ANALYSIS OF ELASTOPLASTIC REISSNER'S PLATES USING THE BOUNDARY ELEMENT METHOD

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Abstract. *The present work consists of a formulation for elastoplastic analysis of plate bending using the Boundary Element Method (BEM). Reissner's theory is considered, which is valid for both thin and thick plates, admitting the occurrence of plastic strains only by flexure. The classical theory of plasticity is considered, with an initial strain procedure, and the yield criterion of von Mises is used. Integral equations are presented for displacements at internal and boundary points and for moments and shear forces at internal points. The numerical implementation is done by using discretized integral equations, employing quadratic boundary elements and constant triangular internal cells, both with linear geometry. It is used an incremental-iterative process to solve the system of equations that governs the elastoplastic problem. Some numerical examples are presented in the end of the work to illustrate the applicability of the formulation. The results are validated by comparing them with results of others researches, obtained from numerical and analytical methods.*

Keywords*: Boundary element method, Reissner's theory, Elastoplastic analysis, Plate bending.*

1 INTRODUCTION

A formulation for elastoplastic plate bending analysis with Reissner's theory by the BEM was presented by Karam & Telles (1992,1998) using the classical theory of plasticity with an initial stress procedure and considering the yield criteria of von Mises and Tresca. Other authors that have also published works using the BEM to analyze elastoplastic plates with Reissner's theory was Ribeiro & Venturini (1998), also considering initial stress procedure and Providakis & Beskos (2000), considering dynamic analysis and initial stress procedure. Supriyono & Aliabadi (2006) have employed the BEM for the analysis of shear deformable plates with combined geometric and material nonlinearities.

A formulation for elastoplastic analysis of Reissner's plate bending by the BEM is developed in the present work, by employing an initial strain procedure, instead of the initial stress procedure used in the previous references. The elastoplastic equations are solved by an incremental-iterative process. Classical theory of plasticity is considered with von Mises' yield criterion (Mendelson, 1968). The initial strain procedure considered here for plate bending with the BEM is similar to that presented in Telles & Brebbia (1981) and Telles (1983) for two- and three-dimensional problems with the BEM.

Plastic strains are considered only due to bending, together with the simplified hypothesis that, when plasticity occurs in a point of the plate, the entire transverse section plastifies simultaneously.

Cartesian index notation is used in the present text, with Greek letters representing indices that vary from 1 to 3 and Roman letters representing ones that vary from 1 to 3.

2 FORMULATION OF REISSNER'S THEORY INCLUDING PLASTIC STRAINS

Reissner's theory for plate bending (Reissner, 1945) is summary presented, in which, bending plastic strains are also included. It is admitted a plate with constant thickness h and subjected to a transverse load q, per unit area. It is also admitted that the plate material is homogeneous and isotropic.

Cartesian coordinates are represented by x_i, with x_α located at the plate middle surface and x_3 in the transverse direction. The domain of the problem, consisting of the plate middle surface, is represented by Ω and the corresponding boundary, by Γ. Generalized displacements are represented by u_i, with u_α being rotations in the $x_\alpha - x_3$ planes and u_3 being transverse displacement.

The total bending strains $\chi_{\alpha\beta}$ and the total shear strains ψ_α are expressed by

$$\chi_{\alpha\beta} = \frac{1}{2}\left(u_{\alpha,\beta} + u_{\beta,\alpha}\right) = \chi^e_{\alpha\beta} + \chi^p_{\alpha\beta} \tag{1}$$

$$\psi_\alpha = u_\alpha + u_{3,\alpha} = \psi^e_\alpha \tag{2}$$

in which $\chi^e_{\alpha\beta}$ and ψ^e_α are the elastic parcels and $\chi^p_{\alpha\beta}$ is the plastic one.

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Moments $M_{\alpha\beta}$ and shear resultants Q_α can be expressed as

$$M_{\alpha\beta} = \frac{D(1-\nu)}{2}\left[2\chi_{\alpha\beta} + \frac{2\nu}{1-\nu}\chi_{\gamma\gamma}\,\delta_{\alpha\beta}\right] + \frac{\nu\, q}{(1-\nu)\lambda^2}\,\delta_{\alpha\beta} - M^{p}_{\alpha\beta} \tag{3}$$

$$Q_\alpha = \frac{D(1-\nu)\lambda^2}{2}\psi_\alpha \tag{4}$$

where $\delta_{\alpha\beta}$ is Kronecker's delta; ν is Poisson's ratio; $\lambda = \sqrt{10}/h$ is a characteristic constant of Reissner's equations and D is the plate bending rigidity, expressed by $D = Eh^3/(12(1-\nu^2))$, with E being Young's modulus.

In Eq. (3), $M^{p}_{\alpha\beta}$ represents the components of plastic moments, defined by

$$M^{p}_{\alpha\beta} = \frac{D(1-\nu)}{2}\left[2\chi^{p}_{\alpha\beta} + \frac{2\nu}{1-\nu}\chi^{p}_{\gamma\gamma}\,\delta_{\alpha\beta}\right] \tag{5}$$

For convenience of the initial strain formulation, the elastic bending strains can be written in terms of the elastic moments, and so, the following expression arises:

$$\chi^{e}_{\alpha\beta} = \frac{1}{D(1-\nu)}\left[M_{\alpha\beta} - \frac{\nu}{1+\nu}M_{\gamma\gamma}\,\delta_{\alpha\beta}\right] - \frac{\nu\, q}{D(1-\nu^2)\lambda^2}\,\delta_{\alpha\beta} \tag{6}$$

In addition, the bending plastic strains can be written as

$$\chi^{p}_{\alpha\beta} = \frac{1}{D(1-\nu)}\left[M^{p}_{\alpha\beta} - \frac{\nu}{1+\nu}M^{p}_{\gamma\gamma}\,\delta_{\alpha\beta}\right] \tag{7}$$

By using Eq. (1), the total bending strains can be written as

$$\chi_{\alpha\beta} = \frac{1}{D(1-\nu)}\left[\left(M_{\alpha\beta} + M^{p}_{\alpha\beta}\right) - \frac{\nu}{1+\nu}\left(M_{\gamma\gamma} + M^{p}_{\gamma\gamma}\right)\delta_{\alpha\beta}\right] - \frac{\nu\, q}{D(1-\nu^2)\lambda^2}\,\delta_{\alpha\beta} \tag{8}$$

3 CONSTITUTIVE EQUATIONS

It is admitted the existence of a yield function F expressed in terms of moments M_{ij} and a hardening parameter κ. During the loading that produces yielding, the moments M_{ij} must stay on the yield surface, so that the following equation is satisfied:

$$F(M_{ij}, \kappa) = f(M_{ij}) - \Psi(\kappa) = M_e - M_o = 0 \tag{9}$$

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where M_e is the equivalent or effective moment, calculated here following von Mises' yield criterion, and M_o is the uniaxial yield moment, calculated by considering that, whenever the equivalent moment, at any point, reaches the yield moment, the entire transverse section plastifies simultaneously, as $M_o = \sigma_o h^2/4$, with σ_o being the uniaxial yield stress of the material.

The equivalent or effective moment M_e is calculated from the following equation:

$$M_e = \sqrt{3J_2^{'}} \tag{10}$$

in which $J_2^{'}$ is the second invariant of the stress deviator stress tensor, given by

$$J_2^{'} = \frac{1}{2} M_{ij}^{'} M_{ij}^{'} \tag{11}$$

with $M_{ij}^{'}$ being the "deviatoric moments" calculated as

$$M_{ij}^{'} = M_{ij} - \frac{1}{3}\delta_{ij} M_{kk} \tag{12}$$

For the solution procedure, it is necessary to calculate the plastic strain differentials or increments. By taking into account the Prandtl-Reuss equations (Mendelson, 1968), adequate relations for the determination of plastic strain increments can be expressed, in this case, as

$$d\chi_{ij}^{p} = M_{ij}^{'} d\gamma \tag{13}$$

in which $d\gamma$ is a positive proportionality factor, that may vary during the plastic deformation process.

It is also defined an equivalent or effective plastic strain increment, as

$$d\chi_e^p = \sqrt{\frac{2}{3} d\chi_{ij}^p \, d\chi_{ij}^p} \tag{14}$$

By squaring both sides of Eqs. (13) and (14) and substituting the last result into the first one, it results

$$\left(\frac{3}{2} d\chi_e^p\right)^2 = 3J_2^{'} \, d\gamma^2 \tag{15}$$

Taking into account the expression of von Mises' criterion for the equivalent moment, given by Eq. (10), the proportionality factor $d\gamma$ can be expressed in terms of M_e and $d\chi_e^p$, as follows:

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$$d\gamma = \frac{3}{2}\,\frac{d\chi_e^p}{M_e} \tag{16}$$

For the application of the BEM with initial strain procedure, the plastic strain increments are calculated with the expressions presented before, admitting that a path of load is defined in order to achieve a certain state of stresses and accumulated plastic strains χ_{ij}^p.

It is considered that small load increments produce additional plastic strains $\Delta\chi_{ij}^p$. Then, the total strains given in Eq. (1) can be written as

$$\chi_{ij} = \chi_{ij}^e + \chi_{ij}^p + \Delta\chi_{ij}^p \tag{17}$$

in which χ_{ij}^e includes the current load increment.

For convenience, a modified total strain can be defined in the form

$$\hat{\chi}_{ij} = \chi_{ij} - \chi_{ij}^p \tag{18}$$

The expression in Eq. (18) is simply

$$\hat{\chi}_{ij} = \chi_{ij}^e + \Delta\chi_{ij}^p \tag{19}$$

or, using Eq. (6),

$$\hat{\chi}_{ij} = \frac{1}{D(1-\nu)}\left[M_{ij} - \frac{\nu}{1+\nu}M_{kk}\,\delta_{ij}\right] - \frac{\nu\,q}{D(1-\nu^2)\lambda^2}\,\delta_{ij} + \Delta\chi_{ij}^p \tag{20}$$

Equation (20) can also be written in deviatoric form, by neglecting the hydrostatic parcel (i.e., $\Delta\chi_{kk}^p = 0$), as follows:

$$\chi_{ij}' = \frac{M_{ij}'}{D(1-\nu)} + \Delta\chi_{ij}^p \tag{21}$$

in which

$$\chi_{ij}' = \hat{\chi}_{ij} - \frac{1}{3}\delta_{ij}\,\hat{\chi}_{kk} \tag{22}$$

By using Eq. (13), one can write Eq. (21) as

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$$\chi'_{ij} = \left(1 + \frac{1}{D(1-\nu)\Delta\gamma}\right)\Delta\chi^{p}_{ij} \tag{23}$$

By squaring both sides of Eq. (23) in a similar manner to that employed to obtain Eq. (16), the following relation arises:

$$1 + \frac{1}{D(1-\nu)\Delta\gamma} = \frac{\chi_{et}}{\Delta\chi^{p}_{e}} \tag{24}$$

in which χ_{et} is the equivalent strain related to the modified total strain tensor $\hat{\chi}_{ij}$ and can be related to deviator strain tensor χ'_{ij}, as

$$\chi_{et} = \sqrt{\frac{2}{3}\,\chi'_{ij}\,\chi'_{ij}} \tag{25}$$

Substituting Eq. (24) into Eq. (23), one obtains

$$\Delta\chi^{p}_{ij} = \frac{\Delta\chi^{p}_{e}}{\chi_{et}}\,\chi'_{ij} \tag{26}$$

From Eq. (26), one can observe that in order to determine the actual magnitudes of plastic strain increments, the equivalent plastic strain increment must be determined. Therefore, substituting the proportionality factor given in Eq. (16) into Eq. (24) and considering that the condition given in Eq. (9) must be satisfied during the entire plastic process, M_e can be substituted by M_o and so, one obtains

$$\Delta\chi^{p}_{e} = \chi_{et} - \frac{2M_o}{3D(1-\nu)} \tag{27}$$

Notice that M_o corresponds to the uniaxial yield moment acting after the application of the current load increment and, therefore, it is still unknown. This term, however, can be approximated by a Taylor's series, truncated about the previous value of M_o (i.e., before the load increment application), as

$$^{k}M_o = {}^{k-1}M_o + {}^{k-1}H'\,\Delta\chi^{p}_{e} + \ldots \tag{28}$$

with H' being the slope of the effective moment versus effective plastic strain curve and it can be determined, for linear hardening, by

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$$H^{'} = \frac{\Delta M_e}{\Delta \chi_e^p} = \frac{IE_T}{1 - \frac{IE_T}{EI}} \tag{29}$$

in which E_T is the tangent modulus.

By substituting Eq. (28) into Eq. (27) and solving for the equivalent plastic strain increment $\Delta\chi_e^p$, one can observe that this increment can be obtained from the uniaxial moment-strain curve, as follows:

$$\Delta\chi_e^p = \frac{3D(1-\nu)\chi_{et} - 2M_o}{3D(1-\nu) + 2H^{'}} \tag{30}$$

in which the values of M_o and $H^{'}$ are calculated before the load increment.

4 INTEGRAL EQUATIONS FOR DISPLACEMENTS IN ELASTOPLASTIC PLATES

Integral equation for the generalized displacements at a source point ξ of the domain Ω can be obtained from a weighted residual procedure or from Betti's second theorem. After taking the limits of the integrals of the resulting equation as point ξ tends to the boundary Γ, three equations of the following form can be written:

$$\begin{aligned} C_{ij}\, u_j(\xi) &= \int_\Gamma u_{ij}^*(\xi,x)\, p_j(x)\; d\Gamma(x) - \int_\Gamma p_{ij}^*(\xi,x)\, u_j(x)\, d\Gamma(x) \\ &+ \int_\Omega \left[u_{i3}^*(\xi,x) - \frac{\nu}{(1-\nu)\lambda^2}\, u_{i\alpha,\alpha}^*(\xi,x) \right] q(x)\; d\Omega(x) + \int_\Omega M_{\alpha\beta i}^*(\xi,x)\; \chi_{\alpha\beta}^p(x)\, d\Omega(x) \end{aligned} \tag{31}$$

which is valid for internal points with $C_{ij} = \delta_{ij}$ and also for boundary points, with $C_{ij} = \delta_{ij}/2$ in case of smooth boundaries. In Eq. (31), u_{ij}^*, p_{ij}^* and $M_{\alpha\beta i}^*$ are terms corresponding to the fundamental solution and represent the response, in a field point x, due to a unit generalized concentrated load applied in direction i of a source point ξ. In addition, the second integral in the right-hand side must be interpreted as a Cauchy's principal value sense.

Expressions for the fundamental solution u_{ij}^* and the corresponding tractions p_{ij}^* used in this work are presented in Van der Weeën (1982) and Karam & Telles (1988), for elastic analysis of Reissner's plates by the BEM.

The expressions obtained for $M_{\alpha\beta i}^*$, which multiply the plastic strains, are presented in Oliveira & Karam (2016).

The domain integral which contains the transverse load in Eq. (31) can be transformed into boundary integral for several kinds of loading. This transformation is done in the present work for a uniformly distributed load, by considering $q(x) = q =$ constant and employing the divergence theorem. So, one obtains

$$C_{ij}(\xi)u_j(\xi)=\int_\Gamma u^*_{ij}(\xi,x)\,p_j(x)\,d\Gamma(x)-\int_\Gamma p^*_{ij}(\xi,x)\,u_j(x)\,d\Gamma(x)$$
$$+\int_\Gamma q\left[v^*_{i,\alpha}(\xi,x)-\frac{\nu}{(1-\nu)\lambda^2}\,u^*_{i\alpha}(\xi,x)\right]n_\alpha(x)\,d\Gamma(x)+\int_\Omega M^*_{\alpha\beta i}(\xi,x)\,\chi^p_{\alpha\beta}(x)\,d\Omega(x) \tag{32}$$

in which v^*_i satisfy the equation $v^*_{i,\alpha\alpha}=u^*_{i,3}$.

Expressions for v^*_i and related derivatives are presented in Van der Weeën (1982) and Karam & Telles (1988).

5 INTEGRAL EQUATIONS FOR MOMENTS AND SHEAR FORCES AT INTERNAL POINTS

Moments and shear forces at internal points are calculated by substituting the displacements given by Eq. (32), with $C_{ij}=\delta_{ij}$, in Eqs. (1) and (2) and then, substituting the resulting expressions in Eqs. (3) and (4). It should be mentioned that the derivatives are taken in this case with respect to the coordinates of point ξ.

For the first three integrals of Eq. (32), the differentiation can be applied directly to the tensors that correspond to the fundamental solution. However, special considerations are necessary for the integral related to the plastic tensor.

By using a procedure similar to that presented in Telles (1983), the following equations are obtained:

$$M_{\alpha\beta}(\xi)=\int_\Gamma u^*_{\alpha\beta k}\,p_k\,d\Gamma-\int_\Gamma p^*_{\alpha\beta k}\,u_k\,d\Gamma+q\int_\Gamma w^*_{\alpha\beta}\,d\Gamma$$
$$+\int_\Omega M^*_{\alpha\beta\gamma\theta}\,\chi^p_{\gamma\theta}\,d\Omega+\frac{\nu q}{(1-\nu)\lambda^2}\,\delta_{\alpha\beta}-\frac{D(1-\nu^2)}{8}\left(2\,\chi^p_{\alpha\beta}+\delta_{\alpha\beta}\,\chi^p_{\theta\theta}\right) \tag{33}$$

$$Q_\beta=\int_\Gamma u^*_{3\beta k}\,p_k\,d\Gamma-\int_\Gamma p^*_{3\beta k}\,u_k\,d\Gamma+q\int_\Gamma w^*_{3\beta}\,d\Gamma+\int_\Omega M^*_{3\beta\gamma\theta}\,\chi^p_{\gamma\theta}\,d\Omega \tag{34}$$

It should be noticed that the last integrals in Eqs. (33) and (34) are considered in Cauchy's principal value sense and that Eq. (33) has a free term containing plastic strains.

Expressions for the tensors $u^*_{i\beta k}$, $p^*_{i\beta k}$ and $w^*_{i\beta}$ are presented in Van der Weeën (1982) and Karam & Telles (1988).

Expressions for the tensor $M^*_{i\beta\gamma\theta}$ are presented in Oliveira & Karam (2016) and can be obtained by considering that they are the terms that multiply the plastic strains when one substitutes the expressions of displacements at internal points and their derivatives into the expressions of stress resultants.

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6 NUMERICAL IMPLEMENTATION

The integral equations for displacements (Eq. (32)) and for stress resultants (Eqs. (33) and (34)) can be written in discretized form.

It is considered that the boundary of the plate is divided into continuous or discontinuous quadratic boundary elements, with linear geometry. In addition, constant triangular internal cells, also with linear geometry, are admitted in the parts of the domain where the existence of plastic strains are expected.

By applying Eq. (32), in discretized form, for all nodal boundary points, one obtains a system with a number of equations equal to three times the number of nodes, which can be written as

$$\mathbf{H}\,\mathbf{U} = \mathbf{G}\,\mathbf{P} + \mathbf{B} + \mathbf{D}\,\boldsymbol{\chi}^{p} \tag{35}$$

in which $\mathbf{U}$ is the vector of nodal displacements, $\mathbf{P}$ is the vector of nodal tractions, $\mathbf{B}$ is the vector that contains the influence of the distributed load, $\boldsymbol{\chi}^{p}$ is the vector that contains the plastic strains at cell points, $\mathbf{H}$ and $\mathbf{G}$ are square matrices generated by the integrals over the boundary elements and $\mathbf{D}$ is the matrix generated by the domain integrals which multiply the plastic strains.

In addition, with the application of Eqs. (33) and (34), in discretized form, to all internal points located at the geometric center of the cells, one obtains

$$\mathbf{M} = \mathbf{G}'\,\mathbf{P} - \mathbf{H}'\,\mathbf{U} + \left(\mathbf{W}' + \mathbf{V}'\right) + \left(\mathbf{D}' + \mathbf{C}'\right)\boldsymbol{\chi}^{p} \tag{36}$$

$$\mathbf{Q} = \mathbf{G}''\,\mathbf{P} - \mathbf{H}''\,\mathbf{U} + \mathbf{W}'' + \mathbf{D}''\,\boldsymbol{\chi}^{p} \tag{37}$$

in which matrices $\mathbf{G}'$, $\mathbf{G}''$, $\mathbf{H}'$ and $\mathbf{H}''$ vectors $\mathbf{W}'$ and $\mathbf{W}''$ contain boundary integrals related to the fundamental solution; $\mathbf{V}'$ contains the free term related to the transverse load; $\mathbf{D}'$ and $\mathbf{D}''$ contain domain integrals that multiply the plastic strains and $\mathbf{C}'$ represents the free term related to the plastic strains.

By considering the boundary conditions, the system represented by Eq. (35) can be reordered to the following form:

$$\mathbf{A}\,\mathbf{y} = \mathbf{f} + \mathbf{D}\,\boldsymbol{\chi}^{p} \tag{38}$$

and, analogously, Eqs. (36) and (37) become, respectively,

$$\mathbf{M} = -\mathbf{A}'\,\mathbf{y} + \mathbf{f}' + \mathbf{D}^{*}\boldsymbol{\chi}^{p} \tag{39}$$

$$\mathbf{Q} = -\mathbf{A}''\,\mathbf{y} + \mathbf{f}'' + \mathbf{D}''\,\boldsymbol{\chi}^{p} \tag{40}$$

with $\mathbf{D}^{*} = \mathbf{D}' + \mathbf{C}'$.

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In Eqs. (38) to (40), $\mathbf{f}$, $\mathbf{f'}$ and $\mathbf{f''}$ are vectors that contain the prescribed values, including the influence of the transverse loading.

The pre-multiplication of Eq. (38) by $\mathbf{A}^{-1}$ leads to

$$\mathbf{y} = \mathbf{K}\boldsymbol{\chi}^{\mathrm{p}} + \mathbf{m} \tag{41}$$

in which $\mathbf{K} = \mathbf{A}^{-1}\,\mathbf{D}$ and $\mathbf{m} = \mathbf{A}^{-1}\,\mathbf{f}$.

The substitution of Eq. (41) into Eqs. (39) and (40) leads to

$$\mathbf{M} = \mathbf{S'}\boldsymbol{\chi}^{\mathrm{p}} + \mathbf{n'} \tag{42}$$

$$\mathbf{Q} = \mathbf{S''}\boldsymbol{\chi}^{\mathrm{p}} + \mathbf{n''} \tag{43}$$

where $\mathbf{S'} = \mathbf{D}^* - \mathbf{A'}\mathbf{K}$, $\mathbf{S''} = \mathbf{D''} - \mathbf{A''}\mathbf{K}$, $\mathbf{n'} = \mathbf{f'} - \mathbf{A'}\mathbf{m}$ and $\mathbf{n''} = \mathbf{f''} - \mathbf{A''}\mathbf{m}$.

It is important to mention that vectors $\mathbf{m}$, $\mathbf{n'}$ and $\mathbf{n''}$ represent the elastic solution to the problem, with $\mathbf{m}$ containing boundary displacements and tractions and vectors $\mathbf{n'}$ and $\mathbf{n''}$ containing, respectively, moments and shear resultants at internal points.

For calculating regular integrals, Gaussian quadrature is used, whereas special procedures are adopted for singular integrals.

For the case of logarithmic singularities of boundary integrals, a quadratic transformation over the integration point coordinates is employed.

The coordinates of the cell points are defined in terms of triangular coordinates. For convenience, a system of polar coordinates (r, ϕ) is defined, centered at the source point ξ . In this case, for singular integrals, Kutt's quadrature (1975) is used.

For the case of quasi-singular integrals, which may occur in the boundary integrals when the source and the field points are very close, a coordinate transformation of the third degree developed by Telles (1987) is employed.

7 SOLUTION TECHNIQUE FOR THE ELASTOPLASTIC PROBLEM

The incremental process starts with the reduction of the maximum equivalent moment $M_e^{\max}$, calculated at the cell points, to the initial yield moment M_o. Then, an initial load factor is calculated by

$$\mu_o = \frac{M_o}{M_e^{\max}} \tag{44}$$

Subsequent values of the load factor for the incremental process are calculated by using the following recursive equation:

$$\mu_i = \mu_{i-1} + \Delta\mu_i \tag{45}$$

where $\Delta\mu_i$ is the increment defined as a percentage β_i in terms of the load at the first yield, as $\Delta\mu_i = \beta_i \ \mu_o$.

In order to perform an incremental procedure, Eqs. (41), (42) and (43) can be written, respectively, as

$$\mathbf{y} = \mathbf{K}\left(\boldsymbol{\chi}^{\mathbf{p}} + \boldsymbol{\Delta\chi}^{\mathbf{p}}\right) + \mu_i \mathbf{m} \tag{46}$$

$$\mathbf{M} = \mathbf{S}'\left(\boldsymbol{\chi}^{\mathbf{p}} + \boldsymbol{\Delta\chi}^{\mathbf{p}}\right) + \mu_i \mathbf{n}' \tag{47}$$

$$\mathbf{Q} = \mathbf{S}''\left(\boldsymbol{\chi}^{\mathbf{p}} + \boldsymbol{\Delta\chi}^{\mathbf{p}}\right) + \mu_i \mathbf{n}'' \tag{48}$$

with $\boldsymbol{\chi}^{\mathrm{p}}$ representing the total accumulated plastic strains up to, but not including, the current plastic strain increment $\boldsymbol{\Delta\chi}^{\mathrm{p}}$, that is calculated iteratively.

For each value of μ_i, the plastic strain increment is calculated iteratively at each considered internal cell point, by following a solution technique for an initial strain formulation, analogous to that presented in Telles (1983).

8 NUMERICAL APPLICATIONS

Two examples are presented to show the validity of the present formulation and the results are compared to ones obtained from other numerical or analytical approaches.

8.1 Example 1: simply supported circular plate

In this example, a simply supported circular plate with radius $a = 10.0\,in$ (254 *mm*) and thickness $h = 1.0\,in$ (25.4 *mm*), subjected to a uniform load q is considered. The material has Poisson's ratio $\nu = 0.24$, Young's modulus $E = 10^4\,ksi$ (6.867x10^4 *MPa*) and uniaxial yield stress $\sigma_0 = 16.0\,ksi$ (109.872 *MPa*). An ideally plastic material ($H^{'} = 0$) is admitted and $\beta_i = 1\%$ is considered.

Due to the plate symmetry, only one quarter of the plate is discretized. In Fig. 1(a), mesh 1 is presented, in which 20 boundary elements and 50 internal cells are employed, and the same discretization was presented in Karam & Telles (1998) for this problem. Fig. 1(b) presents another discretization used, mesh 2, with 36 boundary elements and 162 internal cells, in which smaller elements and cells are used as they approach to the plate center.

This problem was also analyzed, with von Mises' yield criterion, by Armen Jr. *et al* (1970), employing the Finite Element Method (FEM) with a domain discretization into 50 linear triangular elements, and by Hopkins & Wang (1954), by performing a limit analysis.

In the present work with mesh 1, the last value of ρ for which the program converges is 6.62 and it diverges for 6.67; with mesh 2, the last value of ρ for which the program converges is 6.49 and it diverges for 6.54. The same values were obtained by Karam & Telles (1998) for the same discretizations.

The collapse load value of $\rho = 6.50$ was presented in Armen Jr. *et al.* (1970) and the limit load value of $\rho = 6.51$ was obtained in Hopkins & Wang (1954).

In Fig. 2, the load-deflection curves for the point situated at the plate center, obtained with the present work for mesh 1 and mesh 2, can be seen in comparison to those obtained by Karam & Telles (1998) and, in addition, the limit load value calculated by Hopkins & Wang (1954) is presented.

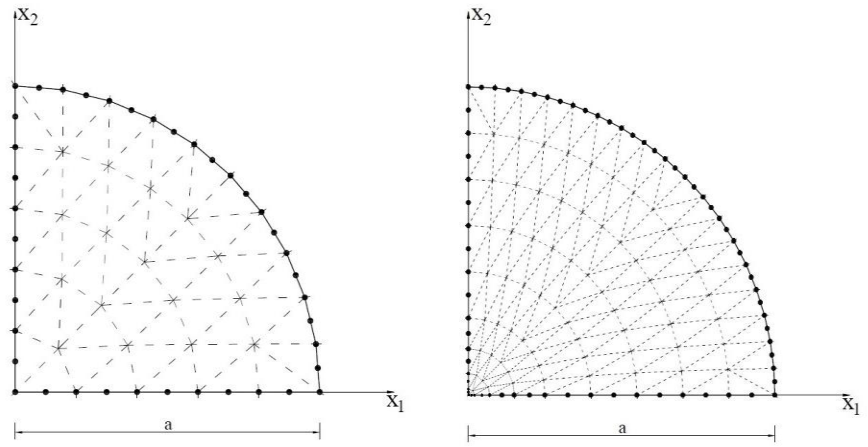
One can observe that the load-deflection curves obtained with the present work are in excellent agreement with the ones obtained in Karam & Telles (1998) and also that the limit load values obtained here are in excellent accordance with those presented in the cited references.

8.2 Example 2: simply supported square plate

A simply supported square plate with side $l = 1.0$ and thickness $h = 0.01$, subjected to a uniform load q is considered. The material is ideally plastic, with Poisson's ratio $\nu = 0.3$, Young's modulus $E = 10.92$ and uniaxial yield stress $\sigma_0 = 1600$. It was adopted $\beta_i = 1\%$.

This example was also analyzed by Owen & Hinton (1980), utilizing the Finite Element Method (FEM), and by Karam & Telles (1992), employing the BEM, and all these works have used initial stress approaches. Only one quarter of the plate was discretized, due to symmetry conditions. Fig. 2(a) presents the discretization used in the present work, consisting of 8 boundary elements and 16 internal cells, and it is the same to that used in Karam & Telles (1992). Fig. 2(b) presents the discretization employed in Owen & Hinton (1980).

Fig. 3 shows the load-deflection curves corresponding to the point situated at the plate center, obtained with the results of the present work in comparison to results obtained in the mentioned references, also using von Mises' yield criterion. One can observe that these results have a very good accordance.



(a) (b)

Figure 1. Discretization for the circular plate with boundary elements and internal cells: (a) mesh 1; (b) mesh 2

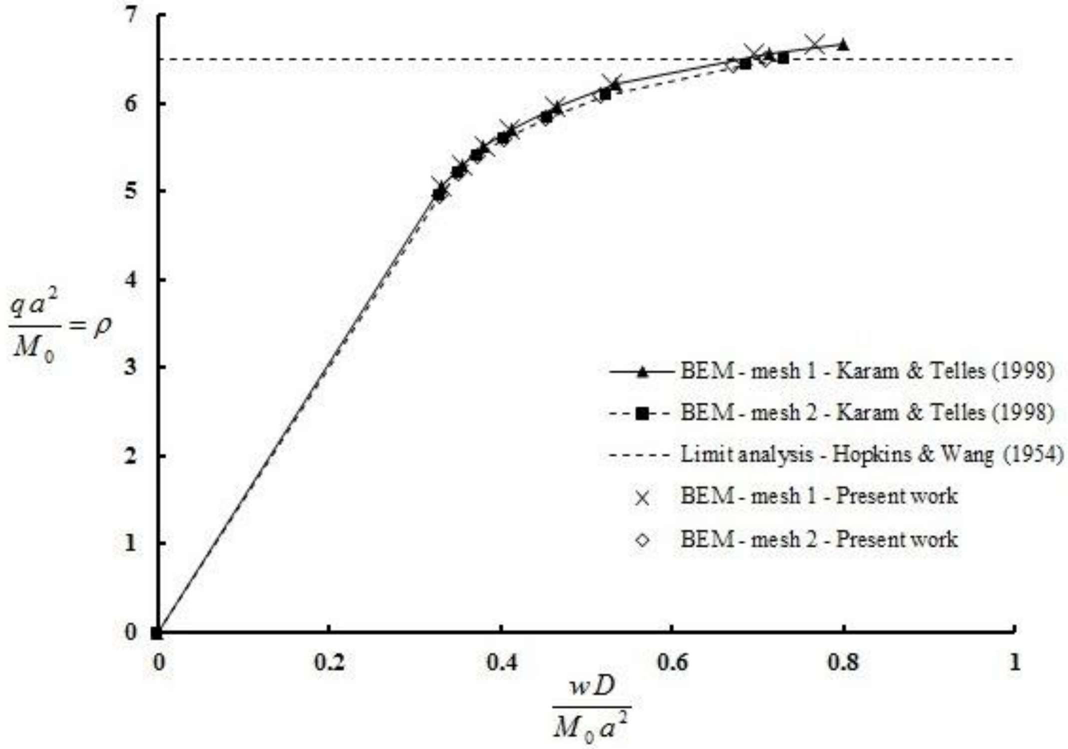


Figure 2. Load-deflection curve for the circular plate

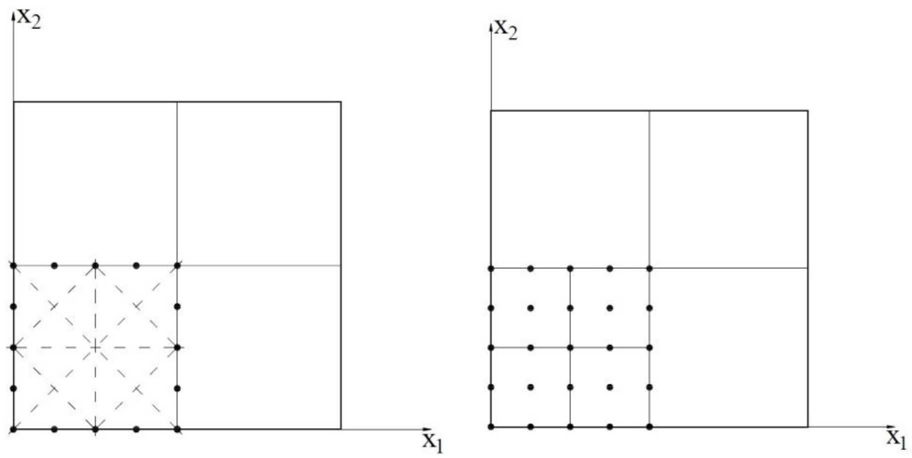


Figure 3. Discretization for the square plate

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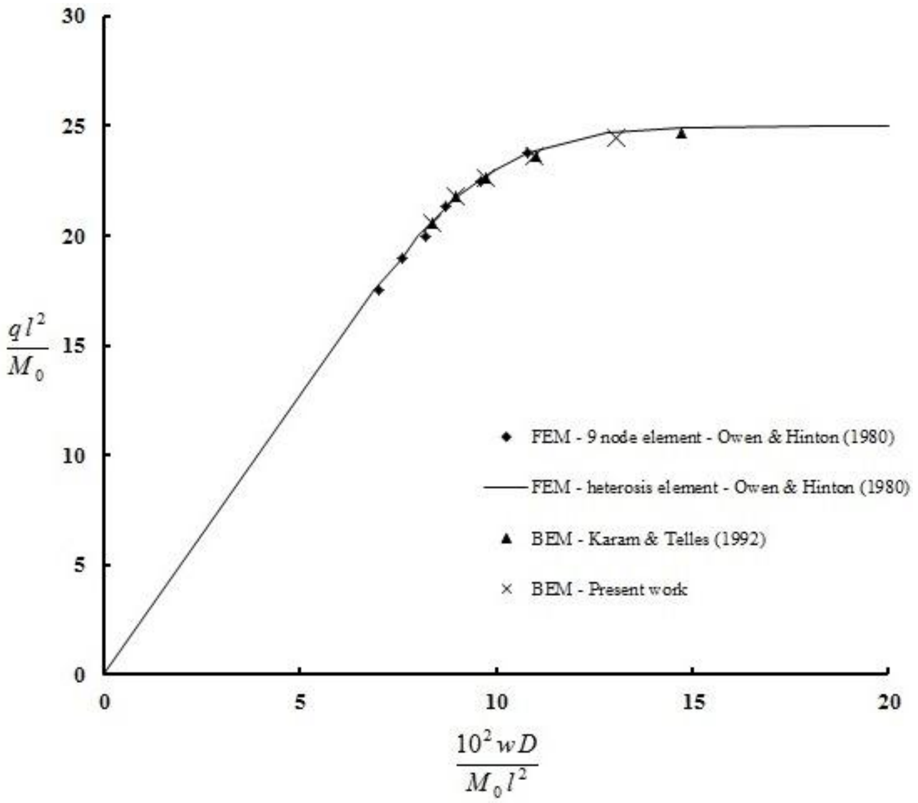

Figure 4. Load-deflection curve for the square plate

9 CONCLUSIONS

A formulation for elastoplastic analysis of Reissner's plates using the BEM was presented in the present work, employing an initial strain approach. An incremental-iterative procedure was considered and von Mises' yield criterion was employed.

The present results, with initial strain approach, were compared with results obtained in others works with initial stress approaches of the BEM and the FEM and also with analytical results. It could be observed the very good accordance of them.

It can be also mentioned that the simplification which considers that the entire transverse section plastifies simultaneously leads to a very good results.

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