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NUMERICAL DEVELOPMENT OF INTEGRATION RULES USING MAPLE SOFTWARE

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Abstract. *Using the Maple Software property to have no limits with the number of decimals and the size of the integers, the present work offers equations of Newton-Cotes Integration until twenty points. It shows Newton-Cotes closed and open integration equations. The new type Newton-Cotes semi-closed or semi-open was proposed. This method has important applications in the integration of tables or discrete functions. An analysis of error in the technique was made. To estimate the error of integration in discrete data, we propose apply different rules or mix several rules. The difference in the result of each equation provides an approximation of the error. Computational routines to generate Newton-Cotes integration rules were presented.*

Keywords: *Numerical Integration, Numerical Methods, Analysis of Error, Maple*

1 INTRODUCTION

The evaluation of integrals, a process known as integration or quadrature, is required in many problems in engineering and science (Sermutlu, 2005). The function f(x), which is to be integrated, may be a known function or a set of discrete data. Some known functions have an exact integral, in which case can be evaluated exactly in closed form. Many known functions, however, do not have an exact integral, and an approximate numerical procedure is required to evaluate. In many cases, the function f(x) is known only at a set of discrete points, in which case an approximate numerical procedure is again required to evaluate (Simos, 2009). Numerical integration (quadrature) equations can be developed by fitting approximating functions (e.g., polynomials) to discrete data and integrating the approximating function:

$$I = \int_{x_1}^{x_N} f(x)dx \cong \int_{x_1}^{x_1+(N-1)h} P(x)dx \tag{1}$$

When the function to be integrated is known at equally spaced points ($\Delta x=h=$constant) and N is number of points with x ranging x_1, x_1+h, x_1+2h, …, $x_1+(N-1)h$. The polynomial can be fit to the discrete data with much less effort, thus significantly decreasing the amount of effort required (Simos, 2008). The resulting equations are called Newton-Cotes formulas (Chapra, 2013).

The distance between the lower and upper limits of integration is called the range of integration. The distance between any two data points is called an increment ($\Delta x=h$). A linear polynomial requires one increment and two data points to obtain a fit. A quadratic polynomial requires two increments and three data points to obtain a fit. And so on for higher-degree polynomials. The group of increments required to fit a polynomial is called an interval (Kalogiratou and Simos, 2003). A linear polynomial requires an interval consisting of only one increment. A quadratic polynomial requires an interval containing two increments. And so on. The total range of integration can consist of one or more intervals. Each interval consists of one or more increments, depending on the degree of the approximating polynomial.

Closed and open forms of Newton-Cotes equations are available. The closed forms are those where the data points at the beginning and end of the limits of integration are known. The open forms have integration limits that extend beyond the range of the data (Wittleveen et al., 2009). The contribution of the article is the equations with the Newton Cotes coefficients ready to be used in the numerical integration and how to find these coefficients. The equations have high precision and numerical stability.

2 NEWTON-COTES CLOSED INTEGRATION EQUATION

The trapezoid rule for a single interval is obtained by fitting a first-degree polynomial to two discrete points (Sermutlu and Eyyubog´lu, 2006). The upper limit of integration is $x_2=x_1+h$, then the integral (I) have the equation:

$$I = \frac{h}{2}(y_1 + y_2) \quad or \quad I = 0.5hy_1 + 0.5hy_2 \qquad Error \approx O(h^2) \tag{2}$$

Simpson's 1/3 rule is obtained by fitting a second-degree polynomial to three equally spaced discrete points. The upper limit of integration is x_3, then

$$\begin{aligned} & I = \frac{h}{3}(y_1 + 4y_2 + y_3) \quad or \quad Error \approx O(h^4) \\ & I = 0.3333333333333333333333333333333333333333hy_1 \\ & \quad +1.3333333333333333333333333333333333333333hy_2 \\ & \quad +0.3333333333333333333333333333333333333333hy_3 \end{aligned} \tag{3}$$

Simpson's 3 /8 rule is obtained by fitting a third-degree polynomial to four equally spaced discrete points. The upper limit of integration is x_4, then:

$$\begin{aligned} & I = \frac{3h}{8}(y_1 + 3y_2 + 3y_3 + y_4) \quad or \quad Error \approx O(h^4) \\ & I = 0.375hy_1 + 1.125hy_2 + 1.125hy_3 + 0.375hy_4 \end{aligned} \tag{4}$$

Boole's rule is obtained by fitting a fourth-degree polynomial to five equally spaced discrete points. The upper limit of integration is x_5, then:

$$\begin{aligned} & I = \frac{2h}{45}(7\mathrm{y}_1 + 32\,\mathrm{y}_2 + 12\,\mathrm{y}_3 + 32\,\mathrm{y}_4 + 7\,\mathrm{y}_5) \quad or \quad Error \approx O(h^6) \\ & I = 0.3111111111111111111111111111111111111111hy_1 \\ & \quad +1.4222222222222222222222222222222222222222hy_2 \\ & \quad +0.5333333333333333333333333333333333333333hy_3 \\ & \quad +1.4222222222222222222222222222222222222222hy_4 \\ & \quad +0.3111111111111111111111111111111111111111hy_5 \end{aligned} \tag{5}$$

Generally, have the closed equations, where N is number of points, *ci* are integer coefficients and *cr* are real coefficients:

$$\begin{aligned} & I = \frac{num}{den} h(ci_1 y_1 + ci_2 y_2 + ci_3 y_3 + ci_4\, y_4 + \ldots + ci_{N-2} y_{N-2} + ci_{N-1} y_{N-1} + ci_N y_N) \\ & or \quad I = h(cr_1\, y_1 + cr_2\, y_2 + cr_3\, y_3 + \ldots + cr_{N-2}\, y_{N-2} + cr_{N-1}\, y_{N-1} + cr_N\, y_N) \end{aligned} \tag{6}$$

The Table 1 and 2 shows of coefficients for higher-order closed integration equations. The Figure 1 shows the generation of rules in Maple 17.0®.

Table 1 : Newton-Cotes closed integration formulas with points=2..14.

Rule	Integer Coefficient	Real Coefficient
Trapezoidal rule Points(N)=2 Segments(N-1)=1 Error≈$O(h^2)$	num=1 den=2 $ci_1=ci_2=1$	$cr_1=cr_2=+0.5000000000000000000000000000000000000000$
Simpson's 1/3 rule Points(N)=3 Segments(N-1)=2 Error≈$O(h^4)$	num=1 den=3 $ci_1=ci_3=1$ $ci_2=4$	$cr_1=cr_3=+0.3333333333333333333333333333333333333333$ $cr_2=+1.333333333333333333333333333333333333333$
Simpson's 3/8 rule Points(N)=4 Segments(N-1)=3 Error≈$O(h^4)$	num=3 den=8 $ci_1=ci_4=1$ $ci_2=ci_3=3$	$cr_1=cr_4=+0.3750000000000000000000000000000000000000$ $cr_2=cr_3=+1.125000000000000000000000000000000000000$
Boole's rule Points(N)=5 Segments(N-1)=4 Error≈$O(h^6)$	num=2 den=45 $ci_1=ci_5=7$ $ci_2=ci_4=32$ $ci_3=12$	$cr_1=cr_5=+0.3111111111111111111111111111111111111111$ $cr_2=cr_4=+1.422222222222222222222222222222222222222$ $cr_3=+0.5333333333333333333333333333333333333333$
Points(N)=6 Segments(N-1)=5 Error≈$O(h^6)$	num=5 den=288 $ci_1=ci_6=19$ $ci_2=ci_5=75$ $ci_3=ci_4=50$	$cr_1=cr_6=+0.3298611111111111111111111111111111111111$ $cr_2=cr_5=+1.302083333333333333333333333333333333333$ $cr_3=cr_4=+0.8680555555555555555555555555555555555556$
Points(N)=7 Segments(N-1)=6 Error≈$O(h^8)$	num=1 den=140 $ci_1=ci_7=41$ $ci_2=ci_6=216$ $ci_3=ci_5=27$ $ci_4=272$	$cr_1=cr_7=+0.2928571428571428571428571428571428571429$ $cr_2=cr_6=+1.542857142857142857142857142857142857143$ $cr_3=cr_5=+0.1928571428571428571428571428571428571429$ $cr_4=+1.942857142857142857142857142857142857143$
Points(N)=8 Segments(N-1)=7 Error≈$O(h^8)$	num=7 den=17280 $ci_1=ci_8=751$ $ci_2=ci_7=3577$ $ci_3=ci_6=1323$ $ci_4=ci_5=2989$	$cr_1=cr_8=+0.3042245370370370370370370370370370370370$ $cr_2=cr_7=+1.449016203703703703703703703703703703704$ $cr_3=cr_6=+0.5359375000000000000000000000000000000000$ $cr_4=cr_5=+1.210821759259259259259259259259259259259$
Points(N)=9 Segments(N-1)=8 Error≈$O(h^{10})$	num=4 den=14175 $ci_1=ci_9=989$ $ci_2=ci_8=5888$ $ci_3=ci_7=-928$ $ci_4=ci_6=10496$ $ci_5=-4540$	$cr_1=cr_9=+0.2790828924162257495590828924162257495591$ $cr_2=cr_8=+1.661516754850088183421516754850088183422$ $cr_3=cr_7=-0.2618694885361552028218694885361552028219$ $cr_4=cr_6=+2.961834215167548500881834215167548500882$ $cr_5=-1.281128747795414462081128747795414462081$
Points(N)=10 Segments(N-1)=9 Error≈$O(h^{10})$	num=9 den=89600 $ci_1=ci_{10}=2857$ $ci_2=ci_9=15741$ $ci_3=ci_8=1080$ $ci_4=ci_7=19344$ $ci_5=ci_6=5778$	$cr_1=cr_{10}=+0.2869754464285714285714285714285714285714$ $cr_2=cr_9=+1.581127232142857142857142857142857142857$ $cr_3=cr_8=+0.1084821428571428571428571428571428571429$ $cr_4=cr_7=+1.943035714285714285714285714285714285714$ $cr_5=cr_6=+0.5803794642857142857142857142857142857143$
Points(N)=11 Segments(N-1)=10 Error≈$O(h^{12})$	num=5 den=299376 $ci_1=ci_{11}=16067$ $ci_2=ci_{10}=106300$ $ci_3=ci_9=-48525$ $ci_4=ci_8=272400$ $ci_5=ci_7=-260550$ $ci_6=427368$	$cr_1=cr_{11}=+0.2683414836192613970391748169525947303725$ $cr_2=cr_{10}=+1.775359414248303137192026080914969803859$ $cr_3=cr_9=-0.8104357062690396023729357062690396023729$ $cr_4=cr_8=+4.549462882796216129549462882796216129549$ $cr_5=cr_7=-4.351551226551226551226551226551226551227$ $cr_6=+7.137646304312970979637646304312970979638$
Points(N)=12 Segments(N-1)=11 Error≈$O(h^{12})$	num=11 den=87091200 $ci_1=ci_{12}=2171465$ $ci_2=ci_{11}=13486539$ $ci_3=ci_{10}=-3237113$ $ci_4=ci_9=25226685$ $ci_5=ci_8=-9595542$ $ci_6=ci_7=15493566$	$cr_1=cr_{12}=+0.2742655400315990593768371546149323927102$ $cr_2=cr_{11}=+1.703408943727954144620811287477954144621$ $cr_3=cr_{10}=-0.4088615497317754262198706643151087595532$ $cr_4=cr_9=+3.186240802744708994708994708994708994709$ $cr_5=cr_8=-1.211958980930335097001763668430335097002$ $cr_6=cr_7=+1.956905244157848324514991181657848324515$
Points(N)=13 Segments(N-1)=12 Error≈$O(h^{14})$	num=1 den=5255250 $ci_1=ci_{13}=1364651$ $ci_2=ci_{12}=9903168$ $ci_3=ci_{11}=-7587864$ $ci_4=ci_{10}=35725120$ $ci_5=ci_9=-51491295$ $ci_6=ci_8=87516288$ $ci_7=-87797136$	$cr_1=cr_{13}=+0.2596738499595642452785309928167071024214$ $cr_2=cr_{12}=+1.884433281004709576138147566718995290424$ $cr_3=cr_{11}=-1.443863565006422149279292136434993577851$ $cr_4=cr_{10}=+6.797986775129632272489415346558203701061$ $cr_5=cr_9=-9.798067646639075210503781932353360924789$ $cr_6=cr_8=+16.65311602683031254459825888397316968746$ $cr_7=-16.70655744255744255744255744255744255744$
Points(N)=14 Segments(N-1)=13 Error≈$O(h^{14})$	num=13 den=402361344000 $ci_1=ci_{14}=8181904909$ $ci_2=ci_{13}=56280729661$ $ci_3=ci_{12}=-31268252574$ $ci_4=ci_{11}=156074417954$ $ci_5=ci_{10}=-151659573325$ $ci_6=ci_9=206683437987$ $ci_7=ci_8=-43111992612$	$cr_1=cr_{14}=+0.2643513483666065097943404821711700018578$ $cr_2=cr_{13}=+1.818389108455209852365937022021678106334$ $cr_3=cr_{12}=-1.010254313749384433908243432052955862480$ $cr_4=cr_{11}=+5.042650005170476813995332513851032369551$ $cr_5=cr_{10}=-4.900009612317529190875751722312568873415$ $cr_6=cr_9=+6.677790334230019869900822281774662727044$ $cr_7=cr_8=-1.392916870155399421272437145453018468891$

Table 2 : Newton-Cotes closed integration formulas with points=15..21.

Rule	Integer Coefficient	Real Coefficient
Points(N)=15 Segments(N-1)=14 Error≈O(h^{16})	num=7 den=2501928000 $ci_1=ci_{15}$=90241897 $ci_2=ci_{14}$=710986864 $ci_3=ci_{13}$=-770720657 $ci_4=ci_{12}$=3501442784 $ci_5=ci_{11}$=-6625093363 $ci_6=ci_{10}$=12630121616 $ci_7=ci_9$ =-16802270373 ci_8 =19534438464	$cr_1=cr_{15}$=+0.25248259702117726809084833776191800883l6 $cr_2=cr_{14}$=+1.9892291256982615007306365331056689O8138 $cr_3=cr_{13}$=-2.1563548587329451526982391179922044ll957 $cr_4=cr_{12}$=+9.7964847461637585094375217832007955464 75 $cr_5=cr_{11}$=-18.535966479051355594565471108680985 22420 $cr_6=cr_{10}$ =+35.337088562100907779920125599137944 81696 $cr_7=cr_9$ =-47.01010285307970493155678340863526048711 cr_8 =+54.65427831975980124128272276420424568573
Points(N)=16 Segments(N-1)=15 Error≈O(h^{16})	num=5 den=688816128 $ci_1=ci_{16}$=35310023 $ci_2=ci_{15}$=265553865 $ci_3=ci_{14}$=-232936065 $ci_4=ci_{13}$=1047777585 $ci_5=ci_{12}$=-1562840685 $ci_6=ci_{11}$=2461884669 $ci_7=ci_{10}$=-2000332805 $ci_8=ci_9$ =1018807605	$cr_1=cr_{16}$=+0.25630949657438915251415251415251415251 42 $cr_2=cr_{15}$=+1.9276106801610777615241900956186670472 38 $cr_3=cr_{14}$=-1.6908435758925784037391180248323105465 96 $cr_4=cr_{13}$=+7.6056406231533533430854859426287997716 57 $cr_5=cr_{12}$=-11.344396722662103521478521478521478521 48 $cr_6=cr_{11}$=+17.870405242601985068502925645782788639 93 $cr_7=cr_{10}$=-14.520078172443720713809999524285238570 95 $cr_8=cr_9$ =+7.3953524285075973134008848294562580276 87
Points(N)=17 Segments(N-1)=16 Error≈O(h^{18})	num=8 den=488462349375 $ci_1=ci_{17}$=15043611773 $ci_2=ci_{16}$=127626606592 $ci_3=ci_{15}$=-179731134720 $ci_4=ci_{14}$=832211855360 $ci_5=ci_{13}$=-1929498607520 $ci_6=ci_{12}$=4177588893696 $ci_7=ci_{11}$=-6806534407936 $ci_8=ci_{10}$=9368875018240 ci_9 =-10234238972220	$cr_1=cr_{17}$=+0.24638315386639209995713090778277346731 89 $cr_2=cr_{16}$=+2.0902590630422424868598399739251141540 45 $cr_3=cr_{15}$=-2.9436231463894084497881572266881946309 27 $cr_4=cr_{14}$=+13.629903822472069523976706602843477346 36 $cr_5=cr_{13}$=-31.601184574227144341609880912021521351 67 $cr_6=cr_{12}$=+68.420239947522362710660251898969608275 55 $cr_7=cr_{11}$=-111.47691389758304030185622328652380179 16 $cr_8=cr_{10}$=+153.44273768862986278347484558363971857 76 cr_9 =-167.61560411466667302334902708385434809 34
Points(N)=18 Segments(N-1)=17 Error≈O(h^{18})	num=17 den=3766102179840000 $ci_1=ci_{18}$=55294720874657 $ci_2=ci_{17}$=450185515446285 $ci_3=ci_{16}$=-542023437008852 $ci_4=ci_{15}$=2428636525764260 $ci_5=ci_{14}$=-4768916800123440 $ci_6=ci_{13}$=8855416648684984 $ci_7=ci_{12}$=-10905371859796660 $ci_8=ci_{11}$=10069615750132836 $ci_9=ci_{10}$=-3759785974054070	$cr_1=cr_{18}$=+0.24959765029771566740858701470106525956 03 $cr_2=cr_{17}$=+2.0321152738643919225309767637809965852 29 $cr_3=cr_{16}$=-2.4466671346505927095010722288794011310 18 $cr_4=cr_{15}$=+10.962745822192869799286980356939204675 83 $cr_5=cr_{14}$=-21.526655871440733171883965534759185552 84 $cr_6=cr_{13}$=+39.972915188944872024218732037662078814 34 $cr_7=cr_{12}$=-49.226312182644877136398667850754858397 42 $cr_8=cr_{11}$=+45.453750211188059702031262877823724384 57 $cr_9=cr_{10}$=-16.971488957751706097692833436513624638 26
Points(N)=19 Segments(N-1)=18 Error≈O(h^{20})	num=3 den=2534852320000 $ci_1=ci_{19}$=203732352169 $ci_2=ci_{18}$=1848730221900 $ci_3=ci_{17}$=-3212744374395 $ci_4=ci_{16}$=15529830312096 $ci_5=ci_{15}$=-42368630685840 $ci_6=ci_{14}$=103680563465808 $ci_7=ci_{13}$=-198648429867720 $ci_8=ci_{12}$=319035784479840 $ci_9=ci_{11}$=-419127951114198 ci_{10}=461327344340680	$cr_1=cr_{19}$=+0.24111742198338402609584766658122316175 01 $cr_2=cr_{18}$=+2.1879738799536850336117411368564461380 54 $cr_3=cr_{17}$=-3.8022858559211843946790557013593596647 87 $cr_4=cr_{16}$=+18.379568138426304850769373420539149988 82 $cr_5=cr_{15}$=-50.143312513574755313556097027380277522 44 $cr_6=cr_{14}$=+122.70604008892478596149538210573150865 06 $cr_7=cr_{13}$=-235.10059536847495715253344620881109160 63 $cr_8=cr_{12}$=+377.57913780141637600410583287944758848 91 $cr_9=cr_{11}$=-496.03830701371746974198481117038013480 80 cr_{10}=+545.98132684196766145335046579754989434 65
Points(N)=20 Segments(N-1)=19 Error≈O(h^{20})	num=19 den=5377993912811520000 ci_1 =ci_{20}=69028763155644023 ci_2 =ci_{19}=603652082270808125 ci_3 =ci_{18}=-926840515700222955 ci_4 =ci_{17}=4301581538450500095 ci_5 =ci_{16}=-10343692234243192788 ci_6 =ci_{15}=22336420328479961316 ci_7 =ci_{14}=-35331888421114781580 ci_8 =ci_{11}=43920768370565135580 ci_9 =ci_{12}=-37088370261379851390 $ci_{10}=ci_{11}$=15148337305921759574	cr_1 =cr_{20}=+0.24387281228282074194661998559277635517 18 cr_2 =cr_{19}=+2.1326520165489292228358723655401997520 74 cr_3 =cr_{18}=-3.2744495594079346437440282617242290000 28 cr_4 =cr_{17}=+15.197125648629170556529342443399424216 44 cr_5 =cr_{16}=-36.543394365405329892940620101854966171 06 cr_6 =cr_{15}=+78.912693677493332913769317497475427314 03 cr_7 =cr_{14}=-124.82458903532563157093669530125594032 64 cr_8 =cr_{13}=+155.16837924505544380673010442819601050 79 cr_9 =cr_{12}=-131.03009159001138647679338608488046040 84 $cr_{10}=cr_{11}$=+53.517801150140585342603473029511757760 26
Points(N)=21 Segments(N-1)=20 Error≈O(h^{22})	num=1 den=82324272054024 ci_1 =ci_{21}=19470140241329 ci_2 =ci_{20}=187926090380000 ci_3 =ci_{19}=-389358194177500 ci_4 =ci_{18}=1985969159340000 ci_5 =ci_{17}=-6208948835889375 ci_6 =ci_{16}=17019387776517504 ci_7 =ci_{15}=-37389734671290000 ci_8 =ci_{14}=68869287574320000 ci_9 =ci_{13}=-105499014813701250 $ci_{10}=ci_{12}$=136324521798440000 ci_{11}=-148192526607280936	cr_1 =cr_{21}=+0.23650546498063206389345700354592724294 52 cr_2 =cr_{20}=+2.2827543528921394997499043254887351900 28 cr_3 =cr_{19}=-4.7295674102285392846208636219418067090 62 cr_4 =cr_{18}=+24.123737869637513288070169416442542819 52 cr_5 =cr_{17}=-75.420634534306609354755299095978640681 32 cr_6 =cr_{16}=+206.73596439879602287062366807734937462 66 cr_7 =cr_{15}=-454.17631687959024595925994404538361156 59 cr_8 =cr_{14}=+836.56114844387109206952122360749434159 38 cr_9 =cr_{13}=-1281.5055898030800930311099629777147055 74 $cr_{10}=cr_{12}$=+1655.9456694494570344170493627037883217 39 cr_{11}=-1800.1073427048578931583234307861809573 63

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```
restart:
for N from 2 by 1 to 105 do
assign(x,array(1..N)): assign(y,array(1..N)):
for j from 1 by 1 to N do x[j]:=0+(j-1)*1: od:
print(`N=`,N,`  Error=Order of h^`,N+(N mod 2));
p:=sort(factor(  int( interp(x,y,xx), xx=x[1]..x[N])  ));
```

Figure 1 : Verification Newton-Cotes closed integration formulas with Maple 17.0®. (Number of points N=2..105)

3 NEWTON-COTES OPEN INTEGRATION EQUATION

In the open integration formulas, the first (y_1) and last (y_N) point does not appear in equation. The midpoints rule for a double interval is obtained by fitting a zero-degree polynomial to three discrete points (Espelid, 2003). The upper limit of integration is $x_3=x_1+2h$, then the integral (I) have the formula:

$$I = 2h(y_2) \quad or \quad I = 2hy_2 \qquad Error \approx O(h^2) \tag{7}$$

For three intervals, the rule is obtained by fitting a first-degree polynomial to four discrete points. The upper limit of integration is $x_4=x_1+3h$, then the integral (I) have the formula:

$$I = \frac{3h}{2}(y_2 + y_3) \quad or \quad I = 1.5hy_2 + 1.5hy_3 \qquad Error \approx O(h^2) \tag{8}$$

For N=5, the rule is obtained by fitting a second-degree polynomial to four equally spaced discrete points. The upper limit of integration is x_5, then

$$\begin{aligned} &I = \frac{4h}{3}(2y_2 - y_3 + 2y_4) \quad or \quad Error \approx O(h^4) \\ &I = 2.666666666666666666666666666666666666667hy_2 \\ &-1.333333333333333333333333333333333333333hy_3 \\ &+2.666666666666666666666666666666666666667hy_4 \end{aligned} \tag{9}$$

For N=6, the rule is obtained by fitting a third-degree polynomial to five equally spaced discrete points. The upper limit of integration is x_6, then:

$$\begin{aligned} &I = \frac{5h}{24}(11\,y_2 + y_3 + y_4 + 11\,y_5) \qquad or \qquad Error \approx O(h^4) \\ &I = 2.291666666666666666666666666666666666667hy_2 \\ &\quad +0.2083333333333333333333333333333333333333hy_3 \\ &\quad +0.2083333333333333333333333333333333333333hy_4 \\ &\quad +2.291666666666666666666666666666666666667hy_5 \end{aligned} \tag{11}$$

Generally, have the open formulas, where N is number of points, *ci* are integer coefficients and *cr* are real coefficients:

$$\begin{aligned} &I = \frac{num}{den} h(ci_2 y_2 + ci_3 y_3 + ci_4\, y_4 + ... + ci_{N-2} y_{N-2} + ci_{N-1} y_{N-1}) \\ &or \qquad I = h(cr_2\, y_2 + cr_3\, y_3 + ... + cr_{N-2}\, y_{N-2} + cr_{N-1}\, y_{N-1}) \end{aligned} \tag{12}$$

The Table 3 and 4 shows of coefficients for higher-order open integration formulas. The Figure 2 shows the generation of rules in Maple 17.0®.

Table 3 : Newton-Cotes open integration formulas with points=3..16.

Rule	Integer Coefficient	Real Coefficient
Midpoint rule Points(N)=3 Segments(N-1)=2 Error$\approx O(h^2)$	num=2 den=1 $ci_2=1$	$cr_2=+2.0000000000000000000000000000000000000000$
Points(N)=4 Segments(N-1)=3 Error$\approx O(h^2)$	num=3 den=2 $ci_2=ci_3=1$	$cr_2=cr_3=+1.5000000000000000000000000000000000000000$
Points(N)=5 Segments(N-1)=4 Error$\approx O(h^4)$	num=4 den=3 $ci_2=ci_4=2$ $ci_3=-1$	$cr_2=cr_4=+2.6666666666666666666666666666666666666667$ $cr_3=-1.3333333333333333333333333333333333333333$
Points(N)=6 Segments(N-1)=5 Error$\approx O(h^4)$	num=5 den=24 $ci_2=ci_5=11$ $ci_3=ci_4=1$	$cr_2=cr_5=+2.2916666666666666666666666666666666666667$ $cr_3=cr_4=+0.2083333333333333333333333333333333333333$
Points(N)=7 Segments(N-1)=6 Error$\approx O(h^6)$	num=3 den=10 $ci_2=ci_6=11$ $ci_2=ci_5=-14$ $ci_4=26$	$cr_2=cr_6=+3.3000000000000000000000000000000000000000$ $cr_3=cr_5=-4.2000000000000000000000000000000000000000$ $cr_4=+7.8000000000000000000000000000000000000000$
Points(N)=8 Segments(N-1)=7 Error$\approx O(h^6)$	num=7 den=1440 $ci_2=ci_7=611$ $ci_3=ci_6=-453$ $ci_4=ci_6=562$	$cr_2=cr_7=+2.9701388888888888888888888888888888888889$ $cr_3=cr_6=-2.2020833333333333333333333333333333333333$ $cr_4=cr_5=+2.7319444444444444444444444444444444444444$
Points(N)=9 Segments(N-1)=8 Error$\approx O(h^8)$	num=8 den=945 $ci_2=ci_8=460$ $ci_3=ci_7=-954$ $ci_4=ci_6=2196$ $ci_5=-2459$	$cr_2=cr_8=+3.894179894179894179894179894179894179894$ $cr_3=cr_7=-8.076190476190476190476190476190476190476$ $cr_4=cr_6=+18.590476190476190476190476190476190476 19$ $cr_5=-20.816931216931216931216931216931216931 22$
Points(N)=10 Segments(N-1)=9 Error$\approx O(h^8)$	num=9 den=4480 $ci_2=ci_9=1787$ $ci_3=ci_8=-2803$ $ci_4=ci_7=4967$ $ci_5=ci_6=-1711$	$cr_2=cr_9=+3.589955357142857142857142857142857142857$ $cr_3=cr_8=-5.631026785714285714285714285714285714286$ $cr_4=cr_7=+9.978348214285714285714285714285714285714$ $cr_5=cr_6=-3.437276785714285714285714285714285714286$
Points(N)=11 Segments(N-1)=10 Error$\approx O(h^{10})$	num=5 den=4536 $ci_2=ci_{10}=4045$ $ci_3=ci_9=-11690$ $ci_4=ci_8=33340$ $ci_5=ci_7=-55070$ $ci_6=67822$	$cr_2=cr_{10}=+4.4587742504409171075837742504409171075 84$ $cr_3=cr_9=-12.885802469135802469135802469135802469 14$ $cr_4=cr_8=+36.750440917107583774250440917107583774 25$ $cr_5=cr_7=-60.703262786596119929453262786596119929 45$ $cr_6=+74.759700176366843033509700176366843033 51$
Points(N)=12 Segments(N-1)=11 Error$\approx O(h^{10})$	num=11 den=7257600 $ci_2=ci_{11}=2752477$ $ci_3=ci_{10}=-6603199$ $ci_4=ci_9=15673880$ $ci_5=ci_8=-17085616$ $ci_6=ci_7=8891258$	$cr_2=cr_{11}=+4.1717988040123456790123456790123456790 12$ $cr_3=cr_{10}=-10.008155450837742504409171075837742504 41$ $cr_4=cr_9=+23.756156305114638447971781305114638447 97$ $cr_5=cr_8=-25.895857583774250440917107583774250440 92$ $cr_6=cr_7=+13.476057925485008818342151675485008818 34$
Points(N)=13 Segments(N-1)=12 Error$\approx O(h^{12})$	num=1 den=1925 $ci_2=ci_{12}=9626$ $ci_3=ci_{11}=-35771$ $ci_4=ci_{10}=123058$ $ci_5=ci_9=-266298$ $ci_6=ci_8=427956$ $ci_7=-494042$	$cr_2=cr_{12}=+5.0005194805194805194805194805194805194 81$ $cr_3=cr_{11}=-18.582337662337662337662337662337662337 66$ $cr_4=cr_{10}=+63.926233766233766233766233766233766233 77$ $cr_5=cr_9=-138.33662337662337662337662337662337662 34$ $cr_6=cr_8=+222.31480519480519480519480519480519480 52$ $cr_7=-256.64519480519480519480519480519480519 48$
Points(N)=14 Segments(N-1)=13 Error$\approx O(h^{12})$	num=13 den=958003200 $ci_2=ci_{13}=348289723$ $ci_3=ci_{12}=-1126407423$ $ci_4=ci_{11}=3371637557$ $ci_5=ci_{10}=-5718293865$ $ci_6=ci_9=6277879038$ $ci_7=ci_8=-2674103430$	$cr_2=cr_{13}=+4.7262539404878814601036823259045481267 70$ $cr_3=cr_{12}=-15.285227125546135962802629469296135962 80$ $cr_4=cr_{11}=+45.752757653627879322323766768211212655 66$ $cr_5=cr_{10}=-77.596630413134319384319384319384319384 32$ $cr_6=cr_9=+85.190140799112153278819945486612153278 82$ $cr_7=cr_8=-36.287294854547458714125380792047458714 13$
Points(N)=15 Segments(N-1)=14 Error$\approx O(h^{14})$	num=7 den=416988000 $ci_2=ci_{14}=329062237$ $ci_3=ci_{13}=-1497122214$ $ci_4=ci_{12}=6058248882$ $ci_5=ci_{11}=-16159538710$ $ci_6=ci_{10}=32215733235$ $ci_7=ci_9=-47966447844$ $ci_8=54874104828$	$cr_2=cr_{14}=+5.5239854839947432540025132617725210317 80$ $cr_3=cr_{13}=-25.132271187660076548965437854326743215 63$ $cr_4=cr_{12}=+101.70015006187228409450631672853895076 12$ $cr_5=cr_{11}=-271.27104609724980095350465720836091206 46$ $cr_6=cr_{10}=+540.80724779849779849779849779849779849 78$ $cr_7=cr_9=-805.21534170767504100837434170767504100 84$ $cr_8=+921.17455129644018532907421796310685199 57$
Points(N)=16 Segments(N-1)=15 Error$\approx O(h^{14})$	num=5 den=172204032 $ci_2=ci_{15}=181146041$ $ci_3=ci_{14}=-737951959$ $ci_4=ci_{13}=2671853466$ $ci_5=ci_{12}=-6013831334$ $ci_6=ci_{11}=9451804423$ $ci_7=ci_{10}=-9336416457$ $ci_8=ci_9=4041701868$	$cr_2=cr_{15}=+5.2596341356281367442081727796013510299 22$ $cr_3=cr_{14}=-21.426674812120543147328861614575900290 19$ $cr_4=cr_{13}=+77.578133187961591979449122306265163408 02$ $cr_5=cr_{12}=-174.61354604054799367299367299367299367 30$ $cr_6=cr_{11}=+274.43621131356552673516959231244945530 66$ $cr_7=cr_{10}=-271.08588424340726238047666619095190523 76$ $cr_8=cr_9=+117.35212645892054374197231340088482945 63$

Table 4 : Newton-Cotes open integration formulas with points=17..23.

Rule	Integer Coefficient	Real Coefficient
Points(N)=17 Segments(N-1)=16 Error≈O(h^{16})	num=16 den=1915538625 $ci_2=ci_{16}=722204696$ $ci_3=ci_{15}=-3892087348$ $ci_4=ci_{14}=18150263624$ $ci_5=ci_{13}=-57468376538$ $ci_6=ci_{12}=137035461016$ $ci_7=ci_{11}=-249560348012$ $ci_8=ci_{10}=355819203336$ $ci_9=-399697102923$	$cr_2=cr_{16}=+6.032389524904516086173934498449489631147$ $cr_3=cr_{15}=-32.50960161035646044464386616062101070919$ $cr_4=cr_{14}=+151.6044699876516454999700149611966190449$ $cr_5=cr_{13}=-480.0185246110607662635881330766692318721$ $cr_6=cr_{12}=+1144.621856035923055323408057094124113524$ $cr_7=cr_{11}=-2084.513210059650976758560532810973728081$ $cr_8=cr_{10}=+2972.066017920155486293052430618568184706$ $cr_9=-3338.566794375132999471623810248148872487$
Points(N)=18 Segments(N-1)=17 Error≈O(h^{16})	num=17 den=62768369664000 $ci_2=ci_{17}=35310023$ $ci_3=ci_{16}=265553865$ $ci_4=ci_{15}=-232936065$ $ci_5=ci_{14}=1047777585$ $ci_6=ci_{13}=-1562840685$ $ci_7=ci_{12}=2461884669$ $ci_8=ci_{11}=-2000332805$ $ci_9=ci_{10}=1018807605$	$cr_2=cr_{17}=+5.776080028330126933659781984296975478633$ $cr_3=cr_{16}=-28.40482276561302211999412175779018812529$ $cr_4=cr_{15}=+120.7857119531877634590652668254079188823$ $cr_5=cr_{14}=-336.0196952465624741067036040581014125988$ $cr_6=cr_{13}=+675.9477281475243925812984454959763601739$ $cr_7=cr_{12}=-957.7617592663299065036554013626324031968$ $cr_8=cr_{11}=+902.0728860329482302483018973318796951954$ $cr_9=cr_{10}=-373.8961288834851104919722644590369458094$
Points(N)=19 Segments(N-1)=18 Error≈O(h^{18})	num=9 den=9529520000 $ci_2=ci_{18}=6912171129$ $ci_3=ci_{17}=-43087461474$ $ci_4=ci_{16}=227788759000$ $ci_5=ci_{15}=-834322842510$ $ci_6=ci_{14}=2317367615100$ $ci_7=ci_{13}=-4988390746282$ $ci_8=ci_{12}=8524579147752$ $ci_9=ci_{11}=-11696802277350$ $ci_{10}=12990970309270$	$cr_2=cr_{18}=+6.528087475654597503336999135318463049555$ $cr_3=cr_{17}=-40.69325141937894038734374868828650341255$ $cr_4=cr_{16}=+215.1313844768676701449810693508172499769$ $cr_5=cr_{15}=-787.9626237827298751668499567659231524778$ $cr_6=cr_{14}=+2188.600111642559121550718189373651558526$ $cr_7=cr_{13}=-4711.204417068016017595849528622637866335$ $cr_8=cr_{12}=+8050.899975000629622478361974160293488025$ $cr_9=cr_{11}=-11046.85445816263568364408700543154324667$ $cr_{10}=+12269.11038367409901023346401497662001864$
Points(N)=20 Segments(N-1)=19 Error≈O(h^{18})	num=19 den=64023737057280000 $ci_2=ci_{19}=21156441141866149$ $ci_3=ci_{18}=-121972899306097215$ $ci_4=ci_{17}=596043470364791516$ $ci_5=ci_{16}=-1967193294708433100$ $ci_6=ci_{15}=4792224378449610000$ $ci_7=ci_{14}=-8635040534820624232$ $ci_8=ci_{13}=11419549616838153340$ $ci_9=ci_{12}=-10248543211438519308$ $ci_{10}=ci_{11}=4175787902007892850$	$cr_2=cr_{19}=+6.278489825356881835928412120617397789995$ $cr_3=cr_{18}=-36.19727921758873480653772631674903694823$ $cr_4=cr_{17}=+176.8848001921393224671383928914101476954$ $cr_5=cr_{16}=-583.7939851280550748211558677720451071767$ $cr_6=cr_{15}=+1422.164143731269979555752198142487591601$ $cr_7=cr_{14}=-2562.577220614401768069350071286648386624$ $cr_8=cr_{13}=+3388.921870115258482018911113388410480087$ $cr_9=cr_{12}=-3041.408233373194120867756294149073483029$ $cr_{10}=cr_{11}=+1239.227414469215032687069842981590396606$
Points(N)=21 Segments(N-1)=20 Error≈O(h^{20})	num=5 den=1247337455364 $ci_2=ci_{20}=1749481500626$ $ci_3=ci_{19}=-12389954060697$ $ci_4=ci_{18}=73278572831682$ $ci_5=ci_{17}=-304672055470086$ $ci_6=ci_{16}=966316491145704$ $ci_7=ci_{15}=-2400158698258188$ $ci_8=ci_{14}=4782407754794376$ $ci_9=ci_{13}=-7751977518223986$ $ci_{10}=ci_{12}=10322815990097148$ $ci_{11}=-11349750778891702$	$cr_2=cr_{20}=+7.012863652504780777619044396407280048932$ $cr_3=cr_{19}=-49.66560575654863142437769429566798286866$ $cr_4=cr_{18}=+293.7399679475580661266111534587995997771$ $cr_5=cr_{17}=-1221.289612365468958918554481275996132751$ $cr_6=cr_{16}=+3873.516693458515541474781051053405349250$ $cr_7=cr_{15}=-9621.128139528889042469653401485523548124$ $cr_8=cr_{14}=+19170.46479374246868509030813848777421471$ $cr_9=cr_{13}=-31074.09901341330118168988869965816949939$ $cr_{10}=cr_{12}=+41379.40356759641848596208767827772804682$ $cr_{11}=-45495.91103066651548985786557791751465496$
Points(N)=22 Segments(N-1)=21 Error≈O(h^{20})	num=7 den=136216903680000 $ci_2=ci_{21}=131721567613331$ $ci_3=ci_{20}=-871503959599375$ $ci_4=ci_{19}=4813298466509865$ $ci_5=ci_{18}=-18345435138969285$ $ci_6=ci_{17}=52322284124735964$ $ci_7=ci_{16}=-113381582504747148$ $ci_8=ci_{15}=188254898608060740$ $ci_9=ci_{14}=-234658964587522740$ $ci_{10}=ci_{13}=203086455887932170$ $ci_{11}=ci_{12}=-81146847108493522$	$cr_2=cr_{21}=+6.768990840221960035672424410814503693761$ $cr_3=cr_{20}=-44.78539412059277986959452226480090257180$ $cr_4=cr_{19}=+247.3488117504173693459774874248558459085$ $cr_5=cr_{18}=-942.7467700665400633484726702605959571904$ $cr_6=cr_{17}=+2688.770475458451912449167189879264182670$ $cr_7=cr_{16}=-5826.524139747866833908641667929593626188$ $cr_8=cr_{15}=+9674.161243248905274191591432303506606912$ $cr_9=cr_{14}=-12058.80259891588794040631066559859129519$ $cr_{10}=cr_{13}=+10436.33464577312869074899236470443900784$ $cr_{11}=cr_{12}=-4170.025264220237589238381372669298365893$
Points(N)=23 Segments(N-1)=22 Error≈O(h^{22})	num=11 den=136073176948800000 $ci_2=ci_{22}=92630057200320343$ $ci_3=ci_{21}=-734938747634165690$ $ci_4=ci_{20}=4810326117267632170$ $ci_5=ci_{19}=-22377003329240798370$ $ci_6=ci_{18}=79806119136903527115$ $ci_7=ci_{17}=-224650147742325069072$ $ci_8=ci_{16}=511409287186029264120$ $ci_9=ci_{15}=-956236430810735088960$ $ci_{10}=ci_{14}=1484914688541488498910$ $ci_{11}=ci_{13}=-1928478457946599022420$ $ci_{12}=2103160001429187403708$	$cr_2=cr_{22}=+7.488107884678219107763794023251961361867$ $cr_3=cr_{21}=-59.41160782200078205336853449914283228910$ $cr_4=cr_{20}=+388.8612618330624445981208858188398831456$ $cr_5=cr_{19}=-1808.931356943816102313267694473241452700$ $cr_6=cr_{18}=+6451.435398147956012301934773154146760206$ $cr_7=cr_{17}=-18160.46101499777111663885000033407847909$ $cr_8=cr_{16}=+41341.74188615452911701408934393529574465$ $cr_9=cr_{15}=-77301.05943565865461982799972793457733459$ $cr_{10}=cr_{14}=+120038.8051504254972433824004775105596634$ $cr_{11}=cr_{13}=-155895.9929729168011329179167912379773400$ $cr_{12}=+170017.0491677866414346941869480738468518$

```
restart:
for N from 3 by 1 to 105 do
assign(x,array(1..N-2)): assign(y,array(1..N-2)):
for j from 1 by 1 to N-2 do x[j]:=0+j*1: od:
print(`N=`,N,`  Error=Order of h^`,N-2+(N mod 2));
```

Figure 2 : Verification Newton-Cotes open integration formulas with Maple 17.0®. With number of points N=3..105.

4 NEWTON-COTES SEMI-CLOSED OR SEMI-OPEN INTEGRATION EQUATION

In the semi-open integration formulas, the last (y_N) point does not appear in equation. In the semi-closed integration formulas, the first (y_1) point does not appear in equation (Xiaoping Zhang et al.,2009). The rule for a double interval is obtained by fitting a zero-degree polynomial to three discrete points. When N is odd, the semi-closed or semi-open integration formulas are same as open rules. The upper limit of integration is x_3=x_1+2h, then the integral (I) have the formula:

$$I = 2h(y_2) \quad or \quad I = 2hy_2 \qquad Error \approx O(h^2) \tag{13}$$

For three intervals, the semi-open rule is obtained by fitting a first-degree polynomial to four discrete points. The upper limit of integration is x_4=x_1+3h, then the integral (I) have the formula:

$$I = \frac{3h}{4}(y_1 + 3y_3) \quad or \quad I = 0.75hy_1 + 2.25hy_3 \quad Error \approx O(h^3) \tag{14}$$

For N=5, the rule is obtained by fitting a second-degree polynomial to four equally spaced discrete points, same of the open formula. The upper limit of integration is x_5, then

$$\begin{aligned} &I = \frac{4h}{3}(2y_2 - y_3 + 2y_4) \quad or \quad Error \approx O(h^4) \\ &I = 2.666666666666666666666666666666666666667hy_2 \\ &-1.333333333333333333333333333333333333333hy_3 \\ &+2.666666666666666666666666666666666666667hy_4 \end{aligned} \tag{15}$$

For N=6, the rule is obtained by fitting a third-degree polynomial to five equally spaced discrete points. The upper limit of integration is x_6, then:

$$I = \frac{5h}{144}(19\mathrm{y}_1 - 10\,\mathrm{y}_2 + 120\,\mathrm{y}_3 - 70\,\mathrm{y}_4 + 85\,\mathrm{y}_5) \qquad or \qquad Error \approx O\left(h^5\right)$$

$$\begin{aligned} I = {} & 0.6597222222222222222222222222222222222222 hy_1 \\ & -0.3472222222222222222222222222222222222222 hy_2 \\ & + 4.166666666666666666666666666666666666667 hy_3 \\ & - 2.430555555555555555555555555555555555556 hy_4 \\ & + 2.951388888888888888888888888888888888889 hy_5 \end{aligned} \tag{16}$$

Generally, have the semi-open formulas, where N is number of points:

$$\begin{aligned} & I = \frac{num}{den} h(ci_1 y_1 + ci_2 y_2 + ci_3 y_3 + ci_4\, y_4 + ... + ci_{N-2} y_{N-2} + ci_{N-1} y_{N-1}) \\ & or \qquad I = h\left(cr_1\, y_1 + cr_2\, y_2 + cr_3\, y_3 + ... + cr_{N-2}\, y_{N-2} + cr_{N-1}\, y_{N-1}\right) \end{aligned} \tag{17}$$

The Table 5 and 6 shows of coefficients for higher-order semi-open integration formulas. The Figure 3 shows the generation of rules in Maple 17.0®.

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Table 5 : Newton-Cotes Semi-Open integration formulas with points=4,6,8,10,12,14,16.

Rule	Integer Coefficient	Real Coefficient
Points(N)=4 Segments(N-1)=3 Error≈$O(h^3)$	num=3 den=4 ci_1=1 ci_2=0 ci_3=3	cr_1=+0.7500000000000000000000000000000000000000 cr_3=+0.0000000000000000000000000000000000000000 cr_3=+2.2500000000000000000000000000000000000000
Points(N)=6 Segments(N-1)=5 Error≈$O(h^5)$	num=5 den=144 ci_1=19 ci_2=-10 ci_3=120 ci_4=-70 ci_5=85	cr_1=+0.6597222222222222222222222222222222222222 cr_2=-0.3472222222222222222222222222222222222222 cr_3=+4.1666666666666666666666666666666666666667 cr_4=-2.4305555555555555555555555555555555555556 cr_5=+2.9513888888888888888888888888888888888889
Points(N)=8 Segments(N-1)=7 Error≈$O(h^7)$	num=7 den=8640 ci_1=751 ci_2=-840 ci_3=8547 ci_4=-11648 ci_5=14637 ci_6=-7224 ci_7=4417	cr_1=+0.6084490740740740740740740740740740740741 cr_2=-0.6805555555555555555555555555555555555556 cr_3=+6.9246527777777777777777777777777777777778 cr_4=-9.4370370370370370370370370370370370370370 cr_5=+11.858680555555555555555555555555555555556 cr_6=-5.8527777777777777777777777777777777777778 cr_7=+3.5785879629629629629629629629629629629630
Points(N)=10 Segments(N-1)=9 Error≈$O(h^9)$	num=9 den=44800 ci_1=2857 ci_2=-4986 ci_3=51966 ci_4=-110322 ci_5=182880 ci_6=-177102 ci_7=129666 ci_8=-50886 ci_9=20727	cr_1=+0.5739508928571428571428571428571428571429 cr_2=-1.0016517857142857142857142857142857142857 cr_3=+10.439598214285714285714285714285714285714 cr_4=-22.162901785714285714285714285714285714286 cr_5=+36.739285714285714285714285714285714285714 cr_6=-35.578526785714285714285714285714285714286 cr_7=+26.048973214285714285714285714285714285714 cr_8=-10.222633928571428571428571428571428571429 cr_9=+4.1639062500000000000000000000000000000000
Points(N)=12 Segments(N-1)=11 Error≈$O(h^{11})$	num=11 den=43545600 ci_1 =2171465 ci_2 =-5199788 ci_3 =58096731 ci_4 =-166532520 ci_5 =353493954 ci_6 =-493861632 ci_7 =509355198 ci_8 =-363089496 ci_9 =191759205 ci_{10}=-61333844 ci_{11}=18686327	cr_1 =+0.5485310800631981187536743092298647854203 cr_2 =-1.3135119966196355085243974132863021751911 cr_3 =+14.675743152006172839506172839506172839506 cr_4 =-42.067573302469135802469135802469135802469 cr_5 =+89.295669229497354497354497354497354497354 cr_6 =-124.75377425044091710758377425044091710758 cr_7 =+128.66758473875661375661375661375661375661 cr_8 =-91.719587191358024691358024691358024691358 cr_9 =+48.440054907958553791887125220458553791887 cr_{10}=-15.493466251469723691945914168136390358613 cr_{11}=+4.7203298840755437977660199882422104644327
Points(N)=14 Segments(N-1)=13 Error≈$O(h^{13})$	num=13 den=201180672000 ci_1 =8181904909 ci_2 =-25042017078 ci_3 =303460165164 ci_4 =-1091975193010 ci_5 =2849201218305 ci_6 =-5161714089948 ci_7 =6998518415616 ci_8 =-7041630408228 ci_9 =5368397527935 ci_{10}=-3000860791630 ci_{11}=1248049610964 ci_{12}=-334728417738 ci_{13}=81322746739	cr_1 =+0.5287026967332130195886809643423400037157 cr_2 =-1.6181784203106747749604892462035319178184 cr_3 =+19.609150858845923330050314177298304282433 cr_4 =-70.561835627678984987186045387103588161790 cr_5 =+184.11120446980612531207769303007398245493 cr_6 =-333.54239501359255823541537827252112966433 cr_7 =+452.23399692694137138581583026027470471915 cr_8 =-455.01983066725217022836070455118074165697 cr_9 =+346.89797568205259797521702283607045511813 cr_{10}=-193.91122369444118369382919647469912020187 cr_{11}=+80.647135638019938615176710414805652900891 cr_{12}=-21.629659486344692197866801041404216007399 cr_{13}=+5.2549566372210944796923632902468881304863
Points(N)=16 Segments(N-1)=15 Error≈$O(h^{15})$	num=5 den=344408064 ci_1 =35310023 ci_2 =-132048240 ci_3 =1737308175 ci_4 =-7509141440 ci_5 =23317670355 ci_6 =-51787057200 ci_7 =87363166155 ci_8 =-113100595200 ci_9 =114119402805 ci_{10}=-89363498960 ci_{11}=54248941869 ci_{12}=-24880511040 ci_{13}=8556919025 ci_{14}=-1970244240 ci_{15}=397602105	cr_1 =+0.5126189931487783050283050283050283050283 cr_2 =-1.9170317684547595261880976166690452404744 cr_3 =+25.221653564418282610246895961181675467392 cr_4 =-109.01518031819371105085390799676513962228 cr_5 =+338.51806610137908966033966033966033966033 cr_6 =-751.82701297028863993149707435421721136013 cr_7 =+1268.3089521823739876195233338090480947620 cr_8 =-1641.9562580276865991151705437419723134010 cr_9 =+1656.7469628847017937419723134008848294562 cr_{10}=-1297.3491085272614290471433328576185719048 cr_{11}=+787.56782345549261006850292564578278863992 cr_{12}=-361.20685954670329670329670329670329670330 cr_{13}=+124.22646156450041773702487988202273916567 cr_{14}=-28.603340716203439417725132010846296560584 cr_{15}=+5.7722531287769150492364778079063793349511

Table 6 : Newton-Cotes Semi-Open integration formulas with points=18,20,22.

Rule	Integer Coefficient	Real Coefficient
Points(N)=18 Segments(N-1)=17 Error≈$O(h^{17})$	num=17 den=1883051089920000 ci_{1} =55294720874657 ci_{2} =-244912369711442 ci_{3} =3489029300972250 ci_{4} =-17585886834501250 ci_{5} =63416259440780110 ci_{6} =-166654158061846266 ci_{7} =336711046842479186 ci_{8} =-532651057910098250 ci_{9} =670227439244428800 ci_{10}=-673987225218482870 ci_{11}=542720673660231086 ci_{12}=-347616418702275846 ci_{13}=175509574710531250 ci_{14}=-68185176240903550 ci_{15}=20014523360265510 ci_{16}=-4031052737981102 ci_{17}=695097885157727	cr_{1} =+0.4991953005954313348171740294021305191205 cr_{2} =-2.211044781196774423415002486137112827295 cr_{3} =+31.49861330583873805806676177046547416918 cr_{4} =-158.7636563802537840385521896397851718251 cr_{5} =+572.5157518371225552605531294537761322006 cr_{6} =-1504.537344853319677900117714932529747345 cr_{7} =+3039.794207901884222712274226089628793920 cr_{8} =-4808.721352778786240060168999028493443543 cr_{9} =+6050.747389779716168605057493946382835272 cr_{10}=-6084.690367695219580800443160819410084548 cr_{11}=+4899.628853201162359464231524784140892313 cr_{12}=-3138.246832267173976985071561791138510715 cr_{13}=+1584.483175231209421948555179007853904973 cr_{14}=-615.5690635800040216043210605232945033063 cr_{15}=+180.6891480246395236371261503536635811768 cr_{16}=-36.39194757513992347706890622822427643121 cr_{17}=+6.275275328925558268476956013699105997754
Points(N)=20 Segments(N-1)=19 Error≈$O(h^{19})$	num=19 den=2688996956405760000 ci_{1} =69028763155644023 ci_{2} =-353947208843214156 ci_{3} =5438538991957452489 ci_{4} =-31293644979684279096 ci_{5} =128605896878516520180 ci_{6} =-390165018822674369064 ci_{7} =918778256758909425228 ci_{8} =-1717150274758012947672 ci_{9} =2590121803284253347498 ci_{10}=-3180795372743080898560 ci_{11}=3195943710049002658134 ci_{12}=-2627210173545633198888 ci_{13}=1761071043128578083252 ci_{14}=-954110145180024206808 ci_{15}=412501439151154330380 ci_{16}=-138949589112759712968 ci_{17}=35595226518134779191 ci_{18}=-6365379507657675444 ci_{19}=957599291114022281	cr_{1} =+0.4877456245656414838932399711855527103437 cr_{2} =-2.500931416824664874149907360722550996191 cr_{3} =+38.42780134095441222912798927464052773436 cr_{4} =-221.1156294534241283897454235960008639451 cr_{5} =+908.7076260428078658921584440557461864750 cr_{6} =-2756.840367547146254441527874975328030624 cr_{7} =+6491.932553822166738924756753801952128196 cr_{8} =-12133.09488606171610139955772962061897389 cr_{9} =+18301.36480637014593133263836498834201619 cr_{10}=-22474.96485191227391420225755605998238030 cr_{11}=+22582.00045421255508488746450211900589583 cr_{12}=-18563.42498955016870428622513715810293701 cr_{13}=+12443.43164455182698901301793847701099491 cr_{14}=-6741.581731892818002066630144404464008849 cr_{15}=+2914.665754902132920269066509970278885252 cr_{16}=-981.7944147736185256780396842594561188171 cr_{17}=+251.5098807506824695028041084827997123780 cr_{18}=-44.97670045977028151661604579808898573441 cr_{19}=+6.766235449922523319821652091802950500339
Points(N)=22 Segments(N-1)=21 Error≈$O(h^{21})$	num=7 den=14983859404800000 ci_{1} =1022779523247467 ci_{2} =-5966218027482930 ci_{3} =98462673861087480 ci_{4} =-636505825186027230 ci_{5} =2937368924847356265 ci_{6} =-10101722474707772328 ci_{7} =27170960245549634640 ci_{8} =-58577829795256960440 ci_{9} =103027050438855916590 ci_{10}=-149446538576972018620 ci_{11}=180038500415174725632 ci_{12}=-180712201906578844740 ci_{13}=151179046691155956690 ci_{14}=-105098354746771143240 ci_{15}=60350973167958502320 ci_{16}=-28329147803950914648 ci_{17}=10710818043854933655 ci_{18}=-3183966521788733730 ci_{19}=723790940733103880 ci_{20}=-116321026020880590 ci_{21}=15512151960713877	cr_{1} =+0.4778112547184455679924133079916924555258 cr_{2} =-2.787234254146951324175841749019345416756 cr_{3} =+45.99874427591187804896400625362066397811 cr_{4} =-297.3560186286105781653736836856735533910 cr_{5} =+1372.248759044328713574769806959153989832 cr_{6} =-4719.215217696328173705208737223935647803 cr_{7} =+12693.44009313908338147729814982840594999 cr_{8} =-27365.76722252499515658028820321249254545 cr_{9} =+48131.08115796670025959799374211490732740 cr_{10}=-69816.84369673698890925674684558022582227 cr_{11}=+84108.47091254089177076793175864383294724 cr_{12}=-84423.20360673035518924412058295396319601 cr_{13}=+70626.21840265571689075329677241793763043 cr_{14}=-49098.73106468978837117819030111459044755 cr_{15}=+28194.12547613585548957753125006150618309 cr_{16}=-13234.50983290264692006301759503279345666 cr_{17}=+5003.766004569320689372409667099014129692 cr_{18}=-1487.451600445568010859823841371125356490 cr_{19}=+338.1329501469220272645360159432774124584 cr_{20}=-54.34161921496169122944278842463475168232 cr_{21}=+7.246802094940405603664837718806196149286

```
restart:
for N from 3 by 1 to 105 do
assign(x,array(1..N-1)): assign(y,array(1..N-1)):
for j from 1 by 1 to N-1 do x[j]:=0+(j-1)*1: od:
print(`N=`,N,`  Error=Order of h^`,N-1);
p:=sort(factor(  int( interp(x,y,xx), xx=0..x[N-1]+1)  )):
```

Figure 3 : Verification Newton-Cotes Semi-Open integration formulas with Maple 17.0®. With N=3..105.

The semi-closed integration formulas are same as semi-open rules. For examples, N=3:

$$I = 2h(y_2) \quad or \quad I = 2hy_2 \qquad Error \approx O(h^2) \tag{18}$$

For N=4:

$$I = \frac{3h}{4}(y_2 + 3y_4) \quad or \qquad I = 0.75hy_2 + 2.25hy_4 \quad Error \approx O(h^3) \tag{19}$$

Generally, have the semi-closed formulas, where N is number of points, *ci* are integer coefficients and *cr* are real coefficients:

$$\begin{aligned} & I = \frac{num}{den} h(ci_2 y_2 + ci_3 y_3 + ci_4\, y_4 + \ldots + ci_{N-2} y_{N-2} + ci_{N-1} y_{N-1} + ci_N\, y_N\,) \\ & or \qquad I = h\left(cr_2\; y_2 + cr_3\; y_3 + \ldots + cr_{N-2}\; y_{N-2} + cr_{N-1}\; y_{N-1} + cr_N\; y_N\;\right) \end{aligned} \tag{20}$$

The semi-open or semi-closed rules can be used on type of improper integral – that is, one with a lower limit of -∞ or an upper limit of +∞. Such integrals usually can be evaluated by making a change of variable that transforms the infinite range to one that is finite (Sung Hee Choi et al., 2003). The following identity serves this purpose and works for any function that decreases toward zero at least as fast $1/x^2$ as x approaches infinity:

$$\int_a^b f(x)dx = \int_{1/b}^{1/a} \frac{1}{w^2} f\left(\frac{1}{w}\right) dw \tag{21}$$

For ab>0. Therefore, it can be used only when a is positive and b is ∞ or when a is -∞ and b is negative. For cases where the limits are from -∞ to ∞, the integral can be implemented in three steps. For example:

$$\int_{-\infty}^{\infty} f(x)dx = \int_{-\infty}^{-A} f(x)dx + \int_{-A}^{A} f(x)dx + \int_{A}^{\infty} f(x)dx \tag{22}$$

where A is a positive number. One problem with using Equation (21) to evaluate an integral is that the transformed function will be singular at one of the limits (Weiwei Sun and Jiming Wu, 2005). The semi-open or semi-closed integration formulas can be used to circumvent this dilemma as they allow evaluation of integral without employing data at the end points of integration interval.

5 ANALYSIS OF ERROR

Let's solve the example problem presented in Equation (23) using Simpson's 1/3 rule (Weiwei Sun and Jiming Wu, 2008). The exact solution is $I_{exact}=e^{0.8}-e^{0.0}=1.2255409$.

$$I = \int_0^{0.8} e^x dx \tag{23}$$

Solving the problems for two increments of h=0.4, the minimum permissible number of increments for Simpson's 1/3 rule, and one interval yields:

$$\begin{aligned} &I(h=0.4) = \frac{0.4}{3}[1+4(1.49182470)+2.22554093] = 1.22571196 \\ &\text{Error} = |1.22571196 - 1.2255409| = 0.0001710 \end{aligned} \tag{24}$$

Breaking the total range of integration into four increments of h=0.2 and two intervals and applying the composite rule yields:

$$\begin{aligned} I(h=0.2) = \frac{0.2}{3}&[1+4(1.22140276)+2(1.49182470) \\ &+4(1.82211880)+2,225540938] = 1.22555177 \\ \text{Error} = |1.&22555177 - 1.2255409| = 0.0000108 \end{aligned} \tag{25}$$

The global error of Simpson's 1/3 rule is $O(h^4)$. Thus, for successive increment havings:

$$Ratio = \frac{0.0001710}{0.0000108} = 15.775 \qquad Ratio = \frac{Error(h)}{Error(h/2)} = \frac{O[(h)^4]}{O[(h/2)^4]} = 2^4 = 16 \cong 15.775 \tag{26}$$

6 MIXING RULES

To estimate the error of integration in discrete function, we can apply different rules or mix several integration formulas of Newton-Cotes (Berriochoa et al.,2007). The difference in the result of each formula provides an approximation of the error. For example, calculate the integral of the points in Table 7:

Table 7: Set of points

x	0.0	0.1	0.2	0.3	0.4	0.5
y=f(x)	1.0000	1.1052	1.2214	1.3499	1.4918	1.6487

Using trapezoidal rule:

$$I = \frac{0.1}{2}[1.0000 + 2(1.1052) + 2(1.2214) + 2(1.3499) + 2(1.4918) + 1.6487] = 0.6493 \quad (25)$$

Using Simpson's 1/3 and 3/8 rule:

$$I = \frac{0.1}{3}[1.0000 + 2(1.1052) + 1.2214] + \frac{3(0.1)}{8}[1.2214 + 3(1.3499) + 3(1.4918) + 1.6487] = 0.6487 \quad (27)$$

Using closed rule with N=6:

$$I = \frac{5(0.1)}{288}[19(1.0000) + 75(1.1052) + 50(1.2214) + 50.(1.3499) + 75(1.4918) + 19(1.6487)] = 0.6487 \quad (28)$$

Using open rule with N=6:

$$I = \frac{5(0.1)}{24}[0(1.0000) + 11(1.1052) + 1.2214 + 1.3499 + 11(1.4918) + 0(1.6487)] = 0.6487 \quad (29)$$

Then, estimation of the error is |0.6487-0.6493|=0.0006 and the best result is 0.6487.

7 CONCLUSION

Newton-Cotes Integration is very important. To improve the error, formulas of higher-order should be applied. To estimate the error of integration for set of points proposed to apply different rules or mix various integration formulas. The difference in the result of each formula provides an approximation of the error of integration. And we can choose the best result as the average of all values or the value that appeared more times using the various formulas of integration. Computational routines to generate Newton-Cotes integration rules were presented with number of points until one hundred. The equations of Newton-Cotes shown in the article can be used in many scientific applications. The equations have high precision and numerical stability.

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