CILAMCE 2017
Florianópolis - Santa Catarina - Brazil

Enriched Techniques applied to Modified Local Green's Function Method in Dynamic Modal Analysis

Paulo de Oliveira Weinhardt

paulo.weinhardt@gmail.com

Marcos Arndt

arndt.marcos@gmail.com

Roberto Dalledone Machado

rdm@ufpr.br

Graduate Program in Numerical Methods in Engineering, Federal University of Paraná

Curitiba, 81.531-980, Paraná, Brasil

Abstract. *The Modified Local Green's Function Method (MLGFM) is an integral technique that is distinguished by the great accuracy in the approximation of dual and flow variables. This method performs a boundary approximation by calculating the projections of the Green's functions on the approximation space basis of the problem. The Finite Element Method can be used to construct an approximation base and then is possible to calculate the corresponding projections. In this article the MLGFM is described in a new approach taking into account the concept of the Partition of Unity Method. Thus, this work presents examples of MLGFM application in modal dynamic analysis with non-polynomial approximation spaces, using enrichment techniques. The accuracy gains of the method and the characteristic aspects of the approximation in terms of frequency spectra are discussed.*

Keywords: *MLGFM, Partition of Unity, GFEM, Dynamic Analysis, Frequency Spectrum*

1 INTRODUCTION

First and foremost, the Modified Local Green's Function Method (MLGFM) is an integral method that is distinguished by the great accuracy in the approximation of dual and flow variables (Barbieri & Machado, 2015). However, unlike the Boundary Element Method (BEM), it is not necessary to directly use the fundamental analytical solution. Instead, a hybrid strategy is used to make the approximation.

In the MLGFM, only projections of the Green's Function (GFs) are required. The Finite Element Method can be used to construct an approximation base and then is possible to calculate the corresponding projections of the GFs (Machado, 1992; Barbieri, 1992; Monken e Silva, 1988). It is interesting to note that, as a consequence of the hybrid formulation, the method presents nodal superconvergence for the flow variables, as highlighted by Barbieri (2005).

In this article the MLGFM is described in a new approach taking into account the concept of the Partition of Unity Method, intending to generalize its concept. It's shown that a the MLGFM may take advantage of a generical method that relies on the Partition of Unity concept. Thus, this work present examples of MLGFM application in modal dynamic analysis with non-polynomial approximation spaces, using enrichment techniques.

Numerical results are presented relying on the fact that MLGFM results for the potential are identical to the FEM results and so modal analysis may present very close approximation behaviour in spectrum terms.

2 PARTITION OF UNITY: PRELIMINARY CONCEPTS

The mathematical theory of the Partition of Unity Method ensures that the local approximation spaces V_i generated by the base of functions v_i present the approximation properties that will be described next (Melenk, 1995).

Let u be the function to be approximated function and $v_i \in V_i$, local approximation spaces V_i that satisfies the following conditions

$$\|u - v_i\|_{L^2(\Omega_i \cap \Omega)} \leq \epsilon_1(i) \tag{1}$$

and

$$\|\nabla(u - v_i)\|_{L^2(\Omega_i \cap \Omega)} \leq \epsilon_2(i) \tag{2}$$

where $\epsilon_1(i)$ e $\epsilon_2(i)$ are dependent values of the i-th cover that limit $(u - v_i)$ and their derivatives according to a convenient norm. This way, the function u_h that approximates u satisfies,

$$\|u - u_h\|_{L^2(\Omega)} \leq C_\infty \sqrt{M \left(\sum_i \epsilon_1^2(i) \right)} \tag{3}$$

and

$$\|\nabla (u - u_h)\|_{L(\Omega)^2} \leq \sqrt{2M \left(\sum_i \left(\frac{C_g}{diam\,(\Omega_i)} \right)^2 \epsilon_1^2(i) + C_\infty^2 \epsilon_2^2(i) \right)} \tag{4}$$

In the Eqs. 3 and 4, M is related to the number of cover overlays on the same point in the domain Ω. In the same expressions, C_∞ and C_g are constants. Lastly, $diam\,(\Omega_i)$ corresponds to the diameter of the cover defined by the approximation functions.

Finally, it can be pointed that the approximation carried out by the Finite Element Method is a particular case of this theory. It is also possible to highlight that the approximation made by the FEM can be extended through these concepts with the enrichment strategy of the Generalized Finite Element Method (GFEM). Thus, the approximation sums up:

$$u_h^e = u_{MEF} + u_{ENR} \tag{5}$$

where u_{MEF} corresponds to the portion described by the classical approximation functions of the FEM and u_{ENR} corresponds to the approximation made by the enrichment functions that aim to incorporate particular aspects of the studied problem.

3 MODIFIED LOCAL GREEN'S FUNCTION METHOD (MLGFM)

The core of MLGFM's theory is the determination Green's function projections on the domain and boundary of the problem, without the knowledge of it. For this purpose, it is necessary to solve two problems associated with the operator under study A.

3.1 Delta Loading applied on the domain

Applying the operator to the Green function (G) related to source and field domain points (P, Q) it results in a Delta loading applied on the domain. The adjunt boundary operator N^* (or Neumann operator) shall be modified by an additional operator N'. The notation * stands for the adjunts operators.

$$A^*G(P,Q) = \delta(P,Q)I \tag{6}$$

$$(N^* + N')G(p,Q) = 0 \tag{7}$$

Projecting the problem described by Eqs. 6 and 7 on the subspace generated by domain interpolation functions ($\Psi(Q)$), it's related the Green's projection in the domain G_d:

$$A^*G_d(P) = \Psi(P) \tag{8}$$

Applying the projection in the Eq. 7, it holds that

$$(N^* + N')G_d(p) = 0 \tag{9}$$

3.2 Delta Loading applied on the boundary

Applying the operator to the Green function (G) related to source and field domain points (P, Q) it results in a Delta loading applied on the domain. The adjunt boundary operator N^* (or Neumann operator) shall be modified by an additional operator $N^{'}$. The notation * stands for the adjunts operators.

$$A^*G(P,Q) = 0 \tag{10}$$

$$(N^* + N^{'})G(p,Q) = \delta(P,Q)I \tag{11}$$

Projecting the problem described by Eqs. 10 and 11 on the subspace generated by boundary interpolation functions ($\Phi(Q)$), it's related the Green's projection in the domain G_b:

$$A^*G_b(P) = 0 \tag{12}$$

Applying the projection in the Eq. 11, it holds that

$$(N^* + N^{'})G_b(p) = \Phi(p) \tag{13}$$

3.3 Discrete Projections

As the Green's Functions projections are more continuous than the domain basis functions, these can be interpolated using this very basis. The interpolation is performed among each discrete projection in the domain and in the boundary, namely G^{DP}, G^{BP}, G^{Dp}, G^{Bp}. The same argument holds for the boundary functions, as its constructions relies on the Trace properties. This way, it's possible write:

- Domain Projections from P in Domain,

$$G_d(P) = \Psi(P)G^{DP} \tag{14}$$

- Boundary Projections from P in Domain,

$$G_b(P) = \Psi(P)G^{BP} \tag{15}$$

- Domain Projections from p in Boundary,

$$G_d(P) = \Phi(p)G^{Dp} \tag{16}$$

- Boundary Projections from p in Boundary,

$$G_b(P) = \Phi(p)G^{Bp} \tag{17}$$

Recovering the Galerking approximation related to the projection techniques in the domain, follows that

$$[K + K^{'}][G^{DP}] = [M] \tag{18}$$

where K is the domain stiffness matrix, $K^{'}$ is the additional stiffness matrix due to the $N^{'}$ operator and M is the domain mass matrix with unitary density.

Analogously for the boundary, follows that

$$[K + K^{'}][G^{BP}] = [m] \tag{19}$$

where K is the domain stiffness matrix, $K^{'}$ is the additional stiffness matrix due to the $N^{'}$ operator and m is the boundary mass matrix with unitary density.

Equations 18 and 19 may be solved at once in a unified extended system of equations:

$$[K + K^{'}][G^{DP}|G^{BP}] = [M|m] \tag{20}$$

Displacements in the domain are then determined as:

$$u = G^{DP}(b) + G^{BP}(f) \tag{21}$$

where b and f are forces applied on the domain and on the boundary, respectively. It is important to note that u corresponds to generalized displacements of the solved discrete problem, that is, coefficients of the linear combination of the interpolation functions present in the approximation basis.

3.4 Eigenproblem in the MLGFM

As described by Barbieri (1992) and Filippin (1992), the eigenproblem derived from the Helmholtz operator with wave velocity c may be solved calculating the following auxiliar matrices:

$$\Lambda_{\Psi\Psi} = \int_{\Omega} [\Psi(Q)]^T [\Psi(Q)] d\Omega_Q \tag{22}$$

$$\Lambda_{d\Phi} = \int_{\partial\Omega} [G_d(p)]^T [\Phi(p)] d\partial\Omega_P \tag{23}$$

$$\Lambda_{d\Psi} = \frac{1}{c} \int_{\Omega} [G_d(P)]^T [\Psi(P)] d\Omega_P \tag{24}$$

$$\Lambda_{\Phi\Phi} = \int_{\partial\Omega} [\Phi(q)]^T [\Phi(q)] d\partial\Omega_Q \tag{25}$$

$$\Lambda_{b\Phi} = \int_{\partial\Omega} [G_b(p)]^T [\Phi(p)] d\partial\Omega_P \tag{26}$$

$$\Lambda_{b\Psi} = \frac{1}{c}\int_{\Omega}[G_b(P)]^T[\Psi(P)]d\Omega_P \tag{27}$$

Additionally, one can define the two following auxiliar matrices:

$$T_K = \Lambda_{d\Phi}[\Lambda_{c\Phi}]^{-1}\Lambda_{\Phi\Phi} \tag{28}$$

and

$$T_M = \Lambda_{d\Phi}[\Lambda_{c\Phi}]^{-1}\Lambda_{c\Psi} \tag{29}$$

Finally, the natural frequencies are obtained from the eigenproblem to be solved given by:

$$\{[\Lambda_{\Psi\Psi} - T_K] + \omega^2[\Lambda_{d\Psi} - T_M]\}\{w\}^D = 0 \tag{30}$$

4 TRIGONOMETRIC ENRICHMENT IN DYNAMIC ANALYSIS

4.1 Bars

For free vibration of a bar problem was proposed by Arndt (2009) a block of enrichment functions to the problem of dynamic analysis with GFEM. This group of functions consists of building a couple of clouds, a sine and a cosine, subordinated cover of enriched node. These clouds are written in the domain of element as two pairs of sine and cosine functions. The basic domain is considered to $\xi \in [0, 1]$.

Sine cloud:

$$\gamma_{1j}(\xi) = sin(\beta_j L_e \xi) \tag{31}$$

$$\gamma_{2j}(\xi) = sin(\beta_j L_e(\xi - 1)) \tag{32}$$

Cosine cloud:

$$\varphi_{1j}(\xi) = cos(\beta_j L_e \xi) - 1 \tag{33}$$

$$\varphi_{2j}(\xi) = cos(\beta_j L_e(\xi - 1)) - 1 \tag{34}$$

Where L_e is the length of the element and $\beta_j = j\pi$ is a hierarchical enrichment parameter proposed by Arndt (2009) to j function levels.

4.2 Euler-Bernoulli Beam

For free vibration of a beam problem was proposed by Arndt (2009) an enrichment function to the problem of dynamic analysis with GFEM. This function consists in a cosine, subordinated cover of enriched node. This function is written in the domain of element considered to $\xi \in [0, 1]$.

$$\gamma_j(\xi) = cos\left[\frac{(j-1)\pi(\xi+1)}{2}\right] - cos\left[\frac{(j+1)\pi(\xi+1)}{2}\right] \tag{35}$$

for $j = 1, 2, ..., J$, where J is the total number of level of enrichments.

5 DYNAMIC ANALYSIS

5.1 Modal Analysis Fixed-Fixed Bar

Consider a bar fixed at both ends as shown in Fig. 1.

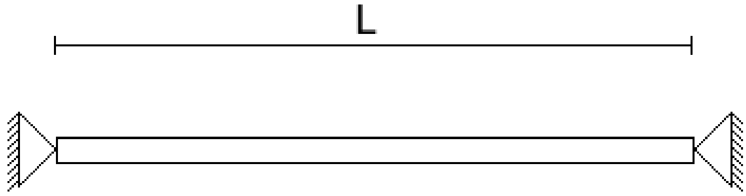

Figure 1: Fixed-Fixed Bar

The model results in the following elliptic eigenproblem, in the strong form:

$$\frac{d^2u}{dx^2} + \omega^2 \frac{\rho}{E} u(x) = 0 \implies A(u) = \left[\frac{d^2}{dx^2} + \omega^2 \frac{\rho}{E}\right](u) = 0 \tag{36}$$

Subject to boundary conditions:

$$\begin{aligned} u(0) &= 0 \\ u(L) &= 0 \end{aligned} \tag{37}$$

Analytical solution is then given by:

$$\omega = \frac{n\pi}{L}\sqrt{\frac{\rho}{E}} \tag{38}$$

This solution is used to normalize approximated solutions such as:

$$\bar{\omega} = \frac{\omega_h}{\omega} \tag{39}$$

Similarly, degrees of freedom (n) are normalized as a function of the total number of degrees of freedom (N) as:

$$\bar{n} = \frac{n}{N} \tag{40}$$

Thus, when plotting $\bar{\omega} \times \bar{n}$, one can compare different methods with different numbers of degrees of freedom.

5.2 Modal Analysis Cantilever Beam

Consider a cantilever beam as shown in Fig. 2.

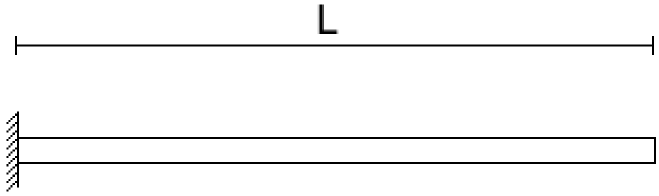

Figure 2: Cantilever Beam

This cantilever beam, with length L, modulus of elasticity E, specific mass ρ, moment of inertia I and cross-sectional area A. This way, the natural frequencies may be obtained as positive roots of the following equation (Arndt, 2009):

$$\cos\left(\sqrt[4]{\frac{\omega^2\rho A}{EI}}L\right)\cosh\left(\sqrt[4]{\frac{\omega^2\rho A}{EI}}L\right)+1=0 \tag{41}$$

A non-dimesionalized frequency parameter λ, calculated as described in Eq. 42 is used to compare the results.

$$\lambda^2=\omega L^2\sqrt{\frac{\rho A}{EI}} \tag{42}$$

6 NUMERICAL RESULTS

The numerical results of modal analysis that will be presented throughout this section have as main objective to show the coincidence of the approximations made by the MLGFM and its domain approximation basis.

The choice of this type of numerical test is based on the fact that the frequency spectrum is an intrinsic aspect of a certain numerical method approximation space, analogous to a fingerprint.

In this way, we will try to expose the coincidence of approximation characteristics numerically with standard finite element frequency spectra.

6.1 Modal Analysis Fixed-Fixed Bar

In this section we will present results referring to the modal analysis of a bar as previously described. The geometric and material parameters were adopted unitary for simplicity, without loss of generality. Each analysis is based on a 20-element uniform mesh.

First, the domain approximation is performed using only a linear finite element mesh. The contour approximation of the elements is simplified in the one-dimensional case, coinciding with the collocation method. Results of the MLGFM are presented along with Linear FEM spectrum.

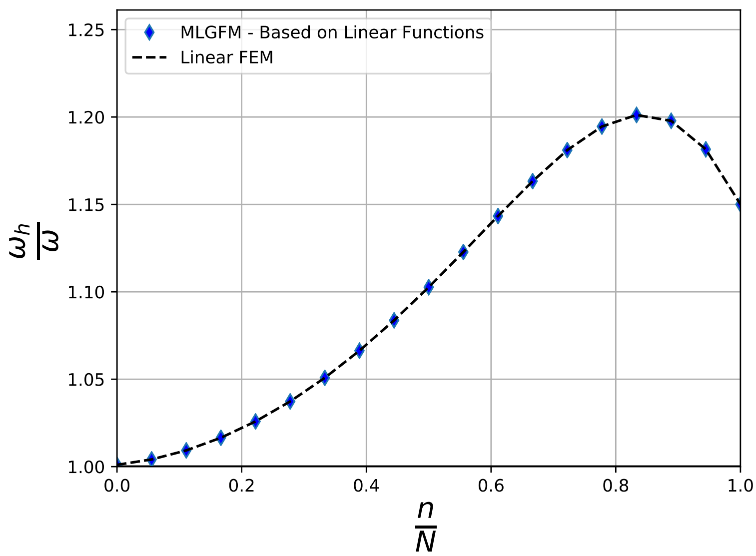

Figure 3: Linear Functions Basis

In a second moment, the trigonometric enrichment of the functions is performed, according to technique previously exposed. Results of the MLGFM are presented along with GFEM spectrum.

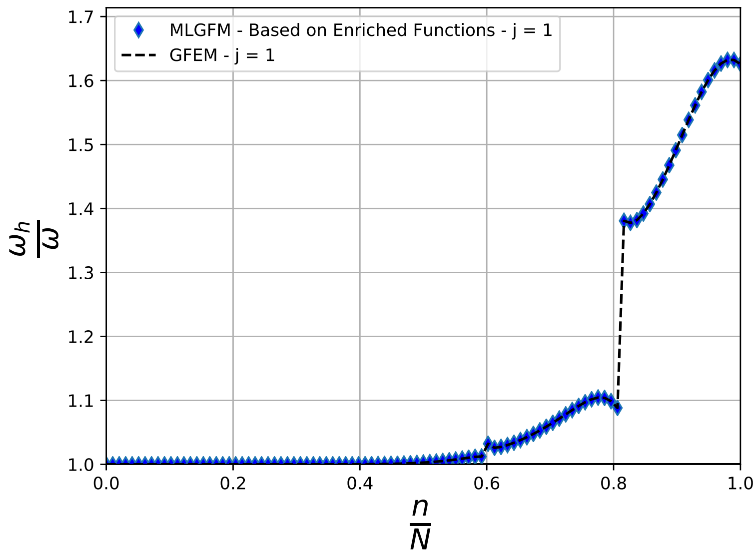

Figure 4: One Level Enriched Functions Basis

Continuing the successive enrichment process, a further level of enrichment is applied according to the appropriate procedure. Results of the MLGFM are presented along with the corresponding GFEM spectrum.

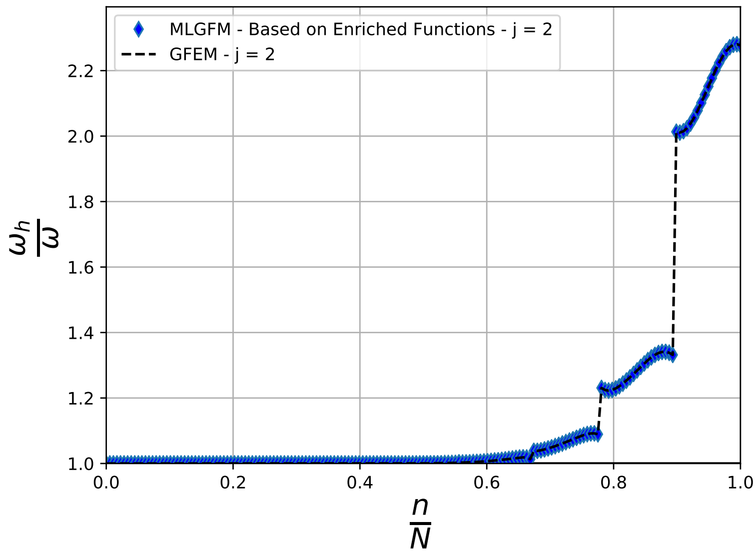

Figure 5: Two Levels Enriched Functions Basis

At the end of the analysis, a third enrichment procedure is carried out, in order to reinforce the argument of the previous tests. Results of the MLGFM are once again presented along with the corresponding GFEM spectrum.

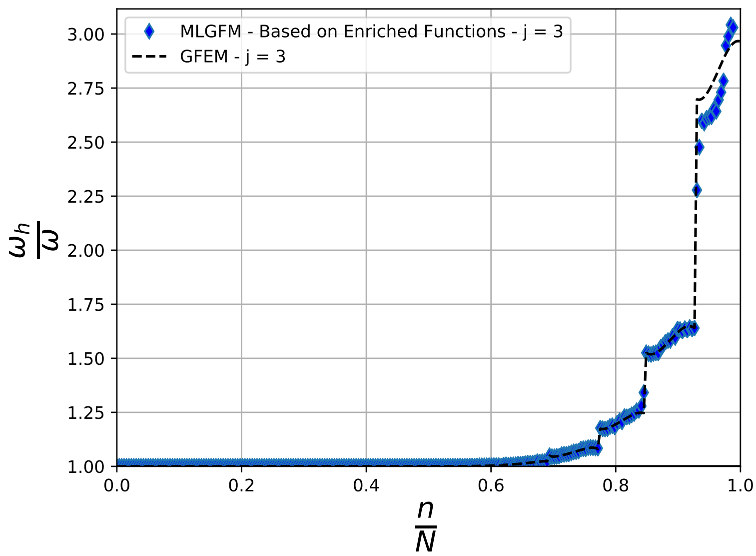

Figure 6: Three Levels Enriched Functions Basis

6.2 Modal Analysis Cantilever Beam

In this section we will present results referring to the modal analysis of a beam as previously described. The geometric and material parameters were adopted unitary for simplicity, without loss of generality. Derived parameters, such as inertia, were calculated accordingly.

First, the domain approximation is performed using only a linear finite element mesh. The contour approximation of the elements is simplified in the one-dimensional case, coinciding with the collocation method. A uniform mesh with 30 elements is used in this example. Results of the MLGFM are presented along with Linear FEM spectrum.

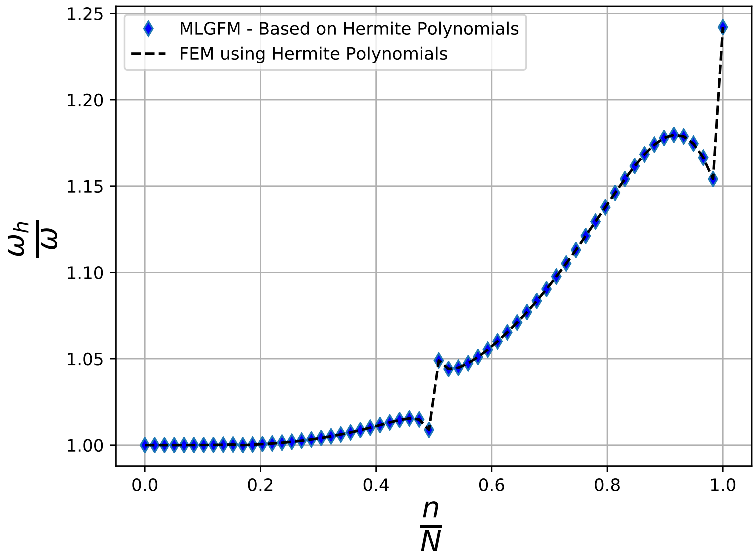

Figure 7: Linear Functions Basis

In a second moment, the trigonometric enrichment of the functions is performed, according to technique previously exposed. A uniform mesh with 30 elements is used in this example. Results of the MLGFM are presented along with GFEM spectrum.

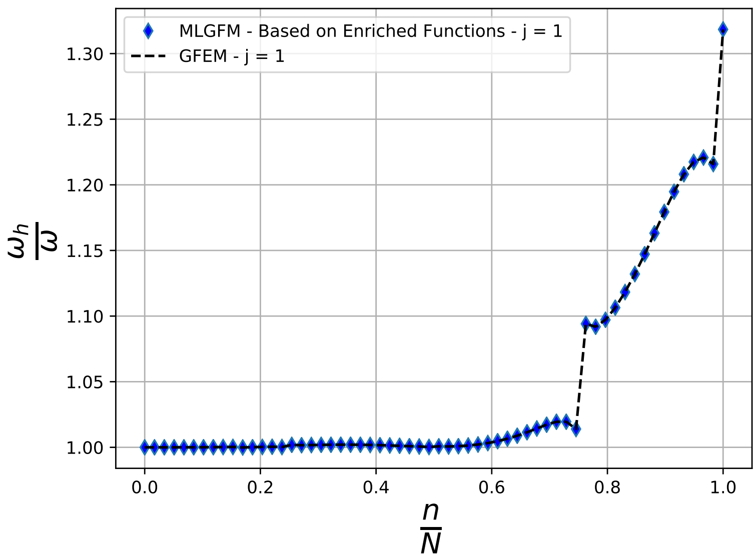

Figure 8: One Level Enriched Functions Basis

Continuing the successive enrichment process, a further level of enrichment is applied according to the appropriate procedure. A uniform mesh with 15 elements is used in this example. Results of the MLGFM are presented along with the corresponding GFEM spectrum.

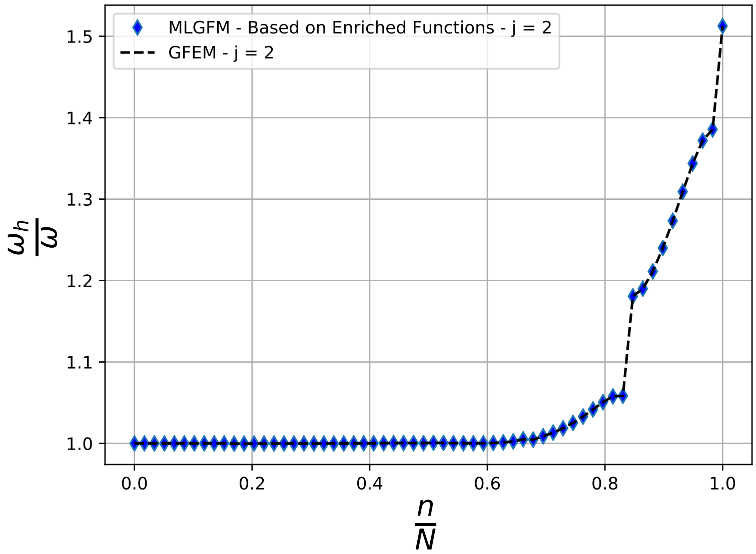

Figure 9: Two Levels Enriched Functions Basis

At the end of the analysis, a third enrichment procedure is carried out, in order to reinforce the argument of the previous tests. A uniform mesh with only 8 elements is used in this final example. Results of the MLGFM are once again presented along with the corresponding GFEM spectrum.

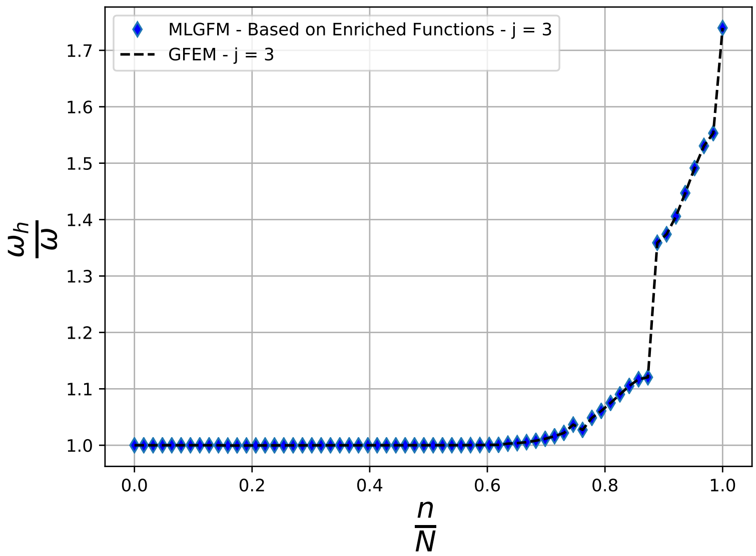

Figure 10: Three Levels Enriched Functions Basis

7 CONCLUDING REMARKS

Numerical tests were performed and the results were plotted. The bar results corroborate the main hypothesis of coincidence of approximation between the MLGFM and the domain approximations, as can be observed in Figs 3, 4 and 5. A little difference in the approximation of the optical branch can be seen in Fig. 6. This difference can be associated with the numerical sensitivity of the inversion required in the process of obtaining the MLGFM auxiliary matrices. This observation does not interfere in the presented analysis since the general behaviour of the approximation is very similar.

Following the analysis for the beam example, one can observe the results also corroborate the main hypothesis of this work, as can be observed in Figs 7, 8 and 9. Normalized frequency results for 3 levels of enrichment show some numerical oscillation near the unit in the acoustic branch of the spectrum, as can be observed in Fig. 10. This characteristic is inherent to the enrichment process and stems from the numerical instability of the method itself. Notably, this numerical instability is also inherited by the MLGFM approach.

Lastly, the numerical results of modal analysis presented highlighted the coincidence of the approximations made by the MLGFM and its domain approximation basis. In this way, future works will bring enrichment strategies using MLGFM in more complex problems whose flow variables are relevant, taking advantage of the differentiated properties of the method. Potential applications are: voltage concentration problem, discontinuity problems and nonlinear dynamics.

8 ACKNOWLEDGEMENTS

The authors respectfully acknowledge CAPES for the financial support.

REFERENCES

Arndt, M., 2009. *O Método dos Elementos Finitos Generalizados Aplicado à Análise de Vibrações Livres de Estruturas Reticuladas*. Doctoral thesis, Universidade Federal do Paraná.

Barbieri, R., 1992. *Desenvolvimento e aplicação do método da função de Green local modificado (MCGFM) para problemas do meio continuo*. Doctoral thesis, Universidade Federal de Santa Catarina.

Barbieri, R., 2005. "Analysis of nodal flux superconvergence for higher order elements". Technical report, CNPq Proc. 302809/2003-0.

Barbieri, R. & Machado, R.D., 2015. "The local formulation for the modified green's function method". *Latin American Journal of Solids and Structures*, Vol. 12, No. 5, pp. 883–904.

Filippin, C.G., 1992. *Desenvolvimento e aplicações do método da função de Green local modificado a equação de Helmholtz*. Master dissertation, Universidade Federal de Santa Catarina.

Machado, R.D., 1992. *Desenvolvimento do método modificado da função de Green local para a solução de placas laminadas de materiais compostos*. Doctoral thesis, Universidade Federal de Santa Catarina.

Melenk, J.M., 1995. *On generalized finite element methods*. Ph.D. thesis, The University of Maryland.

Monken e Silva, L.H., 1988. *Novas formulações integrais para problemas da mecanica*. Doctoral thesis, Universidade Federal de Santa Catarina.