\boldsymbol{I}+\mu[(\nabla\boldsymbol{u})+(\nabla\boldsymbol{u})^T]+\boldsymbol{\sigma}^d$, where $\boldsymbol{\sigma}^d$ is the additional stress tensor due to the dispersed phase. This additional tensor is known as Landau-Batchelor tensor or *Stresslet* (Batchelor, 1970) and is defined as $\boldsymbol{\sigma}^d = \dfrac{n}{N}\sum_{\alpha=1}^{N}\boldsymbol{S}_\alpha$, where N is the total number of dispersed particles and n is its average density in the medium. Moreover, α is a counter for the particles in the flow and $\boldsymbol{S}_\alpha$ is a hydrodynamic dipole that represents the additional stress over the continuous phase due to the particle α. It is important to note that $\boldsymbol{\sigma}^d$ depends on the particle configuration, i.e., its size, shape, orientation and distribution in the flow.

Considering dilute emulsions of equal and very viscous drops, where the viscosity ratio is $\lambda \gg 1$, Oliveira and Cunha (2011) proposed a continuous constitutive model to describe the average mechanical behavior of the equivalent fluid. This model consists of two equations: the first one presents information of the internal scale of the material, where the emulsion is seen as a biphasic medium, and describes how the drop shape varies along the flow; the second defines the additional stress tensor because of the drops as a function of its geometry, making a connection between micro and macroscopic scales. Mathematically,

$$\frac{d\boldsymbol{A}}{dt} = \boldsymbol{W}\cdot\boldsymbol{A} - \boldsymbol{A}\cdot\boldsymbol{W} + \frac{5}{2\lambda}\boldsymbol{E} - \frac{20\sigma}{19\lambda\mu a}\boldsymbol{A}, \tag{4}$$

$$\boldsymbol{\sigma}^d = \phi\mu\left[\left(5 - \frac{25}{2\lambda}\right)\boldsymbol{E} + 4\frac{\sigma}{\lambda\mu a}\boldsymbol{A} + \frac{15}{7}\boldsymbol{\mathcal{F}}[\boldsymbol{A},\boldsymbol{E}]\right], \tag{5}$$

where $\mathcal{F}[\boldsymbol{A},\boldsymbol{E}] = \boldsymbol{A}\cdot\boldsymbol{E} + \boldsymbol{E}\cdot\boldsymbol{A} - (2/3)(\boldsymbol{A}:\boldsymbol{E})\boldsymbol{I}$. Here, σ is the surface tension coefficient and $\boldsymbol{E}$ and $\boldsymbol{W}$ are the strain rate and vorticity tensors, defined as the symmetric and antisymmetric parts of the velocity gradient tensor, respectively. The shape tensor $\boldsymbol{A}$ is a Cauchy-Green deformation tensor and, by definition, is positive, definite, symmetric and traceless. This model is based on harmonic solutions for the Stokes flow with spherical symmetry and more details can be found in Lamb (1932) and Frankel and Acrivos (1970). It is remarkable that the determination of the tensors $\boldsymbol{A}$ and $\boldsymbol{\sigma}^d$ depends essentially on the type of flow and may require simultaneous solutions of the equations, as occurs in the flow through pipes (Poiseuille flow).

2.3 Boundary-layer equations for an emulsion

The theory of laminar boundary-layer for a two-dimensional incompressible flow is well established in the literature, from both the mathematical and the physical point of view (Schlichting, 1979). Based on this and assuming that the Reynolds number of the macroscopic flow is very large, a scale analysis shows that $\delta/L \sim Re_L^{-1/2}$, where δ is the boundary-layer thickness, L is a characteristic length of the problem and $Re_L = Re_o(L/a)$ is the Reynolds number of the macroscopic flow. So, let us consider the classic problem of a free stream $\boldsymbol{u}^\infty = (u^\infty, 0)$ incident over a flat plate, as shown in Fig. 1. In this case, x and y are the Cartesian coordinates parallel and orthogonal to the plate, respectively. The velocity profile in the boundary-layer is two-dimensional, $\boldsymbol{u} = (u, v)$. Physically, the boundary conditions are the absence of slip between the fluid and the wall – $u = v = 0$ for $y = 0$ – and continuity of velocity through the boundary-layer edge – $u = u^\infty$ and $v = 0$ for $y = \delta$ at any position x.

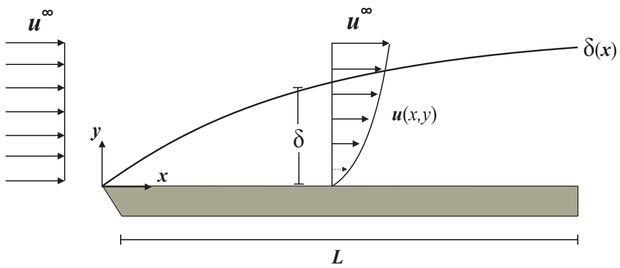


Figure 1: Boundary-layer configuration over a flat plate.

From an operational point of view to treat the emulsion flow, we apply the boundary-layer simplifications directly on the tensors $\boldsymbol{E}$ and $\boldsymbol{W}$ to obtain the dominant components $E_{12} = E_{21} = (1/2)\partial u/\partial y$ and $W_{12} = -W_{21} = -(1/2)\partial u/\partial y$. Using this result, we solve the steady state of Eq. (4) to calculate the deformation tensor $\boldsymbol{A}$. Finally, the additional stress tensor $\boldsymbol{\sigma}^d$ is determined by solving the Eq. (5). So, the most general form of the boundary-layer equations for a steady flow of a dilute emulsion of equal and very viscous drops are

$$
\begin{aligned}
\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} &= 0, && (6)\\
\rho\left(u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y}\right) &= -\frac{\partial p}{\partial x} + \mu_B\frac{\partial^2 u}{\partial y^2} + \mu\phi\left\{\frac{1080}{133}\frac{\sigma}{\lambda^2\mu a}\frac{\dot\gamma\dot\gamma_x}{(c_1^2+\dot\gamma^2)}\left[\frac{\dot\gamma^2}{(c_1^2+\dot\gamma^2)} - 1\right]\right\} && (7)\\
&+ \mu\phi\left\{\frac{100}{19}\frac{\sigma^2}{\lambda^3\mu^2 a^2}\frac{\dot\gamma_y}{(c_1^2+\dot\gamma^2)}\left[1 - \frac{2\dot\gamma^2}{(c_1^2+\dot\gamma^2)}\right]\right\},\\
0 &= -\frac{\partial p}{\partial y} + \mu_B\frac{\partial^2 u}{\partial x\partial y} + \mu\phi\left\{\frac{1580}{133}\frac{\sigma}{\lambda^2\mu a}\frac{\dot\gamma\dot\gamma_y}{(c_1^2+\dot\gamma^2)}\left[1 - \frac{\dot\gamma^2}{(c_1^2+\dot\gamma^2)}\right]\right\} && (8)\\
&+ \mu\phi\left\{\frac{100}{19}\frac{\sigma^2}{\lambda^3\mu^2 a^2}\frac{\dot\gamma_x}{(c_1^2+\dot\gamma^2)}\left[1 - \frac{2\dot\gamma^2}{(c_1^2+\dot\gamma^2)}\right]\right\}.
\end{aligned}
$$

In Eqs. (7) and (8), $\dot\gamma = \partial u/\partial y$ is a measure of the dominant component of the rate of deformation tensor, $\dot\gamma_x = \partial\dot\gamma/\partial x$, $\dot\gamma_y = \partial\dot\gamma/\partial y$ and $c_1 = 20\sigma/(19\lambda\mu a)$. The left side of Eq. (8) vanishes due to the boundary-layer simplifications, since $x \sim L$, $y \sim \delta$, $u \sim u^\infty$, $v \sim u^\infty(\delta/L)$ and $\delta/L \ll 1$. Note that the last two terms of these equations have an elastic character, indicating the possible existence of viscoelastic effects in the boundary-layer of a dilute emulsion. Furthermore, $\mu_B/\mu = 1 + \phi[(5/2) + 25/(4\lambda)]$ is the dimensionless effective viscosity of an emulsion in flows with high shear rates.

3. NUMERICAL METHODOLOGY

3.1 General aspects

The numerical method used to solve the boundary-layer equations presented in Section 2.3 was the Galerkin Finite Element Method (Gresho and Sani, 1998). Within this context, since Ω is the physical domain of the problem and Γ is its contour, the weak form of the conservation equations of motion for an incompressible fluid in the boundary-layer region written in the vectorial form is

$$R_c = \int_\Omega (\nabla \cdot \boldsymbol{u})\, \chi_i \, d\Omega = 0, \tag{9}$$

$$R_m = \int_\Omega \rho\, (\boldsymbol{u} \cdot \nabla \boldsymbol{u}) \cdot \boldsymbol{\phi}_i \, d\Omega + \int_\Omega (\boldsymbol{\sigma} : \nabla \boldsymbol{\phi}_i)\, d\Omega - \int_\Gamma (\boldsymbol{n} \cdot \boldsymbol{\sigma}) \cdot \boldsymbol{\phi}_i \, d\Gamma, \tag{10}$$

where χ_i and $\boldsymbol{\phi}_i$ are the weighting functions adopted to obtain the residual form of the mass and momentum conservation equations, respectively. Moreover, the expansion of the velocity and pressure fields are

$$\boldsymbol{u} = \left(\sum_{i=1}^{n} U_i \phi_i, \sum_{i=1}^{n} V_i \phi_i \right), \tag{11}$$

$$p = \sum_{i=1}^{m} P_i \chi_i. \tag{12}$$

Here, it is important to note that the functions $\boldsymbol{\phi}_i$ used to weigh the momentum equation are the same used in the expansion of the velocity field. On the other hand, the functions χ_i used to weigh the mass conservation equation, which do not contain pressure terms, are the same used in the expansion of the pressure field. Physically, it means that the pressure is adjusted to satisfy the mass conservation principle. The basis functions employed to expand the unknown fields were bi-quadratic Lagrangian interpolating polynomials for velocity and linear discontinuous polynomials for pressure.

The finite element used was bidimensional and quadrilateral of type Q_2/P_{-1}. Accordingly, each element contains nine nodes, each one containing two velocity components. The center of each element, in particular, has also three components of pressure. Therefore, it follows that $n = 9$ and $m = 3$ in Eqs. (11) and (12), such that each element used to discretize the domain of the problem presents 21 degrees of freedom. Isoparametric mapping was employed in order to correlate global and local values of the problem. The residual equations together with the Galerkin formulation for the expansion of the unknown fields gives rise to a system of $2n + m = 21$ nonlinear algebraic equations for each element. The global system is obtained from a connectivity matrix that relates the local degrees of freedom to the global ones. The nonlinear global system was treated using a Newton-Raphson algorithm.

Finally, we address the boundary conditions from the computational point of view. We used a rectangular computational mesh that concentrates points near the wall. In this case, three boundary conditions were used: at the inflow plane (left side) and at the wall (bottom side) the velocity vector was assumed to be known; at the upper plane (upper side) we assumed an outflow condition; lastly, we assumed a pressure condition at the outflow plane (right side).

3.2 Stabilization method

When the problem is dominated by the convective term, as occurs in the boundary-layer region, the Galerkin solution is corrupted by non-physical oscillations. Mathematically, the method loses its best approximation property when the non-symmetric convection operator dominates the diffusion operator in the transport equation, and consequently spurious node-to-node oscillations appear. Moreover, the Galerkin formulation naturally inserts a negative numerical diffusion on the problem and, consequently, the convection term is not discretized using a centered scheme (Donea and Huerta, 2003).

In order to avoid these instabilities, we employed a stabilization technique that consists of an upwind approximation of the convective term. The main ideia of this procedure is to introduce an artificial diffusion term to counterbalance the negative numerical diffusion inherent in the GFEM. Thereby, the weighting functions of the convective term are modified to give more importance to the elements located upstream than the downstream ones. In multiple dimensions, when this artificial diffusion is added only in the flow direction, i.e., in the direction tangent to the velocity vector, the technique evolved is known as Streamline-Upwinding (or SU). So, the SU method can be understood as a standard GFEM with an extra term to counteract this numerical diffusion.

Let us consider the diffusivity tensor

$$\tilde{\nu}_{ij} = \overline{\nu} \frac{u_i u_j}{||\boldsymbol{u}||^2}, \tag{13}$$

where u_i is the component of the velocity vector $\boldsymbol{u}$ in the direction $\boldsymbol{e}_i$ and $\overline{\nu}$ is an artificial diffusion. It is useful to remember that the boundary-layer region is characterized by a competition between two distinct mechanisms of momentum transfer: convection in the direction of flow and diffusion of vorticity due to viscosity. In the case of laminar boundary-layer on a flat plate, as shown in Fig. 1, it is reasonable to admit that the dominant part of these mechanisms operates only in the x and y directions, respectively. With these concepts in mind, we adapted the usual SU formulation and despised the convective effects in the transverse direction to the plate, performing the stabilization considering only the effects in the direction parallel to it. Assuming that the convection occurs only in the x direction, we set $\overline{\nu} = \beta uh/2$, where u is the velocity component and h is the element length, both measured in the x direction (or, in a local frame, the ξ direction, since both coincide) and β is a free parameter that controls the amount of numerical diffusion added, namely *upwinding*. Thus, we set that the optimum value for this parameter is defined as $\beta_o = \coth Pe - Pe^{-1}$, in which Pe is the Peclet adimensional number and expresses the ratio of convective to diffusive transport. This parameter is defined as $Pe = uh/(2\nu)$, where ν is the kinematic viscosity of the fluid, and is calculated separately for each element.

Introducing this concept in the weak formulation of the boundary-layer problem, it can be shown that the weighting functions of the convective term are improved. According to Eq. (10), the weighting function of the first integral term should be rewritten as

$$\tilde{\boldsymbol{\phi}} = \boldsymbol{\phi} + \frac{\overline{\nu}}{||\boldsymbol{u}||^2}(\boldsymbol{u} \cdot \nabla\boldsymbol{\phi}). \tag{14}$$

Note that the stabilization technique is introduced only in the momentum conservation equation and that the new shape function could be used to weigh only its convective term.

4. RESULTS AND DISCUSSION

4.1 Validation of the method

First, we consider a simpler problem in order to validate the GFEM in the context of this work. We evaluated the dimensionless velocity profile in the boundary-layer considering a Newtonian fluid and performing comparisons with parabolic and cubic approximations as well as with the Blasius classical analytical solution for a laminar boundary-layer on a flat plate (Schlichting, 1979). For the simulations, we use water (with kinematic viscosity $\nu = 10^{-6}$ m^2/s) and impose the appropriate value for the free stream $u^\infty = 7 \times 10^{-3}$ m/s to ensure a high Reynolds number in a certain region of the computational domain. This region was selected in the interval 1.0 m < x < 2.0 m, where the local Reynolds number, $Re_x = xu^\infty/\nu$, varies between 7.0×10^3 and 1.4×10^4, which agrees with the laminar flow band and allows the approach $\delta/x \ll 1$ for any x to be used. Figure 2 shows the results obtained for the dimensionless velocity profiles at the position x = 1.2 m on the plate. Thus, it is remarkable that the numerical profile agrees quite well with the analytical solution of the problem in almost all the extent of the laminar boundary-layer thickness. It is worth noting, however, the difference among the numerical profile and the others near the edge of the boundary-layer. This deviation is a consequence of the numerical instabilities of GFEM when applied to predominantly convective problems, and indicates that the principle of stabilization used is not robust enough to perfectly satisfy the boundary condition $u = u_\infty$ for $y = \delta$. Yet, except for the boundary condition at the edge of the boundary-layer, it is believed that the method is able to provide significant results in a first study about the boundary-layer of dilute and monodispersed emulsions.

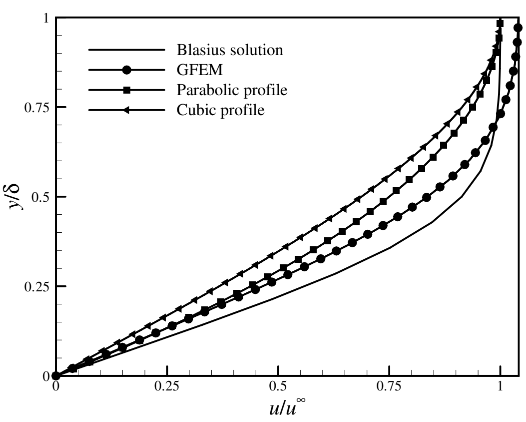
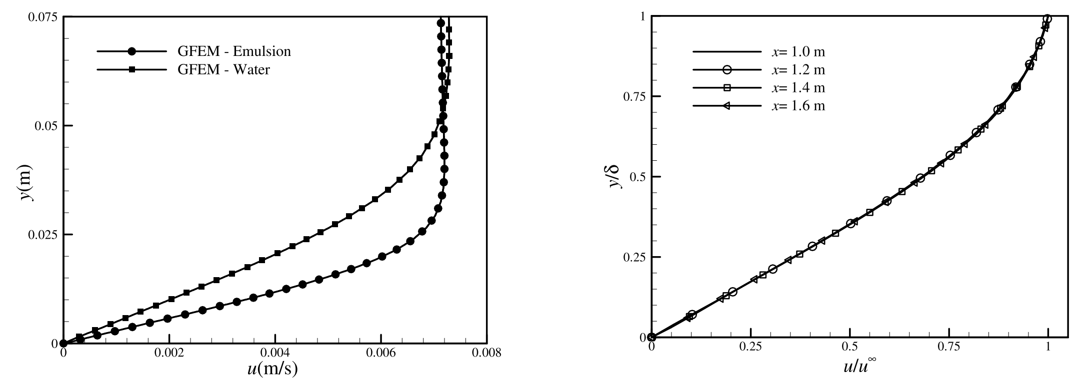


Figure 2: Dimensionless velocity profile: comparison between the numerical solution of GFEM, approximate polynomials profiles and the exact solution due Blasius for a Newtonian fluid.

4.2 Boundary-layer of an emulsion

Considering the model of a dilute and monodispersed emulsion of very viscous drops discussed so far, we use the GFEM to solve the system of partial differential equations that governs the behavior of its boundary-layer, as shown by Eqs. (6), (7) and (8). As parameters for the simulation, we consider oil drops with initial radius $a = 100\ \mu$m dispersed in water, such that the viscosity ratio is $\mathcal{O}(10^2)$. We also adopted $\phi = 30\%$ for the concentration of the dispersed phase, $\sigma = 0.040$ N/m for the surface tension of the drops and $u_\infty = 7 \times 10^{-3}$ m/s for the free stream velocity. The first important result is the probable existence of a similar velocity profile along the length of the plate, indicating a very strong resemblance to the behavior of common Newtonian fluids. Figure 3a shows the dimensionless velocity profiles for four different locations along the plate. Note that all positions for which the velocity profiles were obtained are such that $x > 1.0$ m, so that the Reynolds number is large enough to ensure that the boundary-layer hypothesis is satisfied. Then, we considered a comparison between the velocity profile in the boundary-layer region for a Newtonian fluid (water) and the emulsion under study. Figure 3b shows the results, indicating considerable differences between the two profiles. In this regard, note that the velocity condition at the boundary-layer edge for the emulsion is reached faster, indicating that the boundary-layer thickness, in this case, is smaller. In addition, for the emulsion flow, the velocity gradients near the wall are more intense, and diminish in intensity rapidly as we approach the boundary-layer edge.



(a) (b)

Figure 3: Results for emulsion's boundary-layer. In (a), the possible existence of a similar dimensionless velocity profile inside the boundary-layer region; in (b) comparison of the boundary-layers velocity profiles between the emulsion and a Newtonian fluid.

Regarding to Fig. 3a, the existence of a similar velocity profile indicates the probable existence of a similarity variable for the problem. Besides the strong resemblance to the theory of Newtonian fluids, this possibility also indicates the possible existence of an analytic solution to describe the boundary-layer of emulsions. On the other hand, the phenomena indicated in Fig. 3b are probably associated with non-linearities in the mechanical behavior of the emulsion. Thus, we highlight the viscoelastic nature of this material, as well as its shear thinning behavior.

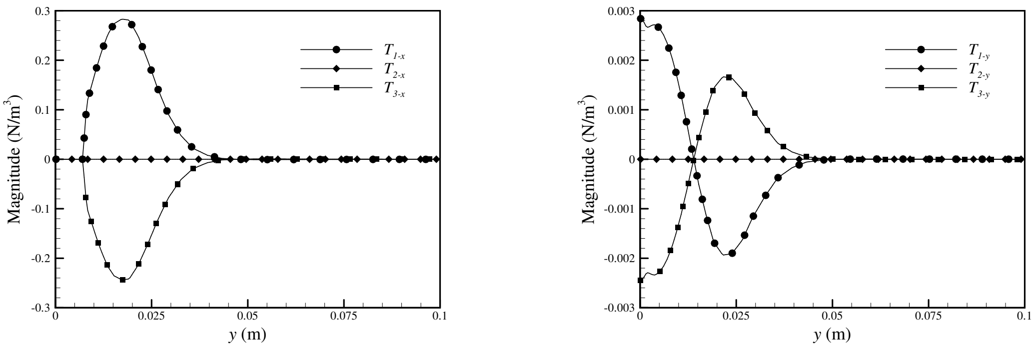
4.3 A simplification for the boundary-layer equations of an emulsion

Although we do not have a dimensionless form for the boundary-layer equations of an emulsion, numerical simulations were held in order to investigate the order of magnitude of the various terms involved in the model. For convenience, pet us rewrite Eqs. (7) and (8), naming the relevant terms, as follows

$$
\begin{aligned}
\rho\left(u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y}\right) &= -\frac{\partial p}{\partial x} + \underbrace{\mu\mu_B\frac{\partial^2 u}{\partial y^2}}_{T_{1-x}} + \underbrace{\mu\phi\left\{\frac{1080}{133}\frac{\sigma}{\lambda^2\mu a}\frac{\dot\gamma\dot\gamma_x}{(c_1^2+\dot\gamma^2)}\left[\frac{\dot\gamma^2}{(c_1^2+\dot\gamma^2)} - 1\right]\right\}}_{T_{2-x}} \\
&+ \underbrace{\mu\phi\left\{\frac{100}{19}\frac{\sigma^2}{\lambda^3\mu^2 a^2}\frac{\dot\gamma_y}{(c_1^2+\dot\gamma^2)}\left[1 - \frac{2\dot\gamma^2}{(c_1^2+\dot\gamma^2)}\right]\right\}}_{T_{3-x}},
\end{aligned}
\tag{15}
$$

$$
\begin{aligned}
0 \quad = \quad & -\frac{\partial p}{\partial y} + \underbrace{\mu\mu_B \frac{\partial^2 u}{\partial x \partial y}}_{T_{1-y}} + \underbrace{\mu\phi \left\{ \frac{1580}{133} \frac{\sigma}{\lambda^2 \mu a} \frac{\dot{\gamma}\dot{\gamma}_y}{(c_1^2 + \dot{\gamma}^2)} \left[1 - \frac{\dot{\gamma}^2}{(c_1^2 + \dot{\gamma}^2)} \right] \right\}}_{T_{2-y}} \\
+ \quad & \underbrace{\mu\phi \left\{ \frac{100}{19} \frac{\sigma^2}{\lambda^3 \mu^2 a^2} \frac{\dot{\gamma}_x}{(c_1^2 + \dot{\gamma}^2)} \left[1 - \frac{2\dot{\gamma}^2}{(c_1^2 + \dot{\gamma}^2)} \right] \right\}}_{T_{3-y}} .
\end{aligned}
\tag{16}
$$

The comparison between the six terms marked in Eqs. (15) and (16) was performed by a post processing of the data generated by the numerical simulation. The points were drawn form a vertical line at $x = 1.0$ m and the results are shown in Fig. 4.



(a) (b)

Figure 4: Comparison between the relevant terms of the linear momentum equation in the boundary-layer region for an emulsion. In (a), the terms for the equation in the x direction; in (b), terms for the equation in the y direction.

From the results shown in Figs 4(a) and 4(b) it can be noted that the second term of the equations in both directions is null. Moreover, it appears that the magnitude of the terms in the y direction is approximately 1% of the terms in the x direction. Therefore, the linear momentum equation in the y direction becomes negligible in relation to the same equation in the x direction. Still based on this analysis, it can be said that the pressure gradient in the direction perpendicular to the plate, $\partial p/\partial y$, can be neglected, since it is in balance with considerably lower terms in magnitude than the ones in balance with the pressure gradient parallel to the plate, $\partial p/\partial x$. Therefore, the pressure term apparently is independent of the y direction, i.e., $p = p(x)$ and its value depends only on the behavior of the flow outside the boundary-layer region. This numerical result can be sustained with reasonable safety, since a similar result can be checked for the flow of an usual Newtonian liquid.

Considering that for the flow on a flat plate we have that u^∞ is constant it follows that $\partial p/\partial x = dp/dx = 0$. Thus, the model for a boundary-layer of a dilute and monodispersed emulsion over a flat plate can be simplified and written only in terms of one equation,

$$
\rho \left(u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} \right) = \mu_B \frac{\partial^2 u}{\partial y^2} + \mu\phi \left\{ \frac{100}{19} \frac{\sigma^2}{\lambda^3 \mu^2 a^2} \frac{\dot{\gamma}_y}{(c_1^2 + \dot{\gamma}^2)} \left[1 - \frac{2\dot{\gamma}^2}{(c_1^2 + \dot{\gamma}^2)} \right] \right\} . \tag{17}
$$

The description of the behavior of the boundary-layer of an emulsion with only the momentum equation in the x direction with the simplification of some terms is analogous to the description of the boundary-layer of a Newtonian fluid, supporting our hypothesis. However, in the case of the emulsion, the existence of viscoelastic terms in the governing equation can be noted, which is beyond the classical balance between convective and diffusive terms inside the laminar boundary-layer for an incompressible linear liquid. However, it is important to regard that a further study should be done in order to write the linear momentum equations in a dimensionless form to prove, in fact, that our simplifications are coherent.

5. CONCLUSION AND FUTURE WORKS

This work dealt with the mechanical and rheological behavior of the boundary-layer of a dilute emulsion of very viscous drops. By considering an differential constitutive model to the emulsion seen as an equivalent fluid and using the boundary-layer hypothesis, we obtained its boundary-layer equations. The governing equations were solved numerically by a Galerkin Finite Element algorithm with an additional stabilization technique.

The numerical results were validated for a Newtonian fluid by comparing with approximate profiles and the analytical solution. For an emulsion, we verified the possible existence of a similar dimensionless velocity profile in the boundary-layer region. For the same conditions, a comparison between the emulsion and a Newtonian fluid was performed. The emulsion differs considerably in this case, presenting a fast variation in velocity gradients through the boundary-layer and a much smaller boundary-layer thickness. We believe that they are a consequence of the viscoelastic nature of the emulsion, as well as their shear thinning effect. Finally, we presented a numerical comparison of the magnitude of each of the terms of the boundary-layer equations for the emulsion. Accordingly, we have proposed a simplification in the boundary-layer model so that it is only described by the equation of linear momentum in the direction parallel to the plate, as in the case of a Newtonian fluid. However, the elastic term still remains in the model.

In the future, we hope to enhance the numerical methodology used, especially in relation to the stabilization procedure and the mesh refinement. We also expect more theoretical and experimental results to confirm the origin of nonlinear effects in the boundary-layer of an emulsion. Furthermore, it is necessary to obtain a dimensionless form for the boundary-layer equations for the emulsion to confirm in an analytical way the relevance of each term for the model, confirming the hypothesis we made in this work.

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8. RESPONSIBILITY NOTE

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