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POTENTIAL AND ELASTOSTATIC ANALYSIS IN NON HOMOGENEOUS TWO-DIMENSIONAL DOMAINS USING THE BOUNDARY ELEMENT METHOD

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Abstract. *This paper aims to present a boundary element analysis of potential and elastostatic problems in non-homogeneous materials implemented in a computational platform developed in MATLAB. Mixed meshes composed by continuous, discontinuous and partially discontinuous isoparametric quadratic elements were used. The nodes that should be shifted from the ends of the elements were defined based on strategies embedded into the pre-processor, which aim to avoid limitations in the spectrum of problems that can be exploited, without considerable increase of the computational cost. The singular integrals of the quadratic elements were treated by using the process of separation in singular and non-singular parts and carried out in a general way to cover elements with and without discontinuities. The Subregions Technique was applied in the treatment of domains that consist of zones of different materials. Some classical test problems have been examined, as well as its graphical results, generated by the post-processor. The numerical results, when compared to available analytical solutions and approximate solutions provided by ANSYS, demonstrated great accuracy when using the Finite Element Method.*

Keywords: *Computational implementation, Boundary element method, Subregions technique, Two-dimensional analysis*

1 INTRODUCTION

The main characteristics of the Boundary Element Method (BEM) is the reduction of the dimension of the problem and, consequently, of the mesh used, since only the boundary must be discretized. This reduction makes the method especially suitable for the analysis of domains composed of regions of different materials. The fact there are no approximations in the domain, the method presents great precision in problems whose variables have a large gradient, such as interfaces of materials with considerable differences in rigidity (Cordeiro, 2015).

Therefore, this study sought to use BEM and the Subregions Technique in the analysis of potential and elastostatic problems in two-dimensional domains composed of regions with different materials.

2 BOUNDARY ELEMENT METHOD FOR POTENTIAL PROBLEMS

Potential problems are those which can be described by a potential function that satisfies the Poisson equation or its homogeneous form: the Laplace equation. These equations have great importance in engineering, as they govern problems related to heat conduction, convective heat transfer, torsion of bars, ideal fluid flow, flow through porous media, electrostatic, among others. The boundary element analysis of potential problems is already well established, being presented in several works of technical literature, such as Brebbia, Telles, and Wrobel (1984), Becker (1992), Beer, Smith, and Duenser (2008) and Kane (1994). The development of this method, according to Brebbia and Dominguez (1992), can be traced back to the researches of Jaswon (1963) and Symm (1963).

In a problem governed by the Laplace equation, which is assumed valid along a domain Ω surrounded by a boundary Γ (Figure 1), the boundary integral equation for boundary points, as seen in Brebbia and Dominguez (1992), is given by:

$$c^i u^i + \int_\Gamma u q^* d\Gamma = \int_\Gamma q u^* d\Gamma \tag{1}$$

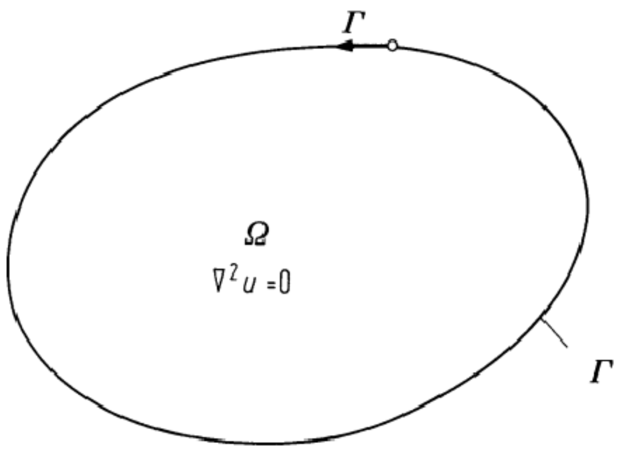

Figure 1: Domain and boundary (Brebbia, Telles and Wrobel, 1984)

where u and q are the potential and its normal derivative, while u^* and q^* represent the fundamental solution applied at a point i and its normal derivative. The fundamental solution for a two-dimensional isotropic domain is:

$$u^* = \frac{1}{2\pi} ln\left(\frac{1}{r}\right) \tag{2}$$

where r is the distance from the source point i to any field point under consideration, as seen in Figure 2.

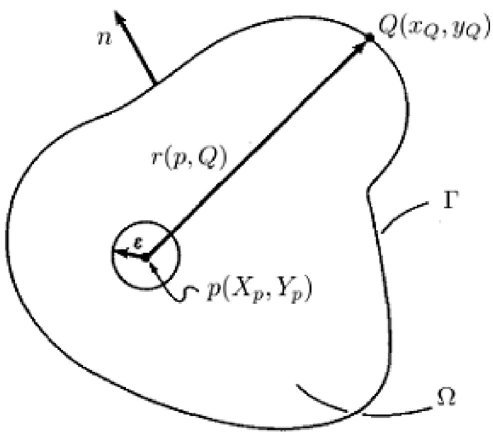

Figure 2: Fundamental solution (Becker, 1992)

Discretizing the boundary Γ in N elements, the Eq. (1) can be represented as:

$$c^i u^i + \sum_{j=1}^{N} \left(\int_{\Gamma_j} u q^* d\Gamma \right) = \sum_{j=1}^{N} \left(\int_{\Gamma_j} q u^* d\Gamma \right) \tag{3}$$

Varying the source point position in such a way that the fundamental solution is applied at each node, a set of equations can be found and expressed in matrix form as:

$$\mathbf{HU} = \mathbf{GQ} \tag{4}$$

Moving all unknown values to the left side of Eq. (4), and prescribed values to the right side, a system of equations can be written as:

$$\mathbf{AX} = \mathbf{F} \tag{5}$$

Once the system is solved and, consequently, all the boundary values are known, it is possible to calculate the potential at any internal point i using the following expression (Brebbia and Dominguez, 1992):

$$u^i = \sum_{j=1}^{N} \left(\int_{\Gamma_j} q u^* d\Gamma \right) - \sum_{j=1}^{N} \left(\int_{\Gamma_j} u q^* d\Gamma \right) \tag{6}$$

and its derivatives in the directions of the Cartesian axes:

$$\begin{aligned} (q_x)^i = \left(\frac{\partial u}{\partial x}\right)^i = \sum_{j=1}^{N} q^j \left(\int_{\Gamma_j} \left(\frac{\partial u^*}{\partial x}\right)^i d\Gamma \right) - \sum_{j=1}^{N} u^j \left(\int_{\Gamma_j} \left(\frac{\partial q^*}{\partial x}\right)^i d\Gamma \right) \\ (q_y)^i = \left(\frac{\partial u}{\partial y}\right)^i = \sum_{j=1}^{N} q^j \left(\int_{\Gamma_j} \left(\frac{\partial u^*}{\partial y}\right)^i d\Gamma \right) - \sum_{j=1}^{N} u^j \left(\int_{\Gamma_j} \left(\frac{\partial q^*}{\partial y}\right)^i d\Gamma \right) \end{aligned} \tag{7}$$

3 BOUNDARY ELEMENT METHOD IN ELASTICITY

The direct formulation of boundary elements for elastostatic was initially presented by Rizzo (1967), following the work of Jawson (1963). The previous formulations were based on the indirect method. The basic integral equation, known as the Somigliana's Identity, was taken to the boundary and discretized in constant elements in a similar way to what had been done for potential problems. Brebbia and Dominguez (1992) used the weighted residual method to deduce the integral equations. Some authors, such as Albuquerque (2001) and Creci Filho (2004), used the Betti's reciprocal theorem to develop this formulation.

In the absence of body forces, the boundary integral equation for elastostatic problems is (Brebbia and Dominguez, 1992):

$$c_{lk}^i u_k^i + \int_{\Gamma} p_{lk}^* u_k d\Gamma = \int_{\Gamma} u_{lk}^* p_k d\Gamma \tag{8}$$

where u_k and p_k are the displacement and traction components, while u_{kl}^* and p_{kl}^* are the Kelvin's fundamental solution in terms of displacement and traction, respectively. This solution represents the response in an infinite elastic domain due to the application of a unit point load. For a two-dimensional isotropic domain, the solution in terms of displacement is:

$$u_{lk}^* = \frac{1}{8\pi\mu(1-\nu)} \left[(3-4\nu) \, ln\left(\frac{1}{r}\right) \delta_{lk} + r_{,l} r_{,k} \right] \tag{9}$$

and in terms of traction is:

$$p_{lk}^* = -\frac{1}{4\pi(1-\nu)\, r} \left[\frac{\partial r}{\partial n} \left[(1-2\nu)\,\delta_{lk} + 2 r_{,l} r_{,k} \right] + (1-2\nu)\left(n_l r_k - n_k r_{,l}\right) \right] \tag{10}$$

where ν is the Poisson's ratio and μ is one of the Lamé's constants.

Discretizing Eq. (8) and applying the resultant expression at each node in a similar way to the one performed for potential problems, it is possible to obtain a system of equations whose solution provides the values of the displacement and traction values along the entire boundary. These values can be used to calculate the internal displacements and stress.

The boundary stresses were calculated by using the process described in Brebbia, Telles and Wrobel (1984). This study defined a local coordinate system composed of an axis in the normal direction and another in the direction tangent to the element. When rotating the physical variables in relation to this new system and applying Hooke's law and the relationship between strains and displacements, the following expressions were obtained:

$$\begin{aligned}
\overline{\sigma}_{11} &= \frac{1}{1-\nu}\left(\nu\overline{\sigma}_{22} + 2\mu\frac{\partial \overline{u}_1}{\partial \overline{x}_1}\right) \\
\overline{\sigma}_{12} &= \overline{p}_1 \\
\overline{\sigma}_{22} &= \overline{p}_2
\end{aligned} \tag{11}$$

4 SUBREGIONS TECHNIQUE

In practical engineering problems, it is common to face situations in which the domain under study is composed of materials with different properties. In the numerical analysis of problems involving non-homogeneous media, several techniques can be used. One of the alternatives is the Subregions Technique that is used in cases which the analyzed media is not completely homogeneous, but can be decomposed into homogeneous regions. This technique considers each subregion both individually and coupled to the others through equations of equilibrium and compatibility imposed on the interface nodes. Figure 3 illustrates this technique.

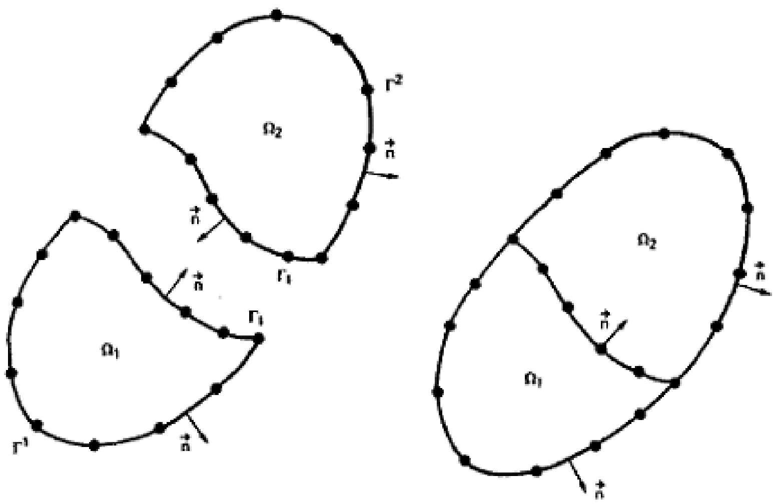

Figure 3: Subregions (Brebbia and Dominguez, 1992)

Rizzo and Shippy (1968) used the boundary integral equation method, considering linear elasticity and suggesting the use of subregions for the treatment of non-homogeneous domains. Venturini (1984) analyzed foundation structures, where non-homogeneous regions were treated by the subregion method, even in cases with different types of physical non-linearity. Monnerat (2008) employed the subregion technique to associate horizontal, vertical and inclined plates. Andrade and Loeffler (2016) compare the classical Subregions Technique to an approach with domain superposition in problems governed by the Laplace equation.

It is necessary, therefore, to list the conditions that must be applied to the interface nodes in order to perform the coupling of the regions. Assuming a heat conduction problem in a medium composed of two domains that are in contact and have perfect thermal conduction between each other, according to Becker (1992), in the absence of heat sources in the interface, the following compatibility condition is valid:

$$u^{(1)} = u^{(2)} \tag{12}$$

and the following equilibrium condition:

$$\left[-K_1 \frac{\partial u}{\partial n}\right]^{(1)} = -\left[-K_2 \frac{\partial u}{\partial n}\right]^{(2)} \tag{13}$$

that can be expressed as:

$$q^{(1)} = -\frac{K_2}{K_1} q^{(2)} \tag{14}$$

being K_1 and K_2 the thermal conductivity of each domain.

Assuming now an elastic problem with a perfect contact situation between two bodies, which means that the contacting surfaces are not allowed to slip due to the infinite friction between them, the following compatibility conditions are valid (Becker, 1992):

$$\begin{aligned} u_x^{(1)} &= u_x^{(2)} \\ u_y^{(1)} &= u_y^{(2)} \end{aligned} \tag{15}$$

and the following equilibrium conditions:

$$\begin{aligned} t_x^{(1)} &= t_x^{(2)} \\ t_y^{(1)} &= t_y^{(2)} \end{aligned} \tag{16}$$

5 COMPUTATIONAL IMPLEMENTATION

If only continuous elements were used, the spectrum of problems that the program could cover would be limited, becoming inappropriate for some situations with discontinuities of physical variables. The use of meshes with only discontinuous elements, however, would unnecessarily increase the computational cost, since the discontinuity should occur only at some points. In order to provide a more effective solution, an intermediate approach was adopted, shifting only the end nodes that are situated in corners of the geometry and those where the discontinuity of variables is prescribed.

In order to facilitate the generation of the quadratic elements, the program classifies them as straight and curved elements. Whenever there is a positive amount of elements associated with a point, the geometry between that point and the next one is considered a line, which will be divided into as many elements as required. When there are no elements associated with a point, then that point and its subsequent two will be taken in the generation of a curved type element. In the transitions between curved and straight elements or between straight elements generated between different pairs of given points, the nodes will be automatically discontinued. To indicate that there is a need to discontinue the end between two curved elements, it is sufficient to request a null amount of elements for the point and the program will use the next two points to generate a curved element, shifting the initial node of the current element and end node of the previous curved element. Figure 4 shows a quarter cylinder discretized using quadratic elements where it is possible to observe that the nodes were shifted from the transitions between curved and straight elements. Thus, all nodes are placed in smooth pieces of the boundary.

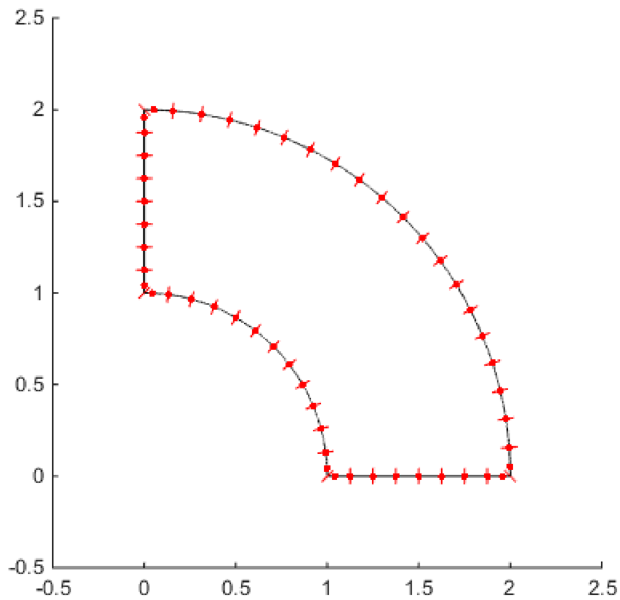

Figure 4: Discretization of a quarter cylinder using quadratic elements

In this work, all the integrals were solved numerically, except for the singular ones. Terms with strong singularities were calculated implicitly. The weak singular integrals were treated by separating them into singular and non-singular parts and separating those parts into terms before and after the singularity. This process, used by Brebbia and Dominguez (1992) to analyze a continuous element, was carried out for a general element, with or without discontinuities. The resulting formulas can be found in Ramos (2017). The singular parts were integrated using Logarithmic Gaussian Quadrature, while the non-singular parts were integrated using Standard Gauss Quadrature.

6 RESULTS

6.1 Hollow cylinder under internal pressure

This first problem is a classical one, adapted from Becker (1992). It was analyzed in order to validate the program. Considering a non-homogeneous hollow cylinder composed of aluminum and steel under internal pressure, as seen in Figure 5:

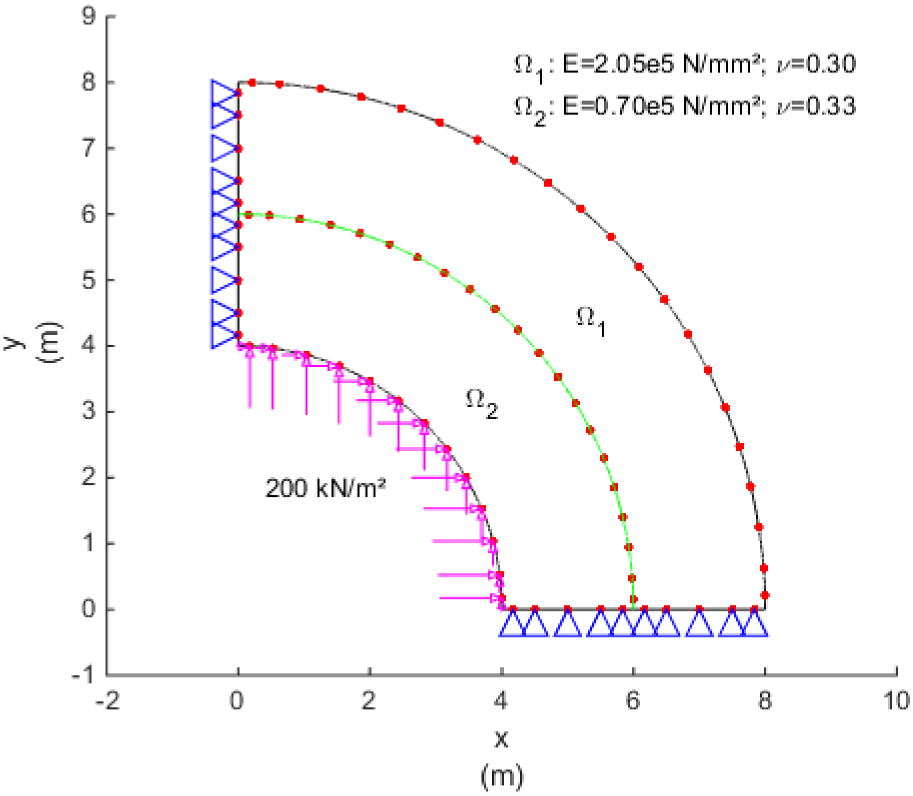

Figure 5: Element mesh and boundary conditions

Assuming that the length of the cylinder is sufficiently large to consider plane strain, the analytical solution of the problem is the so-called Lamé's Solution, which for radial displacements can be found at Timoshenko and Goodier (1951). For a radius r, the solution for the radial displacement u_r is given by an expression as follows:

$$u_r = C_1 r + \frac{C_2}{r} \tag{17}$$

where C_1 and C_2 are constants dependent on the dimensions of the cylinder and the applied pressure.

The results obtained using 44 quadratic elements with nodes shifted from the corners, the exact values, and the errors are presented in Table 1.

Table 1: Radial displacements

Radius (m)	Exact solution (mm)	QUAD(44) (mm)	Error (%)
4.25	0.0139	0.0139	3.35E-03
4.50	0.0131	0.0131	3.92E-03
4.75	0.0124	0.0124	4.15E-03
5.00	0.0117	0.0117	4.34E-03
5.25	0.0112	0.0112	4.52E-03
5.50	0.0106	0.0106	4.68E-03
5.75	0.0102	0.0102	4.65E-03
6.25	0.0095	0.0095	4.83E-03
6.50	0.0093	0.0093	5.13E-03
6.75	0.0091	0.0091	5.19E-03
7.00	0.0089	0.0089	5.24E-03
7.25	0.0087	0.0087	5.28E-03
7.50	0.0086	0.0086	5.32E-03
7.75	0.0085	0.0085	5.85E-03

The results demonstrated great accuracy with the analytical solution, showing errors varying from 0.0033 to 0.0058 percent. Lastly, the graphical output for this problem is presented below:

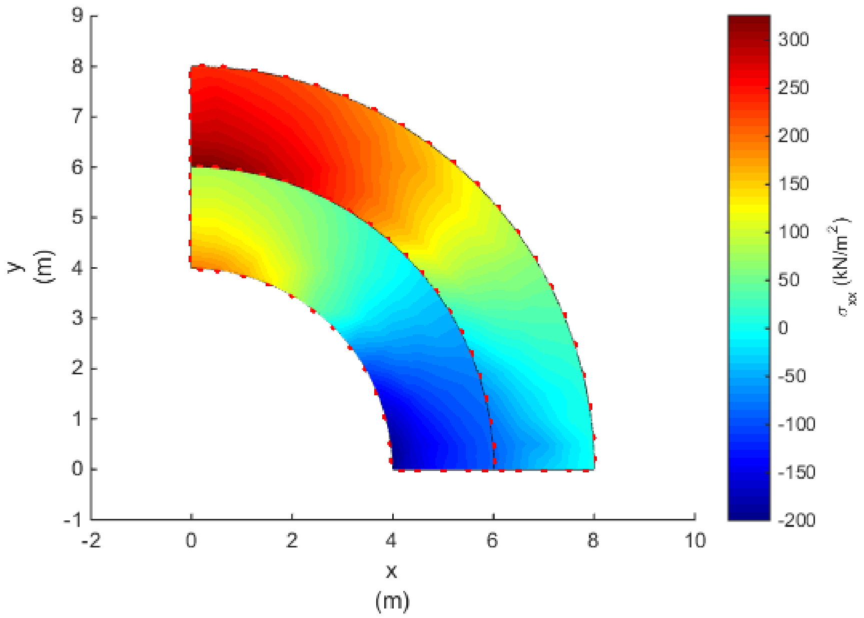

Figure 6: Stress component σ_{xx}

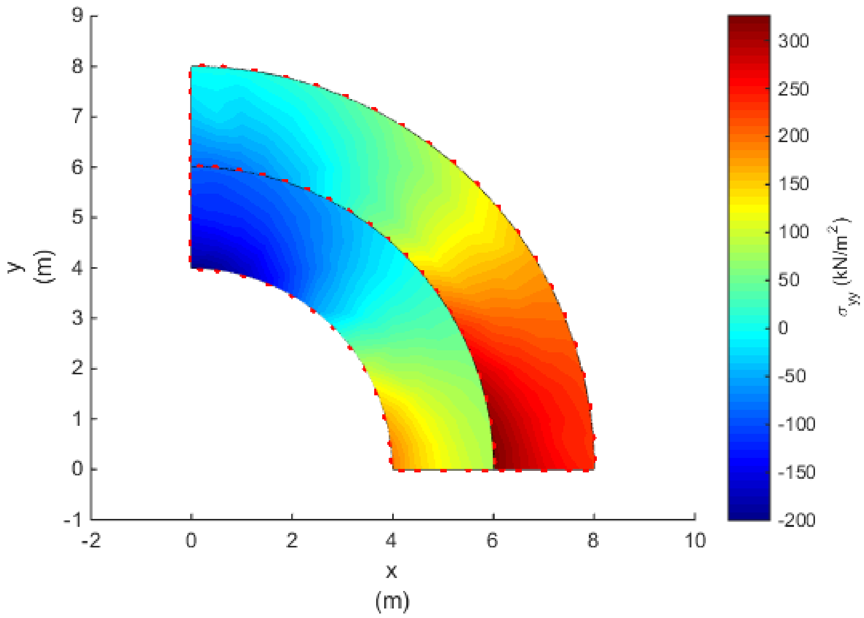

Figure 7: Stress component σ_{yy}

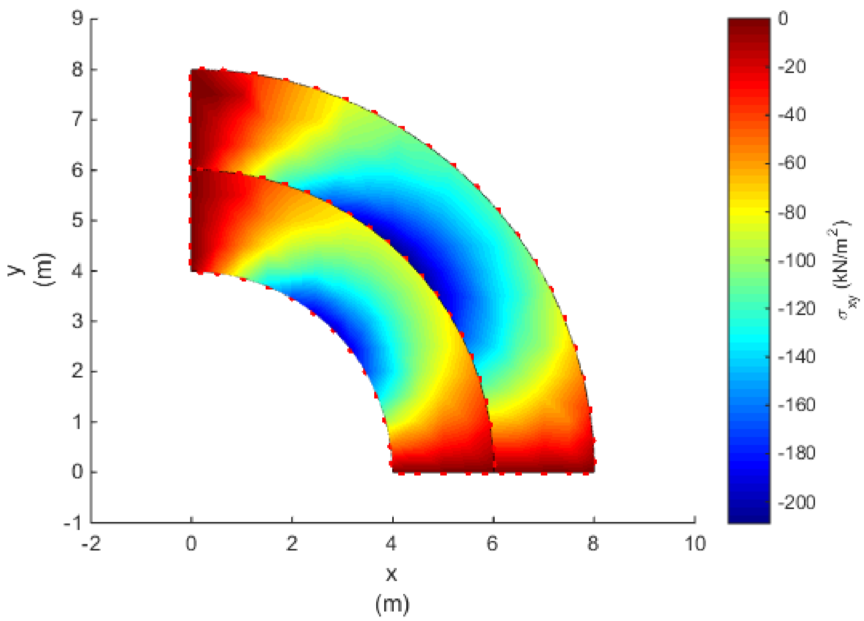

Figure 8: Stress component σ_{xy}

It is possible to notice that σ_{xy} is concentrated in the direction corresponding to 45 degrees, dissipating along the radius. The distribution of the components σ_{xx} and σ_{yy} showed a stress concentration on the inner corners of each region. All components of stress analyzed presented a large relative variation between both sides of the interface.

6.2 Beam with variable section and hole under axial load

After validating the program, its performance was analyzed in a more complex problem. The results were compared to the ones obtained from ANSYS. Considering a beam composed of two regions of different sections. The larger section is composed by steel and has a hole, and the other section is composed by aluminum. The beam is under uniform axial load, as represented in the Figure 9.

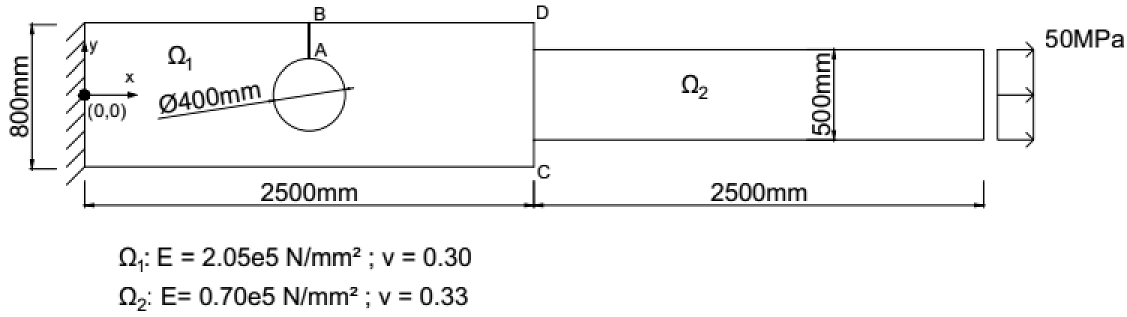

Figure 9: Definition of the problem

The ANSYS analysis was performed using 1260 PLANE182 elements. In the first analysis, the boundary element program employed 272 quadratic elements, shifting the corner nodes. In the second analysis, the same amount of elements were used, but the nodes were not discontinued. The values of the stress component σ_{xx} provided for each analysis are presented below:

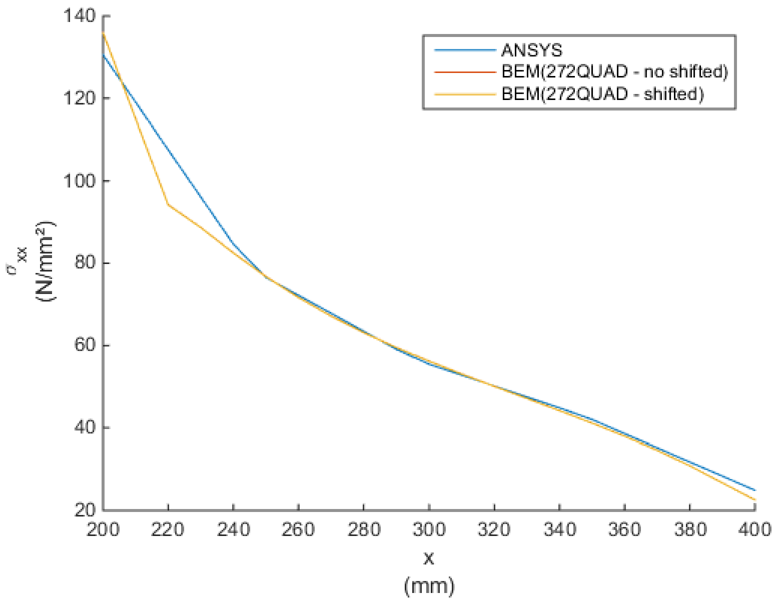

Figure 10: Stress component σ_{xx} along the AB segment

Through the analysis of results for the AB segment (Figure 10), it can be noticed that there is no significant difference between the results obtained from using elements with shifted nodes and elements without them. At the ends of the segment, that is, in the external boundary and in the boundary of the hole, the results were very similar, and so were the results near the middle of the segment. However, the results diverged in some points close to the boundary of the hole. This happened because the results at the ends were calculated using the boundary displacements and tractions, as mentioned in the third section, while in other points the stress components were calculated using the boundary integral equation, which becomes close to singular as the internal point is taken close to the boundary. This fact, along with the stress concentration in the hole, makes the results near the hole worse than those near the external boundary. So, after this first segment was discussed, the second segment was analyzed:

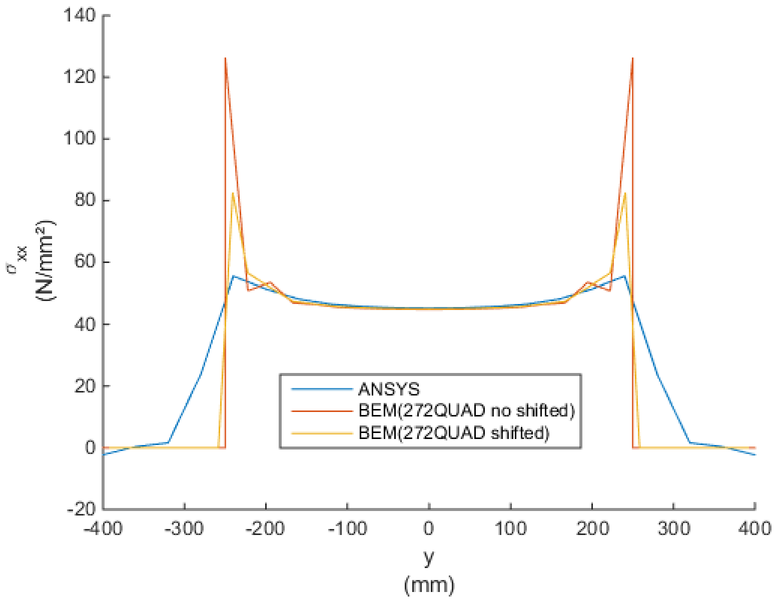

Figure 11: Stress component σ_{xx} along the CD segment

For the CD segment (Figure 11), BEM considers all σ_{xx} values outside the interface as null, while ANSYS shows a small stress growing from the ends of the segment. The absence of tractions in the x direction is a natural condition of the problem, and ANSYS, based in FEM, approximates this type of conditon, while BEM accurately satisfies the essential and natural boundary conditions. The peaks of stress obtained by BEM, as seen on the graphics, were higher than those provided by ANSYS, mainly when the nodes were not shifted. This happened because the variation of the stress near the interface ends is very high. In that way, if the nodes are shifted from the ends, the values obtained are significantly lower.

Both programs provide a graphical representation of the stress distribution. The following figures present the distribution of the stress component σ_{xx} generated by each program:

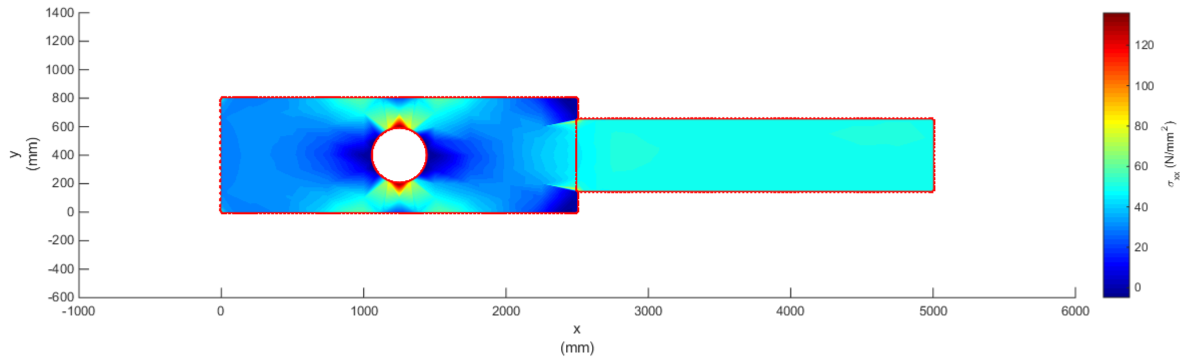

Figure 12: Distribution of the stress component σ_{xx} along the domain (BEM)

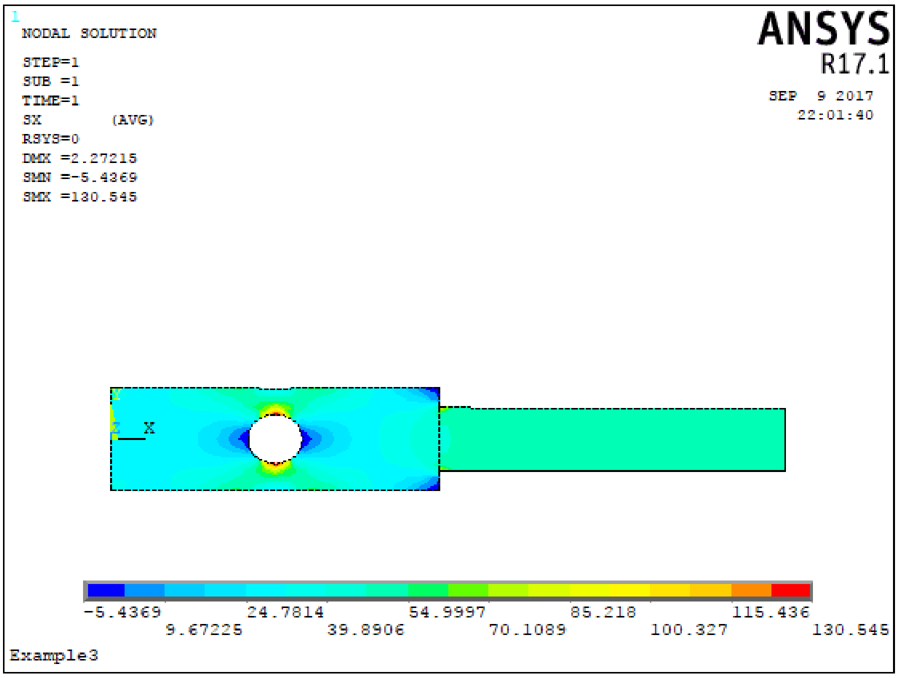

Figure 13: Distribution of the stress component σ_{xx} along the domain (ANSYS)

6.3 Convective heat transfer in a cylinder

The third example is a potential problem that involves two different situations: heat convection in the outer face of the cylinder (representing a linear relationships between potential and heat flux at the boundary as a prescribed condition) and heat conduction between two regions. The problem is shown in Figure 14.

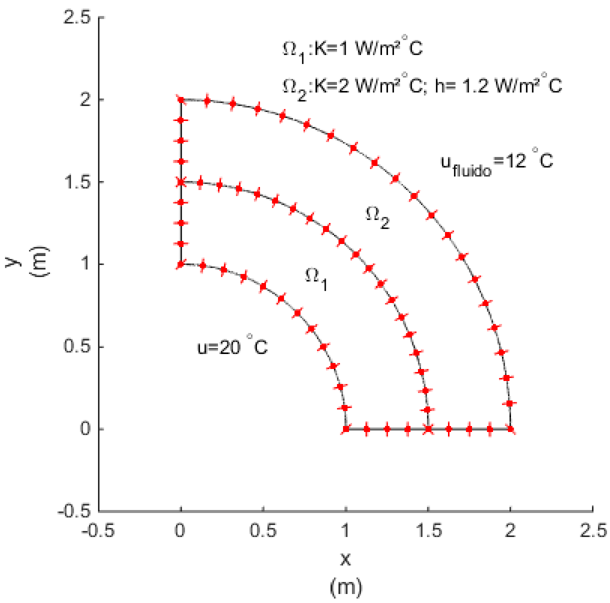

Figure 14: Element mesh and boundary conditions

The analytical solution of the problem was found at Carslaw and Jaeger (1959). The numerical results, provided by the boundary element program using 44 elements with continuous ends and exact values are displayed in Table 2.

Table 2: Temperature

Radius (m)	Exact solution (°C)	QUAD(44) (°C)	Error (%)
1.125	19.0245	19.0245	1.36E-04
1.250	18.1520	18.1519	1.60E-04
1.375	17.3626	17.3626	1.83E-04
1.625	16.3106	16.3105	1.96E-04
1.750	16.0037	16.0037	1.94E-04
1.875	15.7180	15.7180	1.89E-04

Once again, the numerical results approximated the analytical solution in an accurate manner. Hence, the set of problems considered in this work was successfully completed. Finally, it is also important to show the visualization of the temperature distribution and heat flux generated by the program, which can be seen in Figure 15.

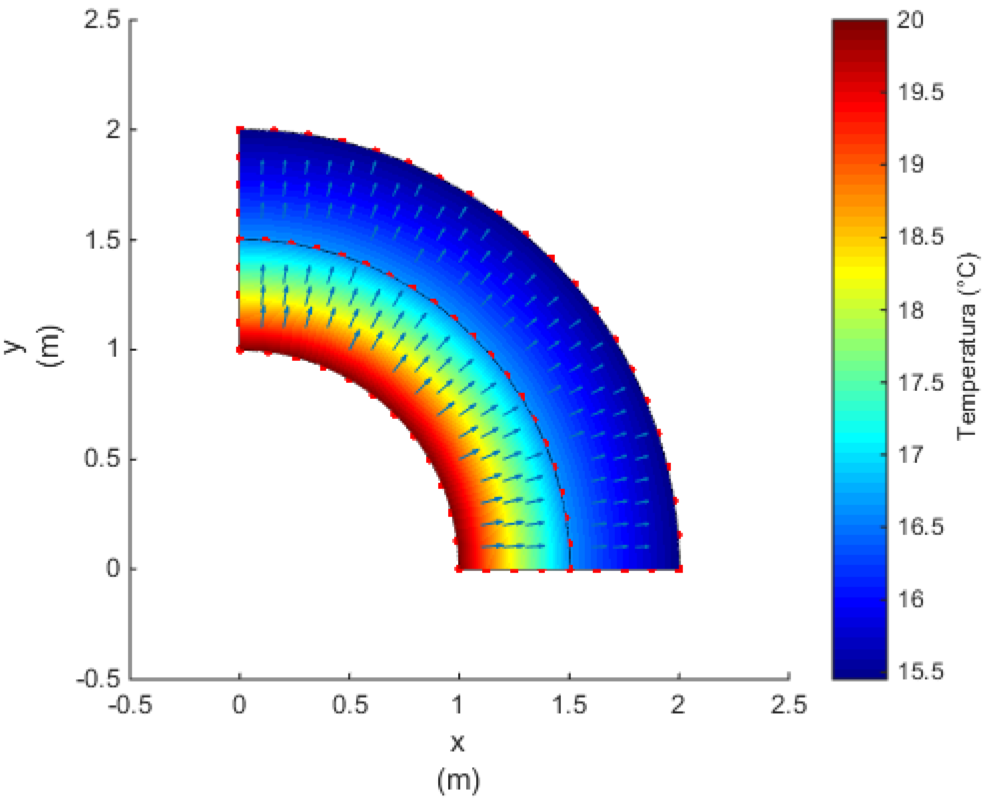

Figure 15: Temperature and heat flux along the domain

7 CONCLUSIONS

The first and third numerical examples were compared with the analytical solution, and showed great accuracy, validating the computational implementation for potential and elastostatic problems. It is important to notice that the first one succeeded in being very well approximated in spite of the presence of stress concentration points and interfaces of materials with considerable differences in rigidity. As mentioned in the introduction, this is a characteristic of the boundary element method.

But perhaps the most interesting observations of this work were motivated by the second problem. Mainly because this example provided an insight into the effect of the singularity occurring when the internal points are taken near the boundary and it showed how this effect becomes more significant in regions with high stress variation.

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