CILAMCE 2017
Florianópolis - Santa Catarina - Brazil

ANALYSIS OF THE NONLINEAR DYNAMIC BEHAVIOUR OF REINFORCED CONCRETE BRIDGE BEAMS WITH VARIABLE REINFORCEMENT DISTRIBUTION THROUGH THE VEHICLE-BRIDGE INTERACTION AND THE DAMAGE MECHANICS

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Abstract. *The concrete's damaging process depends, among other factors, of its interaction with other materials. The applicability of the reinforced concrete is enabled by the proximity between the materials' thermal expansion coefficients, granting, this way, the adhesions and avoiding the bond-slip between the concrete and the steel bars. The beginning of the concrete's fissuration can, however, cause the loss of adhesion between the materials. The material's damaging evolution process depends not only of the stress state, but mainly from the accumulated strain state. Therefore, the distribution of the reinforcement along a bridge's beam highly influences the concrete's damage evolution. For highway and railway bridges' cases, the dynamic loads generated from the passage of vehicles on it can amplify the loss of integrity of the concrete and the strain hardening of the reinforcement. However, the normative codes for bridge's design allow to neglect the structure's dynamic effects, converting them into an equivalent increased static loading. Besides that, the codes suggest a linear elastic behaviour for the materials in the structural analysis. Nevertheless, this method doesn't consider the process of* *damaging and plastification of the concrete and the steel, respectively, neither the dynamic behaviour of the system. The consideration of the damage mechanics on the concrete and plasticity on the steel bars enhance the model complexity, transforming it in a nonlinear and dynamic problem, modifying the structural responses. Additionally, the consideration of a variable distribution of the reinforcement along the beam's length contributes to the analysis of more realistic models. Thus, this work seeks to evaluate the nonlinear dynamic responses of a reinforced concrete bridge through its dynamic interaction between a vehicular model, in which the reinforcement distribution affects the damage evolution. The beams are modelled trough finite elements method, using the ABXDNL 2.7 program and additional routines developed in C++ programming language to consider the reinforcement distribution. It is adopted a constitutive nonlinear dynamic damage model based on the regarded Mazars' damage model, which also regards the effects of the inversion of mechanical solicitations due to vibrations.*

Keywords: *Nonlinear Dynamics, Reinforced Concrete Structures, Damage Mechanics, Dynamic Interaction, Nonlinear Finite Element Method*

1 INTRODUCTION

With the recent changes in material's consumption concept, there has been a bigger concern about its resistance exploitation. Consequently, the structures became more slender and susceptible to nonlinear behaviour, generating increases on the stresses (Machado, 1983).

Although many of the civil engineering problems are solved through static models, the majority structures are submitted to dynamic loadings, and when these are of low intensity when compared to the static ones, their effects can be neglected (Abeche, 2015).

The normative codes related to the structures project design usually suggest the increase in loads and the reduction of materials' strength, so that these structures are able to resist superior requirements than the really needed. Furthermore, for the reinforced concrete structures, the codes allow the consideration of a linear elastic behaviour of the materials in the structural analysis. For the bridge's design case, however, due to the dynamic loads generated by the passage of vehicles over it, the dynamic effects no longer can be neglected. For those cases, although, the codes accept that these effects are simplified, turning them into static loads amplified by a impact factor (Abeche, 2015).

Another factor to be considered is that, when these structures are under solicitations which occasion the beginning of the concrete's cracking, a linear analysis no longer represents an adequate bridge's behaviour. Several engineering problems are of complex nature and need a more elaborated and precise analysis. It is the case of large dimension structures and structural systems subjected to more complex physical effects (Abeche, 2015).

Due to the concrete's complex behaviour, the formulation of a complete constitutive model becomes something difficult. Models have been elaborated based on the elasticity and plasticity theories, and more recently on the damage mechanics, each giving fair responses, provided that the studied situation afford a behaviour consistent with the proposed theory (Pituba, 1998).

The simplified analysis of the interaction between vehicle and bridge problem may not guarantee the structural safety since the bridge's behaviour may change, in consequence of the dynamic effects and the degradation of its component materials.

The present work seeks to contribute for the analysis presented by Abeche (2015) in order to evaluate the dynamic interaction between a vehicular model and an *Euler-Bernoulli* nonlinear finite element beam model affected by damage mechanics, by including the consideration of variable reinforcement distribution.

2 MATHEMATICAL MODELS

In this session it is presented both the constitutive models applied in this work: the *Mazars*' damage model and the plasticity theory, relative to the concrete's and steel's behaviour, respectively. Then, it is discussed the concept of equivalent stiffness.

For this work's development, it is considered the *Euler-Bernoulli*'s beam elements within the Nonlinear Finite Element Method, using *Hermite*'s cubic interpolation functions for the bridge's model. These topics are not further explained in this paper.

2.1 *Mazars*' Constitutive Model

Among many constitutive models based on the damage mechanics and related to the concrete's behaviour, the *Mazars*' constitutive model (1984) presents an adequate representation of some experimental evidence, added the fact that it involves few parameters. When this constitutive model is applied to finite elements of *Euler-Bernoulli* beams, this model uses a scalar variable that quantifies the local state of the material's deterioration.

This model has as fundamental hypothesis (Mazars, 1984):

– Locally, the damage is consequence of stretching;

– The damage is represented by a scalar variable, whose evolution occurs when a reference value for the "equivalent stretch" is surpassed;

– The damage is considered as isotropic, although some experimental analysis indicate an anisotropic behaviour of the concrete;

– The damaged concrete behaves as an elastic medium. Therefore, permanent strain in a unload situation is disregarded.

Therefore, the equivalent deformation, defined as an scalar variable representative of the local state of extension in general constitutive relations, is calculated as shown in Eq. (1):

$$\tilde{\varepsilon} = \sqrt{\langle \varepsilon_1 \rangle_+^2 + \langle \varepsilon_2 \rangle_+^2 + \langle \varepsilon_3 \rangle_+^2} \tag{1}$$

where $\langle \varepsilon_1 \rangle_+$, $\langle \varepsilon_2 \rangle_+$ and $\langle \varepsilon_3 \rangle_+$ represent the positive part of the directional strain components.

The damage can occur due to tensile and compressive states. For the compression, the damage is caused by the concrete's crushing. For the tensile case, the damage represents the concrete's fissuration. The beginning of the fissuration process occurs when the equivalent deformation reaches a reference value, as shown in the Fig. 1:

The damage variable is defined through the linear combination (Perego, 1989):

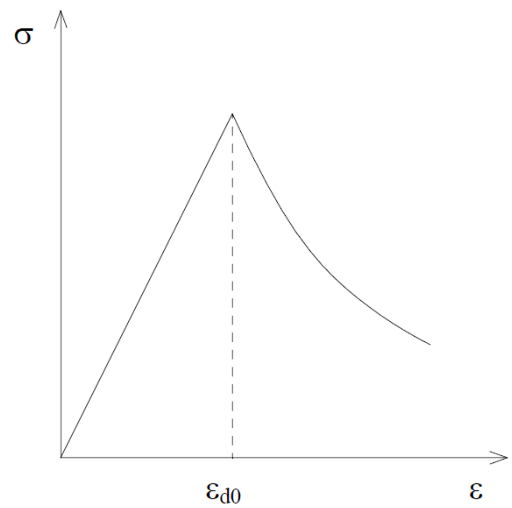

Figure 1: Stress-strain diagram for the Mazars' damage model

$$D(\varepsilon) = \alpha_T D_T + \alpha_C D_C \tag{2}$$

in which α_T and α_C are coefficients obtained through the stain state of each point of the solid, regarding that $\alpha_T + \alpha_C = 1$. For the case of pure tensile stress, it is considered $\alpha_T = 1$. For pure compression stress, $\alpha_C = 1$. D_T and D_C are obtained through the Eqs. (3) and (4). This variables represent the damage associated to the tensile and compressive in uniaxial states, respectively.

$$D_T(\widetilde{\varepsilon}) = 1 - \frac{\varepsilon_{d0}(1 - A_T)}{\widetilde{\varepsilon}} - \frac{A_T}{e^{B_T}(\widetilde{\varepsilon} - \varepsilon_{d0})}. \tag{3}$$

$$D_C(\widetilde{\varepsilon}) = 1 - \frac{\varepsilon_{d0}(1 - A_C)}{\widetilde{\varepsilon}} - \frac{A_C}{e^{B_C}(\widetilde{\varepsilon} - \varepsilon_{d0})}. \tag{4}$$

where A_T, B_T, A_C and B_C are the material's representative parameters that can be obtained from uniaxial assays (Álvares, 1999).

This work uses the damage model proposed by Abeche (2015), based on the *Mazars*' damage model, that adds the consideration of the dynamic inversion due to vibration effects.

2.2 Plasticity Theory

It is adopted a bilinear elastoplastic model which considers the strain hardening by plastic deformation, with the same behaviour when subjected to tensile and compression stresses. The Fig. 2 shows this model.

where σ_{sy} is the yield stress, E_s is the initial Young modulus, ε_{sy} is the yield extension and E_{sy} is the elastic modulus after the yield of the steel bars (Abeche *et al.*, 2016a).

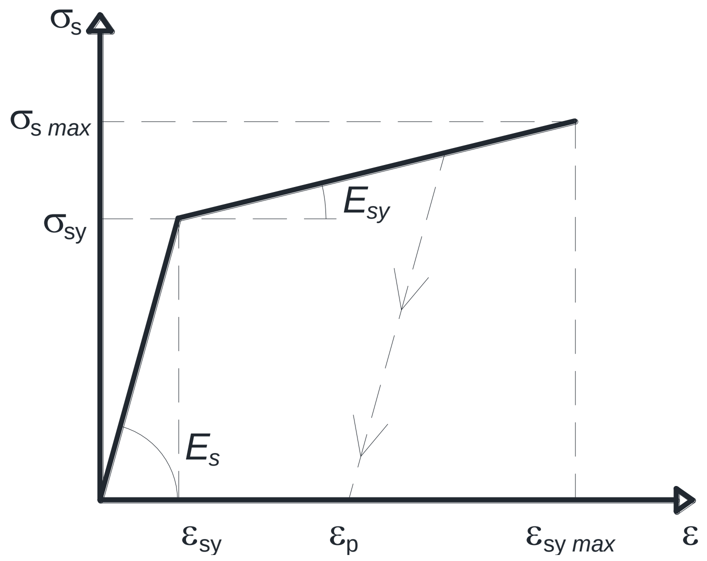

Figure 2: Steel's constitutive model

The steel can suffer plasticity effect upon reaching the yield stress σ_{sy}. Therefore, the material will have a plastic strain ε_p within the ranges $+\varepsilon_{sy} \leq \varepsilon_p \leq +\varepsilon_{sy\ max}$ for a traction case, or $-\varepsilon_{sy} \leq \varepsilon_p \leq -\varepsilon_{sy\ max}$ for a compression case. The stress of this constitutive model can be calculated according to the Eq. (5).

$$|\sigma| = \begin{cases} |\sigma_{sy}| + E_{sy}(\varepsilon_p - |\varepsilon_{sy}|), & |\pm\varepsilon_{sy}| \leq \varepsilon_p \leq |\varepsilon_{sy\,max}| \\ E_s|\varepsilon_{sy}|, & -\varepsilon_{sy} \leq \varepsilon \leq \varepsilon_{sy} \end{cases} \tag{5}$$

2.3 Equivalent Stiffness

In order to consider the beam's cross section, it is used the equivalent stiffness method (Beer & Johnston, 2008). Thereby, it is possible to divide the section into layers and apply the damage for the concrete's case, and the plastification calculations, for the steel's case, in order to modify each layer's stiffness with the suitable precision. The steel bars are modified into layers, with equivalent area and position.

The beam's cross section is divided into n layers, accordingly to the Fig. 3.

The calculation of the equivalent stiffness is written in the Eq. (6).

$$EI_{eqv} = \frac{1}{3} b \sum_{i=1}^{n} E_i (y_i^3 - y_{i-1}^3) \tag{6}$$

where n is the number of layers, b the beam's constant width, E_i the elastic modulus of the i^{th} layer, considering different materials and danification or plastification degrees. y_i and y_{i-1} are the coordinates of the y axis division (Abeche, 2015).

The refinement can be applied considering the necessity of detecting the damage in each section: those which are more susceptible to the damage effects (more distant of the

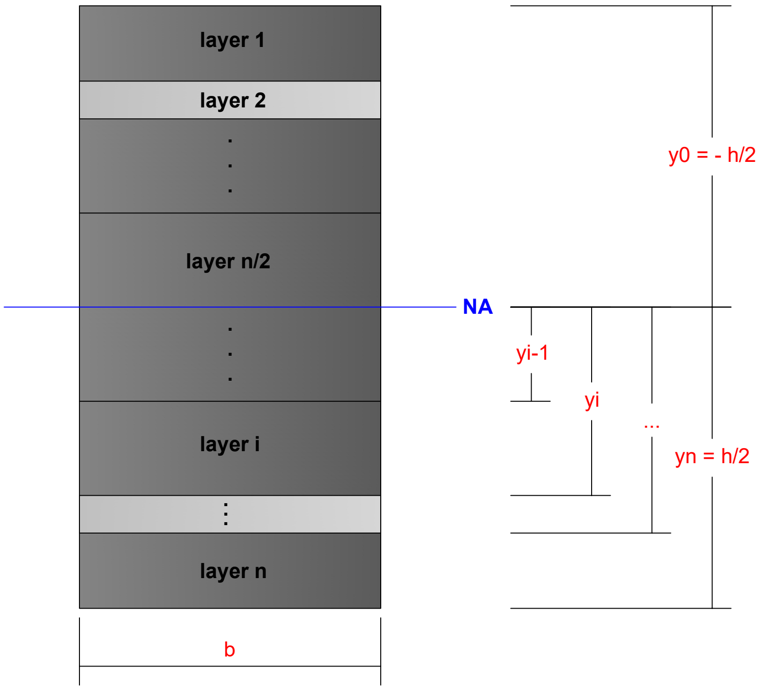

Figure 3: Beam's cross section divided into n layers (Abeche, 2015)

neutral axis), may need a better local refinement. The Fig. 4 exemplifies how the same solid can have different degrees of refinement.

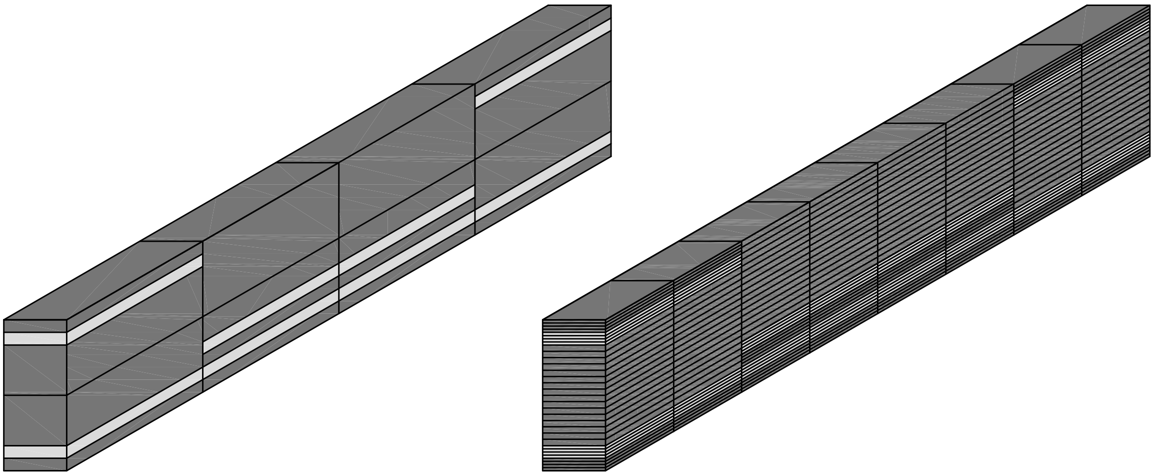

Figure 4: Different degrees refinement

3 Nonlinear Dynamics

The solution for obtaining dynamic responses in a nonlinear system, trough Finite Element Method, is the combined appliance of incremental formulations, iterative methods solution and time integration algorithms (Bathe, 1996).

Due to the physical nonlinearities of the bridge and its dynamic interaction with the vehicular model, when taken into account the damage mechanics and the plasticity, the movement equation must be solved iterative and incrementally, in order to consider the bridge's stiffness loss in the dynamic responses. Therefore, the *Newton-Raphson*'s and *Newmark*'s methods are used in conjunction, seeking the nonlinear dynamic equilibrium. Several researches involve the utilization of iterative methods. Among these, the works of Abeche (2015) and Machado (1983) can be highlighted due to its application in dynamic problems.

In a case of deterioration of the material, the forces are no longer linearly dependent of the displacements. This model considers that the damage directly affects the stiffness of the system and consequently the structural damping (Abeche *et al.*, 2016b). The stiffness matrix, thus, becomes as described in Eq. (7).

$$[K_B] = [K_B(\{u_B(\overrightarrow{x},t)\})], \quad EI = f(\overrightarrow{x},t) \tag{7}$$

It can be noted that the stiffness matrix becomes dependent of the displacements which are a functional of time and position vector.

The damping matrix, according to the *Rayleigh*'s method, becomes as written in the Eq. (8).

$$[C_B(\{u_B(\overrightarrow{x},t)\})]^{(i-1)}_{t+\Delta t} = (\tilde{\alpha}_B)^{(i-1)}_{t+\Delta t}[M_B]^{(i-1)}_{t+\Delta t} + (\tilde{\beta}_B)^{(i-1)}_{t+\Delta t}[K_B(\{u_B(\overrightarrow{x},t)\})]^{(i-1)}_{t+\Delta t} \tag{8}$$

where $\tilde{\alpha}_B$ and $\tilde{\beta}_B$ are *Rayleigh*'s coefficients for calculating the damping matrix, $t+\Delta t$ is the time increment and $i-1$ is the iteration step.

The damping matrix, due to its relation to the stiffness matrix, also becomes dependent of the time and the position vector.

The nonlinear dynamic equation, for a damaged beam, is shown in the Eq. (9).

$$\begin{aligned}&\left(\frac{[M_B]^{(i-1)}_{t+\Delta t}}{\beta\Delta t^2} + \frac{\gamma[C_B]^{(i-1)}_{t+\Delta t}}{\beta\Delta t}\right)\{u_B\}^{(i)}_{t+\Delta t} + [K_B]^{(i-1)}_{t+\Delta t}\{\Delta u\}^{(i)} = \{F^{ext}_B\}_{t+\Delta t} + \\ &\frac{[M_B]^{(i-1)}_{t+\Delta t}}{\beta\Delta t^2}(\{u_B\}_t + \{\dot{u}_B\}_t\Delta t + \left(0,5-\beta\right)\{\ddot{u}_B\}_t\Delta t^2)^{(i)} + [C_B]^{(i-1)}_{t+\Delta t}(\frac{\gamma}{\beta\Delta t} + \\ &\left(\frac{\gamma}{\beta}-1\right)\{\dot{u}_B\}_t + \left(\frac{\gamma}{2\beta}-1\right)\{\ddot{u}_B\}_t\Delta t)^{(i)} - \{F^{int}_B\}^{(i-1)}_{t+\Delta t}\end{aligned} \tag{9}$$

where β and γ are the *Newmark* method parameters determined for the numerical integration accuracy and stability, i is the present iteration, $\{F^{ext}_B\}_{t+\Delta t}$ the external forces vector, $\{F^{int}_B\}_{t+\Delta t}$ the internal forces vector and $\{\Delta u\}$ the incremental displacements vector. $[M_B]$, $[C_B]$, $[K_B]$ are the global matrices of mass, damping and stiffness, $\{u_B\}$, $\{\dot{u}_B\}$ and $\{\ddot{u}_B\}$ are the global vectors of displacements, velocities and accelerations, respectively.

It is used the ABXDNL 2.7 program (Abeche, 2015) for solving the Eq. (9), considering the damage mechanics and the plasticity theory.

4 NUMERICAL ANALYSIS

The objective of this paper is to compare the linear dynamic responses of a reinforced concrete bridge, in which a single degree of freedom vehicle passes through it, and its non-linear dynamic responses, considering the damage mechanics and the plasticity theory.

For the numerical analysis, it is used the structural design of a real bridge presented by Gorges (2012).

The bridge is composed by a symmetrical double overhanging reinforced concrete beam. The reinforcement distribution is variable through the length. There are 5 different cross section configurations.

The reinforcement distribution and their geometry parameters are shown in Fig. 5 and Table 1, respectively.

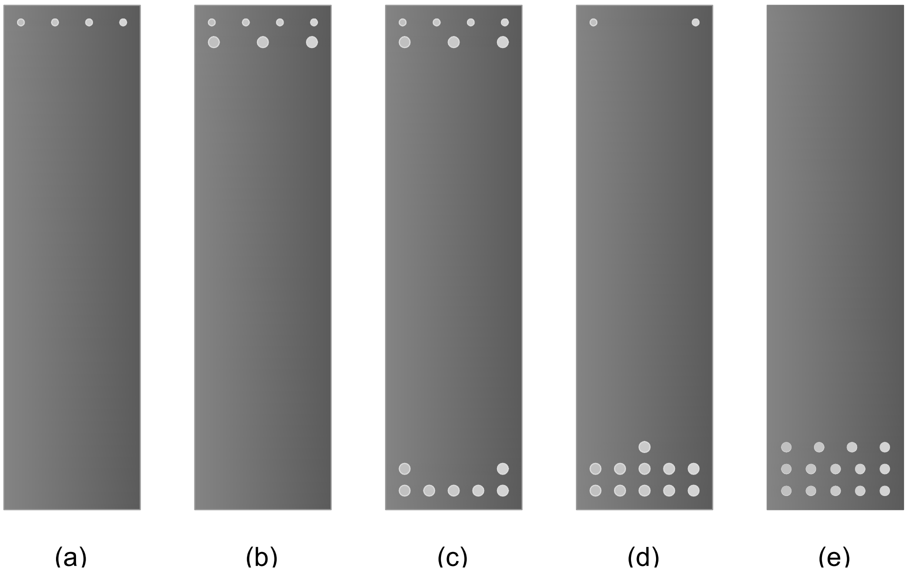

Figure 5: Beam's cross-section configuration: (a) Section 1, (b) Section 2, (c) Section 3, (d) Section 4, (e) Section 5

Table 1: Cross-section data

Section 1	Section 2	Section 3	Section 4	Section 5
$A_i = 0\ cm^2$	$A_i = 0\ cm^2$	$A_i = 87,97\ cm^2$	$A_i = 138,23\ cm^2$	$A_i = 175,93\ cm^2$
$A_s = 19,64\ cm^2$	$A_s = 57,33\ cm^2$	$A_s = 57,33\ cm^2$	$A_s = 9,82\ cm^2$	$A_s = 0\ cm^2$
$A_C = 0,923\ m^2$	$A_C = 0,919\ m^2$	$A_C = 0,910\ m^2$	$A_C = 0,910\ m^2$	$A_C = 0,907\ m^2$
$\rho = 0,213\%$	$\rho = 0,624\%$	$\rho = 1,596\%$	$\rho = 1,627\%$	$\rho = 1,939\%$
$m = 2.231\ kg/m$	$m = 2.251\ kg/m$	$m = 2.298\ kg/m$	$m = 2.300\ kg/m$	$m = 2.315\ kg/m$

The cross-sections transformed into material layer equivalent stiffness are presented in the Fig. 6.

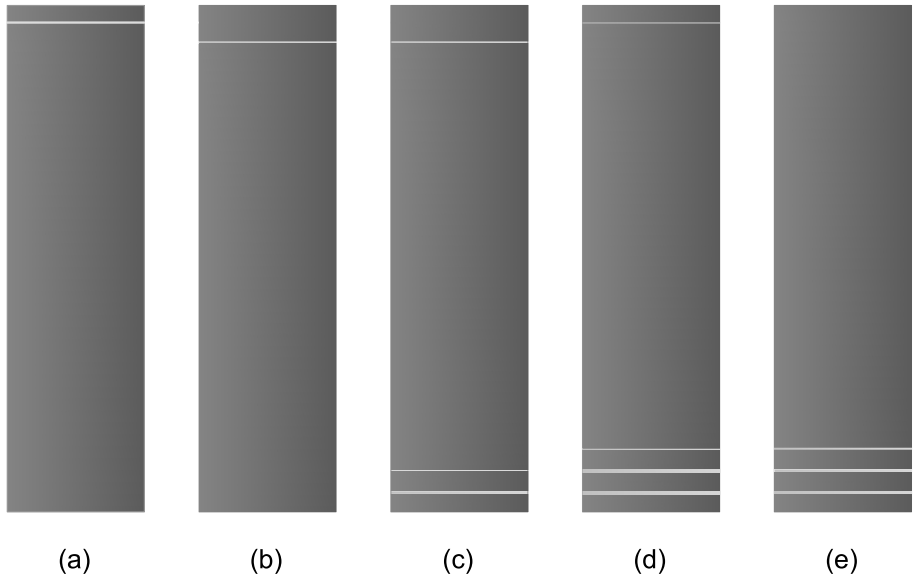

Figure 6: Beam's discretized configuration: (a) Section 1, (b) Section 2, (c) Section 3, (d) Section 4, (e) Section 5

The vehicle is a system composed by wheel mass (m_1), suspended mass (m_2), a spring with stiffness coefficient (k) and a damper with damping coefficient (c). The vehicle crosses the bridge with constant velocity.

The beam is discretized into 14 finite elements, each element with one of those 5 cross-section configurations presented in Fig. 6. The allocation of these cross-section configurations and its interaction with the vehicular model is shown in Fig. 7.

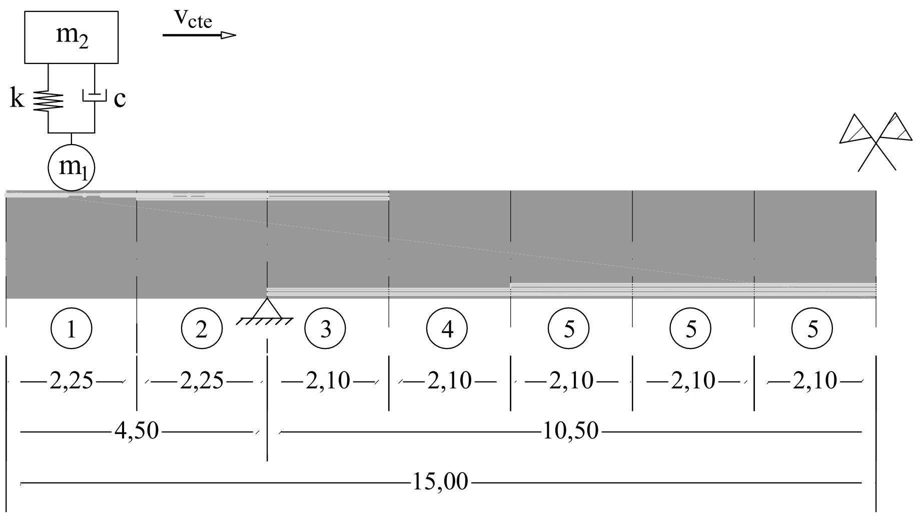

Figure 7: Bridge-vehicle system and beam's different cross section distribution

The input data for the bridge, the vehicle, the constitutive models and the *Newmark* parameters are presented in the Table 2.

Table 2: Bridge, vehicle, damage, plastcity and *Newmark* input data

Bridge	Vehicle	Damage parameters	Plasticity parameters	*Newmark* parameters
$L_B = 30\ m$	$m_1 = 4.400\ kgf$	$A_T = 0,995$	$E_s = 210\ GPa$	$\gamma = 0,5$
$b = 0,5\ m$	$m_2 = 13.200\ kgf$	$B_T = 30.000$	$\nu_s = 0,3$	$\beta = 0,25$
$h = 1,85\ m$	$k = 9.120\ kN/m$	$A_C = 1,2$	$\varepsilon_{sy} = 10‰$	2.800 Time steps
$I = 0,2638\ m^4$	$c = 96\ kNs/m$	$B_C = 1.050$		$dt = 0,001543\ s$
$E_C = 29,43\ GPa$	$v = 50\ km/h$	$\varepsilon_{d0} = 5 \cdot 10^{-5}$		$\zeta = 0,025$
$\nu_C = 0,2$	1.400 Time steps			
14 Elements				

The layer refinement adopted for this problem is as presented in Table 3. The difference in each discretization is a consequence of the different reinforcement distribution in each section.

Table 3: Discretization into layers by section

Section 1	Section 2	Section 3	Section 4	Section 5
40 layers	60 layers	100 layers	100 layers	80 layers

The outermost zones, next to the reinforcements, present a higher degree of refinement. In the case of double or triple reinforcement layers, the concrete zone between those also presents a higher degree of discretization. This local refinement is applied in order to obtain better responses from the concrete behaviour, since the damage begins at the furthermost layers, relatively to the neutral axis. However, the zone next to the neutral axis may not need such refinement, which can be useful for reaching a better computational performance.

The responses obtained in this analysis consists in 2800 time steps, which can be converted to $4,32\ s$. The initial 1400 time steps consist in the time necessary for the vehicle to cross the bridge; to it were added 1400 time steps, in order to capture the damping responses.

The vertical displacement in the midspan node, considering the linear static, linear dynamic and the nonlinear dynamic responses are shown in the Fig. 8.

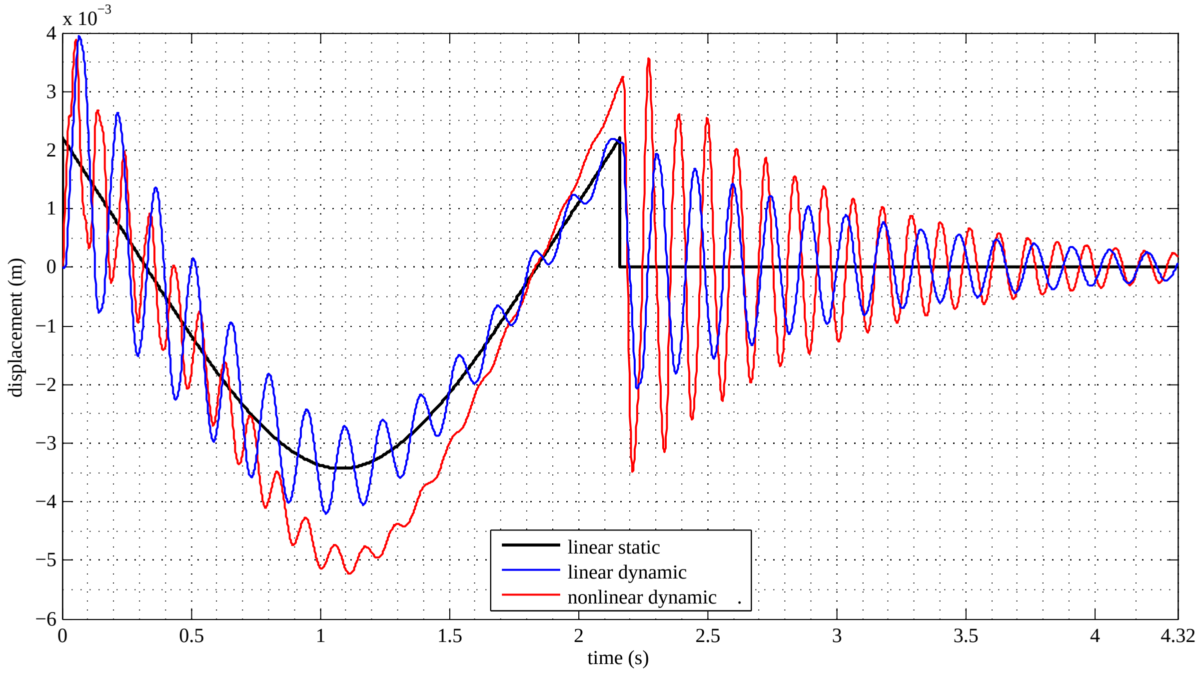

Figure 8: Dynamic response of displacement at midspan

It is notable that the linear dynamic response oscillates around the linear static. The nonlinear dynamic response presents differences in the amplitude and in the frequency of the oscillation. Both linear and nonlinear dynamic responses of velocity at the same node are presented in the Fig. 9.

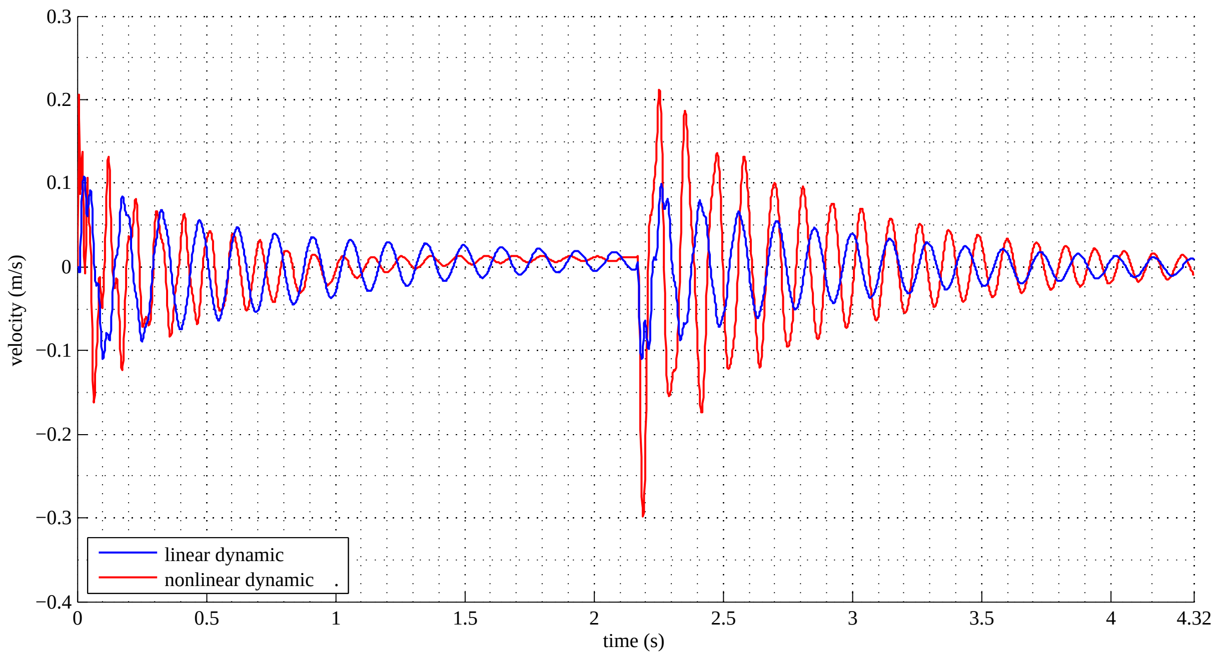

Figure 9: Dynamic response of velocity at midspan

It is observed the occurrence of velocity peaks in the periods right after the entrance of the vehicle on the bridge. The amplitude and the frequency of the velocity oscillation also varies when comparing the linear and the nonlinear cases.

The final damaged beam's configuration is presented in the Fig. 10.

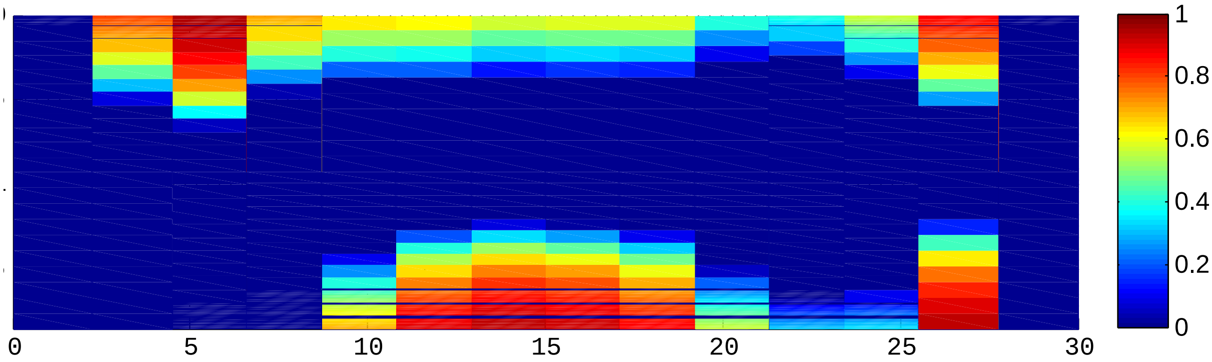

Figure 10: Beam's final damaged configuration

In order to better visualize the damage in each layer, it is presented a distorted image of the beam's damage configuration in the Fig. 11.

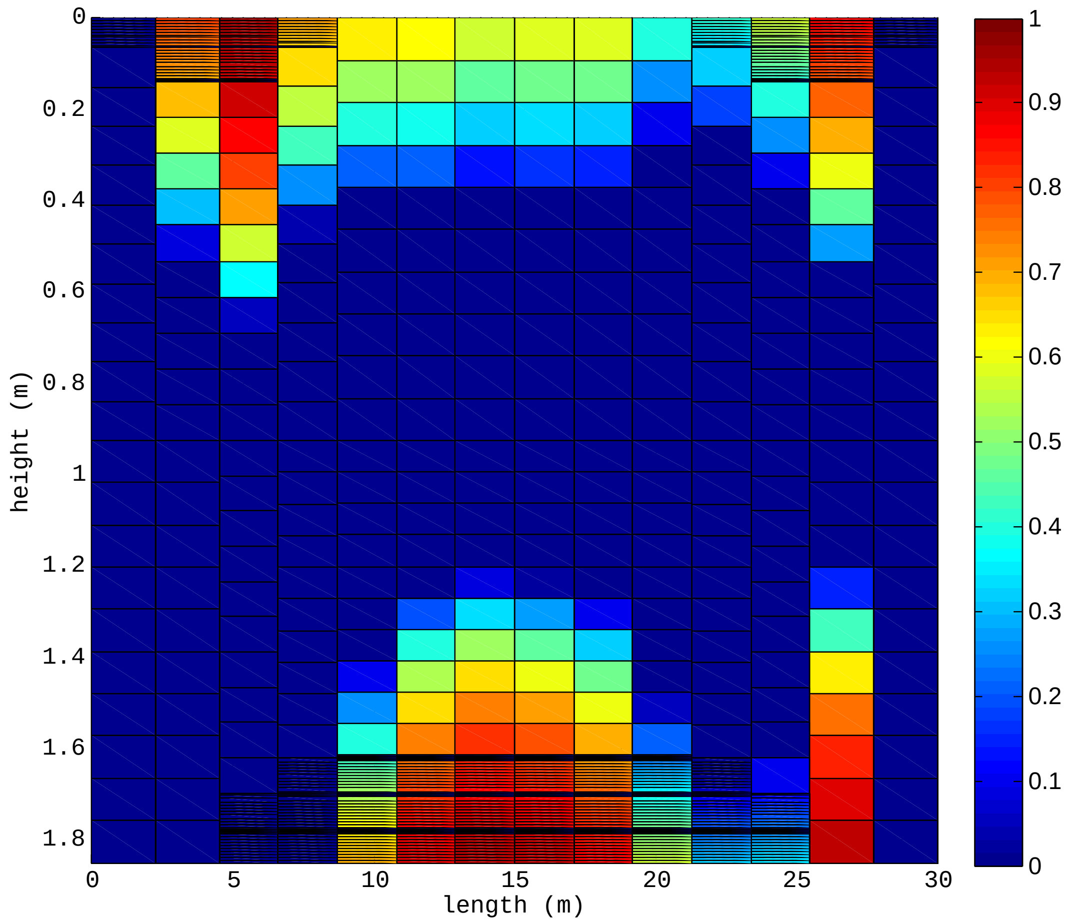

Figure 11: Amplified view of the beam's final damaged configuration

As can be observed, the damage is not applied to the reinforcement layers.

The variation on the 10 first natural frequencies, when comparing the nonlinear with the linear dynamics, is presented in the Table 4.

Table 4: Beam's natural frequencies

Natural frequencies of vibration	Linear dynamic analysis (rad/s)	Nonlinear dynamic analysis (rad/s)	Variation (%)
1^{st}	43,07973	34,91265	-18,96%
2^{nd}	137,43212	118,63736	-13,68%
3^{rd}	211,83561	181,04129	-14,54%
4^{th}	304,69787	255,26927	-16,22%
5^{th}	508,50130	428,56316	-15,72%
6^{th}	821,10405	687,86553	-16,23%
7^{th}	1210,71524	1018,37434	-15,89%
8^{th}	1590,61608	1362,14268	-14,36%
9^{th}	1810,05515	1578,41333	-12,80%
10^{th}	2043,87320	1769,60522	-13,42%

Due to the loss in the structure's stiffness, caused by the damage on the concrete, the natural frequencies suffered a reduction of almost 19%, for the first natural frequency case.

5 CONCLUSIONS

When comparing the linear static with the linear dynamic displacement responses, it can be noted that the second one oscillates around the first during the vehicle's passage over the bridge. When considering the nonlinear dynamic responses, however, it can be noted that due to the stiffness loss caused by the concrete's damage, the mid-span node presents a substantial increase in the vertical displacement.

Another important observation is the reduction of the natural frequencies of the bridge, due to the concrete's degradation. Usually, the bridge's analysis involves the investigation of its natural frequencies in order to avoid the design of structures with natural frequencies close enough to those originated by the dynamic loads, which may cause the phenomenon of resonance. When the resonance frequencies are reached, the structures have its dynamical responses amplified. Therefore, the consideration of the concrete's damage is of great importance because it alters the frequencies that the resonance phenomena occurs.

It is also notable that, although the beam presents a symmetrical geometry, the final damaged configuration does not present symmetries. Analysing the 2^{nd} and the 13^{th} elements, their final damaged configuration presents an evident difference in the bottom layers. This occurrence may be due to the lack of refinement in the bottom layers, once there is no reinforcement next to it. For a more precise damage configuration and, consequently, dynamic responses, it would be advised to increase the degree of refinement in these specific areas, and proceed a new computational analysis, in order to improve the results.

Acknowledgements

Acknowledgements to CAPES for supporting this research.

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