Boundary element method to analysis nonlinear in elasticity

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Abstract. *This paper presents a proposal for solving plane elastic problems. The kernels presents on the fundamental solution for displacements and tractions are weakly singular and can be integrated easily. Only the boundary is discretized and are used continuos quadratic elements. The domains integrals are transformed in boundary integral using the radial integration method (RIM). Results for internal displacements under residual stress are presented and comparated with a stress residual.*

Keywords: *Boundary element method, residual stress, radial integration method.*

1 INTRODUCTION

The boundary element method is a established numercial technique to deal an enormous number of engineering complex problems which have the formulation described by partial differential equations. Problems elasto-plascity using the BEM has atracted the attention of many researchers during the past years, proving to be a particularly adequate field of applications for that technique. The first work using BEM applied em elasto-plascity problems was proposed by Mukherjee (1977) presented the expressions for creep and Fracture problems, Bui (1978) presented the expressions three-dimensional thermoelastoplastic problems, Telles and Brebbia (1979) e (1980) presenteds expressions for plasticity problems and Banerjee and Davies (1986) presented the formulation the analisys of two and three dimensional problems of elasto-plasticity. The kernels of Kelvin fundamental solution for displacement and traction are weakly singular and can be integrated easily. The equilibruim equations was obtained through of Hooke's Law and in followed aplicated the Betti's theorem (Kane, 1994). Domain integrals which appear are exactly transformed into boundary integrals by a radial integration method (RIM) proposed by Gao (2002).

2 Basics equations

On the formulations bellow the Latin indices take values of 1 and 2.

1. The relationship between the total stress (σ_{ij}) and the strain (ε) is give by Hooke's law:

$$\sigma ij = \lambda\delta_{ij}\varepsilon_{kk} + 2G\varepsilon_{ij} \tag{1}$$

in which λ and G are elastic constants.

2. The elasto-plastic deformation can be decompost in a parcel elastic part and other plastic part as follows:

$$\dot{\varepsilon} = \dot{\varepsilon}^p + \dot{\varepsilon}^e \tag{2}$$

3. The strain-displacement relationship:

$$\dot{t}_i = \dot{\sigma}_{ij}n_j \tag{3}$$

$$\dot{t}_i^* = \dot{\sigma}_{ij}^*n_j \tag{4}$$

4. The strain-displacement relationship:

$$\dot{\varepsilon}_{ij} = (\dot{u}_{i,j} + \dot{u}_{j,i})/2 \tag{5}$$

$$\dot{\varepsilon}_{ij}^* = (\dot{u}_{i,j}^* + \dot{u}_{j,i}^*)/2 \tag{6}$$

5. Differential equations of equilibrium:

$$\dot{\sigma}_{ij,j} + b_i = 0 \tag{7}$$

$$\dot{\sigma}_{ij,j}^* + b_i^* = 0 \tag{8}$$

where b is the body force. More details in Gao and Davies (2002).

3 Betti's theorem

Considering a domain Ω with a boundary Γ associated with the total strain σ_{ij} and σ_{ij}^* and your related strains ε_{ij} and ε_{ij}^*. Using the Eq. (1) and multiplying both sides by ε_{ij}^*, we obtain:

$$\dot{\sigma}^e_{ij}\varepsilon^*_{ij} = \lambda\delta_{ij}\dot{\varepsilon}_{kk}\varepsilon^*_{ij} + 2G\dot{\varepsilon}_{ij}\varepsilon^*_{ij} \tag{9}$$

Using the properties of Kronecker delta, we obtain:

$$\dot{\sigma}^e_{ij}\varepsilon^*_{ij} = \lambda\dot{\varepsilon}_{kk}\varepsilon^*_{mm} + 2G\dot{\varepsilon}_{ij}\varepsilon^*_{ij} \tag{10}$$

So,

$$\dot{\sigma}^e_{ij}\varepsilon^*_{ij} = \sigma^*_{ij}\dot{\varepsilon}_{ij} \tag{11}$$

Using the Betti's theorem on the Eq. (11), the following integral results:

$$\int_\Omega \dot{\sigma}^e_{ij}\varepsilon^*_{ij}d\Omega = \int_\Omega \sigma^*_{ij}\dot{\varepsilon}_{ij}d\Omega \tag{12}$$

Substuting the Eq. (2) to (11) into Eq. (12) and integration by parts, the hand-left is given by:

$$\int_\Omega \dot{\sigma}^e_{ij}\varepsilon^*_{ij} = \int_\Gamma u^*_i d\Gamma + \int_\Omega u^*_i b_i d\Omega + \int_\Omega \varepsilon^*_{ij}\dot{\sigma}^p_{ij}d\Omega \tag{13}$$

Similarly, the right-hand side of Eq. (11) can be integrated, we obtain:

$$\sigma^*_{ij}\dot{\varepsilon}_{ij} = \int_\Gamma t^*_i\dot{u}_i d\Gamma + \int_\Omega b^*_i\dot{u}_i d\Omega \tag{14}$$

By substuting the Eq. (13) and (14) into Eq. (11), we obtain:

$$\int_\Gamma u^*_i d\Gamma + \int_\Omega u^*_i b_i d\Omega + \int_\Omega \varepsilon^*_{ij}\dot{\sigma}^p_{ij}d\Omega = \int_\Gamma t^*_i\dot{u}_i d\Gamma + \int_\Omega b^*_i\dot{u}_i d\Omega \tag{15}$$

4 Boundary integral equation

The boundary integral equation for a region Ω bounded by a boundary Γ, can be express by direct formulation of the boundary equations for elasto-plastic (Gao and Davies, 2002) how can be seen bellow:

$$\begin{aligned} c_{ij}(P)\dot{u}_j(P) + \int_\Gamma T_{ij}(P,Q)\dot{u}_j(Q)D\Gamma(Q) = \int_\Gamma U_{ij}(P,Q)\dot{t}_j(Q)d\Gamma(Q) + \\ \int_\Omega E_{ijk}(P,q)\dot{\sigma}^p(q)d\Omega \end{aligned} \tag{16}$$

where $c_{ij} = 0,5\delta_{ij}$ for smooth boundary and $c_{ij} = \delta_{ij}$ for internal points. T_{ij} is the fundamental solution for tractions (t) and U_{ij} is the fundamental solution for displacements (u). The term E_{ijk} of the Eq. (16) can be written as follows:

$$E_{ijk} = \frac{-1}{8\pi(\beta-1)(1-\nu)Gr^{\beta-1}}\left\{(1-a\nu)(r_{,k}\delta_{ij} + r_{,j}\delta_{ik}) - r_{,i}\delta_{jk} + \beta r_{,i}r_{,j}r_{,k}\right\} \tag{17}$$

where,

$$r_{,i} = \frac{r_i}{r}$$

and

$$r_i = x_i(Q) - x_i(p)$$

where, $\beta = 3$ in three dimensions and $\beta = 2$ in plane-strain ($\alpha = 2$ and $\beta = 2$), r is the distance beteween the source point p and the field point Q.

5 Fundamental solution

The Kelvin fundamental solutions for displacements (U_{ij}) and tractions (T_{ij}) are given by:

$$U_{ij} = \frac{1}{8\pi(1-\nu)G}\left[-(3-4\nu)\delta_{ij} lnr + r_{,i}r_{,j}\right] \tag{18}$$

for $2D$ problems, and

$$U_{ij} = \frac{1}{16\pi(1-\nu)Gr}\left[(3-4\nu)\delta_{ij} + r_{,i}r_{,j}\right] \tag{19}$$

for $3D$ problems, and

$$T_{ij} = \frac{-1}{4\alpha\pi(1-\nu)r^{\alpha}}\left\{\frac{\partial r}{\partial n}\left[(1-2\nu)\delta_{ij} + (\alpha+1)r_{,i}r_{,j}\right] - (1-2\nu)(r_{,i}n_j - r_{,j}n_i)\right\} \tag{20}$$

where $\alpha = 1$ for $2D$ and $\alpha = 2$ for $3D$ problems.

Some useful quantities related to the distance r are give by:

$$r = \sqrt{r_i r_i} \tag{21}$$

$$r_i = x_i - x_i^p \tag{22}$$

$$\frac{\partial r}{\partial n} = r_{,i}n_i \tag{23}$$

$$r_{,i} = \frac{\partial r}{\partial x_i} = \frac{r_i}{r} \tag{24}$$

$$r_{,i}r_{,i} = 1 \tag{25}$$

The kernels of displacements and tractions are found in Gao (2003).

6 Domain integral

The Eq. (16) presents domain integral on your formulation. The domain integral can be transformed into equivalent boundary integral using the radial integration method (RIM). The domain integral of Eq. (16) can be expressed in polar co-ordinate of following form:

$$\int_{\Omega} E_{ijk}(P,q)\dot{\sigma}^p(q)d\Omega = \int_{\Omega} E_{ijk}(P,q)\dot{\sigma}^p(q)rdrd\phi \tag{26}$$

The term of domain integral $E_{ijk}(P,q)\dot{\sigma}^p(q)$ of the Eq. (16) is called of $f(x_1,x_2)$ and can be written as:

$$\int_{\Omega} f(x_1,x_2)d\Omega = \int_0^{2\pi}\int_0^{r} f(x_1,x_2)rdrd\phi = \int_0^{2\pi} F(\phi)d\phi \tag{27}$$

where

$$F(\phi) = \int_0^r f(r,\phi) r dr \tag{28}$$

The term $d\phi$ can be express in the form cartesian co-ordinate system:

$$d\phi = \frac{r_i n_i}{r^2(Q)} d\Gamma = \frac{1}{r(Q)} \frac{\partial r}{\partial n} d\Gamma \tag{29}$$

The symbol Q indicates that the field point is located on the boundary. On the Eq. (17) the value β is 2. Substituting the domain integral of Eq. (16), we obtain:

$$\begin{aligned} \int_\Omega E_{ijk}(P,q)\dot{\sigma}^p(q) d\Omega = \int_\Gamma \frac{1}{r}\frac{\partial r}{\partial n}\frac{K}{r^{-1}}\{(1-a\nu)(r_{,k}\delta_{ij} + r_{,j}\delta_{ik}) - r_{,i}\delta_{jk} \\ +\beta r_{,i} r_{,j} r_{,k}\}\dot{\sigma}^p(q) d\Gamma \end{aligned} \tag{30}$$

where K is given by:

$$K = \frac{-1}{8\pi(\beta-1)(1-\nu)G}$$

More details can be found in Gao and Davies (2000).

7 Results

The code implement presented a formulation to problems elastoplasticity and thermoelasticity. The results to thermoelascity was shown in Gao (2003).

Consider a plate unitary subjected to tensile displament, under uniformily distribuited load with amplitude $q = 1N/m$. The plate have the following properties: E=$1N/m^2$ and Poisson $\nu = 0$. The plate present a side clamped and three free as shown the Figure 1.

The plate was discretized into 4, 8, 12 and 16 quadratic boundary elements, 8, 16, 24 and 32 doundary nodes and 1 internal point. Was used a stress residual give by:

$$\dot{\sigma}^p = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

The computed displacements in the internal points is shown in Table (1).

Table 1: Displacements into internal points

Boundary elements	Displacements (u_x)	Time (s)
4	1,000153	0,289077
8	1,000076	0,288737
12	1,000051	0,327817
16	1,000039	0,367943

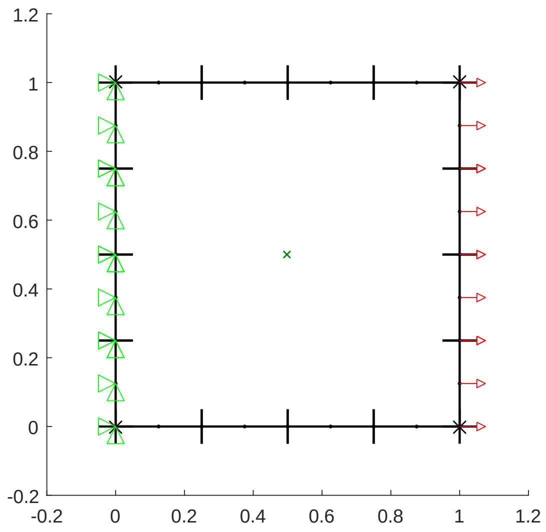

Figure 1: Plate unitary.

The displacements u_x can be see on Figure 2.

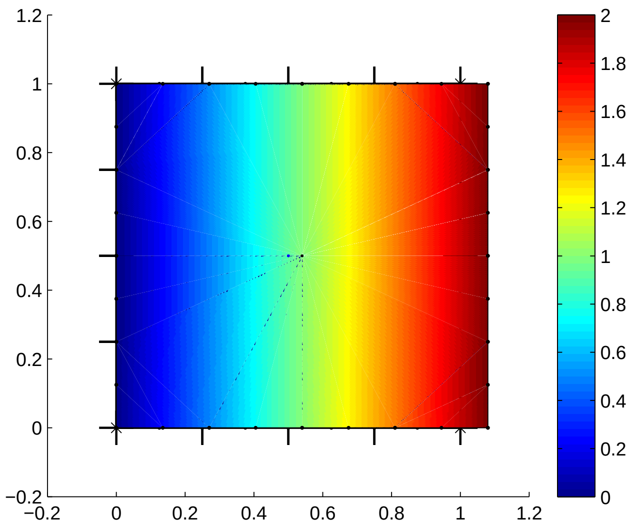

Figure 2: Displacements on direction x.

As can be seen in Table 1, results are in good agreement considering the number of the boundary elements and low processing time on a Dell Inspiron, Core i3, 4GB, DDR4, 2400 MHz. The displacements on direction y are equals zero. The code implement was validated the stress residual give by:

$$\dot{\sigma}^p = \begin{bmatrix} -1 & 0 \\ 0 & 0 \end{bmatrix}$$

The results are shown in Table 2.

Table 2: Displacements into internal points

Boundary elements	Displacements (u_x)	Time (s)
16	0,000000	0,367943

8 Conclusions

This paper presented a boundary element formulation for elasto-plascity problems for the computation of displacements in plates under uniformly distribuited load. Domain integral included in these equations was evaluated using the radial integration method (RIM). Was used of this work quadrature Gauss with 10 integrations points.

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