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THE INFLUENCE OF THE STABILIZATION TERM ON DG AND HYBRID DG FORMULATIONS FOR POISSON PROBLEM

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Abstract.*Discontinuous Galerkin (DG) methods have been successfully applied to solve many problems in engineering due to their well suited formulation for* p*-adaptivity techniques, as well as their good accuracy and flexibility. Their disadvantages of having much more degrees of freedom when compared to continuous Galerkin methods for higher dimension spaces can be overcome by hybridization associated with stabilization techniques to achieve coercivity of the bilinear form involved. The extra stabilization terms are, at least in the general setting, usually necessary for existence and uniqueness of solution of most DG and hybrid DG methods. In 1974 Bevilacqua et al. proposed a symmetric DG formulation for a heterogeneous Poisson problem with an interface of discontinuity based on the minimization of a functional in which the interface conditions are weakly imposed. Numerical experiments with discontinuous trigonometric basis functions show that this formulation is stable despite the fact that no stabilization term is added. In this work we investigate the stability behavior of the formulation presented by Bevilacqua et al. and how the edge stabilization term influences the associated DG and hybrid DG methods in a more general setting of finite element meshes and polynomial approximations. We conduct numerical experiments with the heterogeneous Poisson problem confronting the results of DG and hybrid DG methods with and without stabilization terms.*

Keywords: *Discontinuous Galerkin, Hybridization, Stabilization, Finite Elements*

1 INTRODUCTION

Discontinuous Galerkin methods have been proposed, applied and analyzed over the last decades for elliptic (Baumann and Oden (1999); Brezzi et al. (2006)), parabolic (Rivière (2008); Arnold (1982)) and hyperbolic (Klieber and Rivière (2006); Kaya and Rivière (2005)) problems. Choosing these methods over classical continuous Galerkin methods is justified by the natural properties of robustness, local conservation and flexibility for implementing h and p-adaptivity strategies of finite element spaces consisting of discontinuous piecewise polynomials. The disadvantage of such methods compared to continuous Galerking methods is that they usually have a much more complex computational implementation and much larger number of degrees-of-freedom for higher dimension spaces.

Hybridizations of DG methods establish a natural connection between DG and hybrid methods, and have been used to derive new stabilized finite element methods with reduced computational cost (Bernardo Cockburn and Sayas (2010); Cockburn et al. (2009)), but still keeping the robustness and flexibility of DG methods. The hybridization technique inserts boundary integrals into the bilinear forms, usually with opposite sign to the interior integrals. Therefore, coercivity of such bilinear forms is usually proven by adding a positive symmetric consistent term, called the stabilization term, to bound these negative boundary terms.

Bevilacqua et al. (1974) presented a symmetric DG formulation for a second order problem, applied their formulation to a case test and showed that the method could be stable even if no stabilization term is added. In this work we investigate the properties of their formulation for a general number of elements, using the traditional space of piecewise polynomial functions. We also check how their results can be extended to a hybrid DG formulation for the same problem. In particular, we provide numerical evidence that the symmetric and the nonsymmetric formulations with no extra stabilization term can lead to the optimal rates of convergence of the stabilized methods.

The remainder of the paper is organized as follows: in Section 2 we present some preliminary notations and introduce our model problem. In Section 3 we recall a DG and a hybrid DG formulation for our model problem, presenting their properties and discussing the influence of the stabilization term in the numerical analysis. In Section 4 we conduct p and h convergence tests to determine the influence of the stabilization term in these formulations. Concluding remarks are presented in Section 5.

2 PRELIMINARIES

In this section we present some notations and definitions commonly adopted in broken function spaces associated with DG and hybrid methods. We also introduce our model problem consisting of a heterogeneous Poisson problem with an interface of discontinuity in the diffusion coefficient.

2.1 Notations and Definitions

Let $H^m(\Omega)$ denote the usual Sobolev space equipped with the norm $\|\cdot\|_{m,\Omega}=\|\cdot\|_m$ and the associated semi-norm $|\cdot|_{m,\Omega}=|\cdot|_m$, with $m \geq 0$. For $m = 0$, we set $L^2(\Omega) = H^0(\Omega)$ as the space of square integrable functions and $H^1_0(\Omega)$ the subspace of functions in $H^1(\Omega)$ with zero trace on $\partial\Omega$. In addition, we set $L^2_0(\Omega) = \left\{q \in L^2(\Omega) : \int_\Omega q\, d\mathbf{x} = 0\right\}$.

For simplicity, we restrict our presentation to the two dimensional case ($d = 2$), and we consider

$\mathcal{T}_h = \{K\} :=$ the union of all elements K of the partition of the domain Ω

as a finite element partition of the domain Ω. Let

$\mathcal{E}_h = \{e;\, e$ is an edge of K for all $K \in \mathcal{T}_h\}$

denote the set of all edges e of all elements K,

$\mathcal{E}_h^0 = \{e \in \mathcal{E}_h;\, e$ is the interior edge$\}$

be the set of interior edges, and

$\mathcal{E}_h^\partial = \{e \in \mathcal{E}_h; e \subset \partial\Omega\}$

be the set of edges of $\mathcal{E}_h$ on the boundary of Ω. We assume that the domain Ω is polygonal and $\mathcal{T}_h$ is a regular partition of Ω. Thus, there exists $c > 0$ such that $h \leq ch_e$, where h_e is the diameter of the edge $e \in \partial K$ and h, the mesh parameter, is the element diameter. For each element K we associate a unit outward normal vector $\mathbf{n}_K$.

To establish a connection between the discontinuous Galerkin methods and the related hybrid methods we recall some definitions normally used to formulate and analyze DG methods. Given the elements $K^+, K^- \in \mathcal{T}_h$ sharing the edge e, we define the corresponding outward unit normal vectors $\mathbf{n}_{K^+}$ and $\mathbf{n}_{K^-}$ on e belonging to K^+ and K^- respectively. For a scalar function q and a vector-valued function $\mathbf{v}$, such that are smooth inside each element $K^\pm$, we denote by $(\mathbf{v}^\pm, q^\pm)$ the traces of $(\mathbf{v}, q)$ on e taken from the interior of $K^\pm$, respectively. We define the following average and jump operators on $e \in \mathcal{E}_h^0$

$$\langle q \rangle = \frac{1}{2}(q^+ + q^-); \quad \langle \mathbf{v} \rangle = \frac{1}{2}(\mathbf{v}^+ + \mathbf{v}^-); \tag{1}$$

$$[\![q]\!] = q^+\mathbf{n}_{K^+} + q^-\mathbf{n}_{K^-}; \;\; [\![\mathbf{v}]\!] = \mathbf{v}^+ \cdot \mathbf{n}_{K^+} + \mathbf{v}^- \cdot \mathbf{n}_{K^-}. \tag{2}$$

For boundary edges $e \in \mathcal{E}_h^\partial$, we define

$$\langle q \rangle = q, \;\; [\![q]\!] = q\,\mathbf{n}; \quad \langle \mathbf{v} \rangle = \mathbf{v}, \;\; [\![\mathbf{v}]\!] = \mathbf{v} \cdot \mathbf{n}, \tag{3}$$

where $\mathbf{n}$ is the unit outward normal vector on Γ. Based on the these definitions, we have the following identities

$$\sum_{K \in \mathcal{T}_h} \int_{\partial K} q(\mathbf{v} \cdot \mathbf{n}_K)\, ds = \sum_{e \in \mathcal{E}_h} \int_e \langle q \rangle [\![\mathbf{v}]\!] ds + \sum_{e \in \mathcal{E}_h^0} \int_e [\![q]\!] \cdot \langle \mathbf{v} \rangle\, ds, \tag{4}$$

(see Arnold et al. (2002); Brezzi et al. (2005)). It is also worth noting that

$$\sum_{K \in \mathcal{T}_h} \int_{\partial K} (p - \hat{p})\,(q - \hat{q})\, ds = \sum_{e \in \mathcal{E}_h} \frac{1}{2} \int_e [\![p]\!] \cdot [\![q]\!]\, ds + \sum_{e \in \mathcal{E}_h} 2 \int_e (\langle p \rangle - \hat{p})\,(\langle q \rangle - \hat{q})\, ds, \tag{5}$$

(see Arruda et al. (2013); Faria et al. (2014)), where $\hat{p}$ and $\hat{q}$ are scalar functions uniquely defined on each edge $e \in \mathcal{E}_h$.

2.2 Model Problem

Let $\Omega \subset \mathbb{R}^n$ be an open bounded domain, for $n = 1, 2$, or 3, and $\Gamma = \partial\Omega$ be its Lipchitz boundary. We are interested on solving the second order elliptic model problem

$$\begin{cases} \mathbf{u} = -\kappa\nabla p, & \text{in } \Omega \\ \nabla\cdot\mathbf{u} + \lambda p = f, & \text{in } \Omega \\ p = 0, & \text{on } \Gamma \end{cases}, \tag{6}$$

where $\mathbf{u}$ is the velocity field, p is the pressure and κ is the hydraulic conductivity tensor. We may allow κ to be heterogeneous over Ω, but we fix $\kappa = \kappa_K I$ for each element K, where κ_K is constant on K.

3 NUMERICAL FORMULATIONS

In this section we recall classical stabilized DG and hybrid DG formulations for our model problem 6 in its primal form.

3.1 A DG formulation with two elements

First we consider a simple mesh $\mathcal{T}_h$ consisting of only two elements: $\mathcal{T}_h = \{K_1, K_2\}$, for which only one edge is in $\mathcal{E}_h^0 = \{e\}$. In order to obtain a variational DG formulation for our problem, we use the classical primal approach: we replace the second equation of (6) in the first one, and multiply the residual by a test function q_i living in a space V yet to be defined. After summing both residuals we get

$$\int_{K_1} \left(-\nabla\cdot(\kappa_1\nabla p_1) + \lambda p_1 - f_1\right) q_1 \, d\mathcal{X} + \int_{K_2} \left(-\nabla\cdot(\kappa_2\nabla p_2) + \lambda p_2 - f_2\right) q_2 \, d\mathcal{X} = 0, \tag{7}$$

where $\kappa_i = \kappa_K I$ is constant, with $\kappa_i \neq \kappa_j$ for $i \neq j$, and $\phi_i = \phi|_{K_i}$ for $\phi = p, q, f$. Using *Green's theorem* and integration by parts on the equation above, we have

$$\int_{K_1} (\kappa_1\nabla p_1\cdot\nabla q_1 + \lambda p_1\, q_1 - f_1 q_1)\, d\mathcal{X} + \int_{K_2} (\kappa_2\nabla p_2\cdot\nabla q_2 + \lambda p_2\, q_2 - f_2 q_2)\, d\mathcal{X} \\ - \int_{\partial K_1} \kappa_1\nabla p_1\cdot\mathbf{n}_1\, q_1\, ds - \int_{\partial K_2} \kappa_2\nabla p_2\cdot\mathbf{n}_2\, q_2\, ds = 0. \tag{8}$$

Applying the boundary conditions from (6) to the integrals in ∂K_i we obtain the simpler form

$$\sum_{i=1}^{2} \left(\int_{K_i} (\kappa_i\nabla p_i\cdot\nabla q_i + \lambda p_i\, q_i)\, d\mathcal{X} - \int_e \kappa_i\nabla p_i\cdot\mathbf{n}_i\, q_i\, ds \right) = \sum_{i=1}^{2} \int_{K_i} f_i q_i\, d\mathcal{X}. \tag{9}$$

To simplify the sum on i above for the integrals on e, we use definitions (1)-(3) and identity (4) to write

$$\sum_{i=1}^{2} \int_{K_i} (\kappa_i\nabla p_i\cdot\nabla q_i + \lambda p_i\, q_i)\, d\mathcal{X} - \int_e \left([\![\kappa\nabla p]\!]\langle q\rangle + \langle\kappa\nabla p\rangle\cdot[\![q]\!] \right) ds = \sum_{i=1}^{2} \int_{K_i} f_i q_i\, d\mathcal{X}. \tag{10}$$

We notice at this point that because of the equations in (6), both p and $\kappa\nabla p$ are differentiable, and therefore continuous. Because $\kappa\nabla p$ is continuous, the definition of $[\![\kappa\nabla p]\!]$ shows us that $[\![\kappa\nabla p]\!] = 0$, leading to the variational form

$$\sum_{i=1}^{2} \int_{K_i} (\kappa_i \nabla p_i \cdot \nabla q_i + \lambda p_i\, q_i)\; d\mathcal{X} - \int_e \langle \kappa\nabla p\rangle \cdot [\![q]\!]\, ds = \sum_{i=1}^{2} \int_{K_i} f_i q_i \, d\mathcal{X}. \tag{11}$$

Following the usual approach in the development of DG methods for this class of second order problems, we add to the above formulation a consistent term, using the fact that $[\![p]\!] = 0$. Such term makes the formulation symmetric, as follows:

$$\sum_{i=1}^{2} \int_{K_i} (\kappa_i \nabla p_i \cdot \nabla q_i + \lambda p_i\, q_i)\; d\mathcal{X} - \int_e (\langle \kappa\nabla p\rangle \cdot [\![q]\!] + \langle \kappa\nabla q\rangle \cdot [\![p]\!])\; ds = \sum_{i=1}^{2} \int_{K_i} f_i q_i \, d\mathcal{X}. \tag{12}$$

3.2 The Bevilacqua-Rojas-Feijóo formulation

In Bevilacqua et al. (1974), the authors presented a slightly different discontinuous Galerkin formulation that numerically approximates the solution to our model problem. From the minimization problem

$$\begin{cases} \textit{min} & J(p) = \frac{1}{2}\int_\Omega \kappa(\nabla p)^2 + 2\lambda p^2 - 2pf\, d\mathcal{X}, & \\ \text{subject to} & p = 0, & \text{on } \Gamma_D \\ & (-\kappa\nabla p)\cdot \mathbf{n} = 0, & \text{on } \Gamma_N \end{cases}, \tag{13}$$

the functional $J(p)$ was modified to weakly impose the interface conditions, yielding the minimization problem

$$\begin{cases} \textit{min} & I(p) = J(p) - B(p), & \\ \text{subject to} & p = 0, & \text{on } \Gamma_D \\ & (-\kappa\nabla p)\cdot \mathbf{n} = 0, & \text{on } \Gamma_N \end{cases}, \tag{14}$$

where

$$B(p) = \int_e \langle \kappa\nabla p\rangle \cdot [\![p]\!]\, ds. \tag{15}$$

To see why the functional $I(p)$ provides a variational formulation with the characteristics mentioned, we take its first variation, leading to

$$dI(p;q) = dJ(p;q) - dB(p;q) = 0, \tag{16}$$

where

$$dJ(p;q) = \sum_{i=1}^{2} \int_{K_i} (\kappa_i \nabla p_i \cdot \nabla q_i + \lambda p_i\, q_i - f_i q_i)\; d\mathcal{X}, \tag{17}$$

and

$$dB(p;q) = \int_e \left(\langle \kappa \nabla p \rangle \cdot [\![q]\!] + \langle \kappa \nabla q \rangle \cdot [\![p]\!] \right) \, ds. \tag{18}$$

Using integration by parts on equation (16), we get

$$\begin{aligned} dI(p;q) = & \int_{K_1} \left(\nabla \cdot (-\kappa_1 \nabla p_1) + \lambda p_1 - f_1 \right) \, q_1 \, d\mathcal{X} \\ & + \int_{K_2} \left(\nabla \cdot (-\kappa_2 \nabla p_2) + \lambda p_2 - f_2 \right) \, q_2 \, d\mathcal{X} \\ & + \int_e [\![\kappa \nabla p]\!] \langle q \rangle \, ds - \int_e \langle \kappa \nabla q \rangle \cdot [\![p]\!] \, ds = 0 \end{aligned} \tag{19}$$

The above formulation assures that the solution p must satisfy

$$-\nabla \cdot (\kappa \nabla p) + \lambda p = f \quad \text{in} \quad K \in \mathcal{T}_h,$$

as well as

$$[\![\kappa \nabla p]\!] = 0 \quad \text{and} \quad [\![p]\!] = 0 \quad \text{on} \quad e \in \mathcal{E}_h^0,$$

which means that the continuity of $\kappa \nabla p$ and p is imposed by the variational formulation itself, not by the space where p is chosen to live at.

Remark 3.1. *The DG formulation* (12) *and the Bevilacqua-Rojas-Feijóo formulation* (16) *are identical (given the definitions of $J(p)$ and $B(p)$), despite the fact that they were developted using different approaches.*

3.3 A DG formulation for general meshes

In this section we present the DG formulation introduced in the previous sections extended to a general mesh $\mathcal{T}_h = \bigcup_K K$. In order to do that, we observe that our model problem (6) may be written in the form

$$\begin{cases} \mathbf{u} = -\kappa \nabla p, & \text{in } K \in \mathcal{T}_h \\ \nabla \cdot \mathbf{u} + \lambda p = f, & \text{in } K \in \mathcal{T}_h \\ p = 0, & \text{on } e \in \mathcal{E}_h^\partial \\ [\![\mathbf{u}]\!] = 0, & \text{on } e \in \mathcal{E}_h^0 \\ [\![p]\!] = 0, & \text{on } e \in \mathcal{E}_h^0 \end{cases} , \tag{20}$$

from which we can obtain the weak form

Find $p \in V$ such that for all $q \in V$ it is valid that

$$\sum_{K \in \mathcal{T}_h} \int_K (\kappa \nabla p \cdot \nabla q + \lambda p \, q) \, d\mathcal{X} - \sum_{e \in \mathcal{E}_h^0} \int_e \langle \kappa \nabla p \rangle \cdot [\![q]\!] \, ds = \sum_{K \in \mathcal{T}_h} \int_K f \, q \, d\mathcal{X}, \tag{21}$$

with V being the space defined by

$$V = \left\{ q \in H^1(K), K \in \mathcal{T}_h; \, q|_e = 0, e \in \mathcal{E}_h^\partial \right\}. \tag{22}$$

Even though we can show several properties for formulation (21) in the infinite dimension space V, such formulation has not been proven stable in finite dimension spaces.

Remark 3.2. *The elements $K \in \mathcal{T}_h$ must be chosen so that their boundaries coincide with the discontinuities of κ. This is so because, as we noticed, $[\![\kappa \nabla p]\!] = [\![\boldsymbol{u}]\!] = 0$, forcing ∇p to be discontinuous where κ is discontinuous. Since V is such that all its functions are $H^1(K)$ for all K, then the only edges where those functions can be discontinuous are the edges of ∂K.*

The authors in Bevilacqua et al. (1974) approximated the problem (14) for q and p continuous on K_i, $i = 1, 2$ and discontinuous on e, in a finite dimension space of trigonometric functions satisfying the boundary conditions. Numerical results show that their formulation is stable at least for a mesh with two elements (K_1 and K_2). This might not be true for a general finite element mesh. In general, their formulation is not *stable*. However, stability can be recovered by adding an extra term to $I(p)$ as follows. Replacing $I(p)$ by $K(p)$ given by

$$K(p) = I(p) + \frac{\beta}{2} \int_e [\![p]\!] \cdot [\![p]\!] \, ds, \tag{23}$$

we obtain the variational formulation

$$\begin{aligned} dK(p;q) = {} & \sum_{i=1}^{2} \int_{K_i} (\kappa_i \nabla p_i \cdot \nabla q_i + \lambda p \, q) \, d\mathcal{X} - \sum_{i=1}^{2} \int_{K_i} f_i q_i \, d\mathcal{X} \\ & - \int_e (\langle \kappa \nabla p \rangle \cdot [\![q]\!] + \langle \kappa \nabla q \rangle \cdot [\![p]\!]) \, ds + \frac{\beta}{2} \int_e [\![p]\!] \cdot [\![q]\!] \, ds = 0. \end{aligned} \tag{24}$$

The amount of the extra stability is controlled by the stabilization parameter $\beta = \beta_0 / h$. The constant β_0 is a positive one, and varies according to K. This formulation is well suited for numerical purposes, since stability can be proven on very general finite dimensional spaces, see Arnold et al. (2002); Rivière (2008).

We then extend formulations (12) and (16) to a general mesh $\mathcal{T}_h$, applying the stabilization technique showed above, and find a numerical solution to our model problem by solving the following stabilized discontinuous Galerkin method:

Find $p_h \in V_h^k$ such that for all $q_h \in V_h^k$, we have

$$a_{SDG}(p_h, q_h) = f(q_h), \tag{25}$$

where the bilinear and linear forms are

$$\begin{aligned} a_{SDG}(p_h, q_h) = {} & \sum_{K \in \mathcal{T}_h} \int_K (\kappa \nabla p_h \cdot \nabla q_h + \lambda p_h \, q_h) \, d\mathcal{X} - \sum_{e \in \mathcal{E}_h^0} \int_e \langle \kappa \nabla p_h \rangle \cdot [\![q_h]\!] \, ds \\ & + s \sum_{e \in \mathcal{E}_h^0} \int_e \langle \kappa \nabla q_h \rangle \cdot [\![p_h]\!] \, ds + \frac{\beta}{2} \sum_{e \in \mathcal{E}_h^0} \int_e [\![p_h]\!] \cdot [\![q_h]\!] \, ds \end{aligned} \tag{26}$$

and

$$f(q_h) = \sum_{K \in \mathcal{T}_h} \int_K f \, q_h \, d\mathcal{X} \tag{27}$$

respectively. Here the symmetrization parameter s may assume values $s = -1$ (*symmetric*), $s = 1$ (*non-symmetric*) or $s = 0$ (*incomplete*), see Sun and Wheeler (2006). The space V_h^k is defined as

$$V_h^k = \left\{ q_h \in L^2(\Omega);\, q_h|_K \in \mathbb{S}_k(K),\, \forall K \in \mathcal{T}_h^0;\, q_h|_K = 0,\, \forall K \in \mathcal{T}_h^\partial \right\},$$

with $\mathbb{S}_k = \mathbb{P}_k$ being the space of polynomial functions of degree at most k for simplex elements, or $\mathbb{S}_k = \mathbb{Q}_k$ the product space of polynomial functions of degree at most k in each space variable associated to quadrilateral elements. For the space V_h^k, we define the energy norm

$$||p||^2 = \sum_{K\in\mathcal{T}_h} \int_\Omega \left(|\nabla p|^2 + |p|^2\right) \, d\mathcal{X} + \sum_{e\in\mathcal{E}_h^0} \frac{1}{h^\gamma} \int_e |[\![p]\!]|^2 \, ds. \tag{28}$$

For the SDG formulation (25), we can prove the following properties:

Consistency: Consistency of problem (25) is easily proven by taking into account that the solution to the original problem (6) also satisfies (20). Because of the conditions imposed on $\mathcal{E}_h$, all the boundary terms vanish, and using Green's theorem and integration by parts on (25), we arrive to the final expression

$$\sum_{K\in\mathcal{T}_h} \int_\Omega (-\nabla\cdot(\kappa\nabla p_h) + \lambda p_h - f)\, q_h \, d\mathcal{X} = 0,$$

which proves the consistency.

Local Conservation: Choosing $q_h = 1$ on K, and $q_h = 0$ for every other element in $\mathcal{T}_h$, we can see from Eq. (25) that

$$\int_K \lambda p_h \, d\mathcal{X} - \int_{\partial K} \kappa\nabla p_h \cdot \mathbf{n}_K \, ds + \frac{\beta}{2} \int_{\partial K} p_h \cdot \mathbf{n}_K \, ds = \int_K f q_h \, d\mathcal{X}. \tag{29}$$

Therefore, the local conservation of the method depends on the stabilization parameter β.

Stability: Using the energy norm defined on Eq. (28), we can show that

$$\begin{aligned} a_{SDG}(q_h, q_h) \geq & \sum_{K\in\mathcal{T}_h} \int_K \left(\kappa_{min}|\nabla q_h|^2 + \lambda|q_h|^2\right) \, d\mathcal{X} \\ & + (s-1) \sum_{e\in\mathcal{E}_h^0} \int_e \langle\kappa\nabla q_h\rangle \cdot [\![q_h]\!] \, ds + \frac{\beta}{2} \sum_{e\in\mathcal{E}_h^0} \int_e |[\![q_h]\!]|^2 \, ds, \end{aligned} \tag{30}$$

where κ_{min} is the smallest eigenvalue in modulus of κ over Ω. From the above equation, we can promptly see that the method is stable if $s = 1$: we take $m = \min\{\kappa_{min}, \lambda, \beta_0/2\}$, and write

$$a_{SDG}(q_h, q_h) \geq m \sum_{K\in\mathcal{T}_h} \int_K |\nabla q|^2 \, d\mathcal{X} + \frac{m}{h} \sum_{e\in\mathcal{E}_h^0} \int_e |[\![q]\!]|^2 \, ds = m||q||.$$

If $s = 0$ or $s = -1$, then we have to rely on trace inequalities, see Arnold et al. (2002); Rivière (2008)

$$\int_{\partial K} |q_h|^2 \, ds \leq C_k \frac{1}{h_K} \int_K |q_h|^2 \, d\mathcal{X}, \qquad \forall q_h \in \mathbb{S}_k(\partial K), \tag{31}$$

$$\int_{\partial K} |\nabla q_h \cdot \mathbf{n}_K|^2 \, ds \leq \tilde{C}_k \frac{1}{h_K} \int_K |\nabla q_h|^2 \, d\mathcal{X}, \qquad \forall q_h \in \mathbb{S}_k(\partial K). \tag{32}$$

These inequalities allow us to write

$$\begin{aligned}\sum_{e\in\mathcal{E}_h^0}\int_e \langle\kappa\nabla q_h\rangle\cdot[\![q_h]\!]\,ds &= \sum_{e\in\mathcal{E}_h^0}\int_e \langle\kappa\nabla q_h\rangle\cdot\mathbf{n}_e[\![q_h]\!]\cdot\mathbf{n}_e\,ds\\ &\leq \sum_{e\in\mathcal{E}_h^0}\int_e |\langle\kappa\nabla q_h\rangle\cdot\mathbf{n}_e|\,|[\![q_h]\!]|\,ds\\ &\leq \frac{h}{2\delta}\sum_{e\in\mathcal{E}_h^0}\int_e |\langle\kappa\nabla q_h\rangle\cdot\mathbf{n}_e|^2\,ds + \sum_{e\in\mathcal{E}_h^0}\frac{\delta}{2h}\int_e |[\![q_h]\!]|^2\,ds\\ &\leq \frac{\hat{C}_k}{2\delta}\kappa_{min}\sum_{K\in\mathcal{T}_h}\int_K |\nabla q_h|^2\,d\mathcal{X} + \sum_{e\in\mathcal{E}_h^0}\frac{\delta}{2h}\int_e |[\![q_h]\!]|^2\,ds,\end{aligned}$$

with $\delta > 0$ coming from Young's inequality. Using the above on (30) proves that for δ big enough, we have

$$a_{SDG}(q_h,q_h) \geq \tilde{m}||q||,$$

where

$$\tilde{m} = \min\left\{\kappa_{min}(1+\frac{(s-1)}{2\delta}\tilde{C}_k),\lambda,\frac{\beta_0+\delta(s-1)}{2}\right\},$$

and β_0 chosen to compensate the value of δ, keeping $\tilde{m}$ positive.

Continuity: The continuity of the bilinear form $a_{SDG}(\cdot,\cdot)$ in (25) is usually proven in the norm

$$|||p|||^2 = ||p||^2 + \sum_{K\in\mathcal{T}_h} h^2\int_\Omega |\nabla^2 p|^2\,d\mathcal{X},$$

valid for the space

$$V(h) = V_h^k + H^2(\Omega) \subset H^2(\mathcal{T}_h).$$

The infinite dimension space $V(h)$ allows us to use the inequality

$$\int_{\partial K} |\nabla q\cdot\mathbf{n}_K|^2\,ds \leq C_K\left(\frac{1}{h_K}\int_K |\nabla p|^2\,d\mathcal{X} + h_K\int_K |\nabla^2 p|^2\,d\mathcal{X}\right), \tag{33}$$

(see Arnold et al. (2002) and Agmon and (U.S.) Theorem 3.10) which is used to bound the boundary terms involving averages and jumps as follows:

$$\begin{aligned}\sum_{e\in\mathcal{E}_h^0}\int_e \langle\kappa\nabla p\rangle\cdot[\![q]\!]\,ds &\leq \sum_{e\in\mathcal{E}_h^0}\int_e |\langle\kappa\nabla p\rangle\cdot\mathbf{n}_e|\,|[\![q]\!]|\,ds\\ &\leq \sum_{e\in\mathcal{E}_h^0} h^{1/2}\int_e |\langle\kappa\nabla p\rangle\cdot\mathbf{n}_e|\,ds\left(\sum_{e\in\mathcal{E}_h^0}\frac{1}{h}\int_e |[\![q]\!]|^2\,ds\right)^{1/2}\\ &\leq C_\Omega\kappa_{max}\left(\sum_{K\in\mathcal{T}_h}\int_K |\nabla p|^2\,d\mathcal{X} + \sum_{K\in\mathcal{T}_h} h^2\int_K |\nabla^2 p|^2\,d\mathcal{X}\right)^{1/2}\\ &\cdot\left(\sum_{e\in\mathcal{E}_h^0}\frac{1}{h}\int_e |[\![q]\!]|^2\,ds\right)^{1/2} \leq C_\Omega\kappa_{max}|||p|||\left(\sum_{e\in\mathcal{E}_h^0}\frac{1}{h}\int_e |[\![q]\!]|^2\,ds\right)^{1/2}.\end{aligned}$$

Therefore, given p_h in V_h and q in $V(h)$, we have

$$|a_{SDG}(p_h, q)| \leq \max\{\kappa_{max}, \lambda, \beta_0\} |||p_h||| \, |||q||| + C_\Omega \kappa_{max} |||p_h||| \left(\sum_{e \in \mathcal{E}_h^0} \frac{1}{h} \int_e |[\![q]\!]|^2 \, ds \right)^{1/2}$$
$$+ C_\Omega \kappa_{max} |||q||| \left(\sum_{e \in \mathcal{E}_h^0} \frac{1}{h} \int_e |[\![p_h]\!]|^2 \, ds \right)^{1/2} \leq M |||p_h||| \, |||q|||,$$

where

$$M = 3 \max\{\kappa_{max}, \lambda, \beta_0, C_\Omega \kappa_{max}\}, \tag{34}$$

with κ_{max} being the largest eigenvalue in modulus of κ over Ω. This proves the continuity of the bilinear form $a_{SDG}(\cdot, \cdot)$ in $V(h)$.

3.4 A hybrid discontinuous Galerkin method

Hybrid discontinuous Galerkin methods are developed by inserting new variables, traditionally called *Lagrange multipliers*, defined only at the edges of the elements. Those variables are commonly identified with the trace of the main variables of the problem. Choosing the same order of polynomial approximations for both the Lagrange multipliers and the main variables for both methods, for high order approximations, a discontinuous Galerkin method has more degrees of freedom than a hybrid discontinuous Galerkin method after the static condensation of the element degrees of freedom.

Choosing the Lagrange multiplier to be associated with the trace of the pressure field, we may rewrite our main problem (20) as

$$\begin{cases} \mathbf{u} = -\kappa_i \nabla p, & \text{in } K \in \mathcal{T}_h, \\ \nabla \cdot \mathbf{u} + \lambda p = f_i, & \text{in } K \in \mathcal{T}_h, \\ \hat{p} = p, & \text{on } e \in \mathcal{E}_h, \\ \hat{p} = 0, & \text{on } e \in \mathcal{E}_h^\partial, \\ [\![\mathbf{u}]\!] = 0, & \text{on } e \in \mathcal{E}_h^\partial, \end{cases} \tag{35}$$

where $\hat{p}$ is uniquely defined on each edge $e \in \mathcal{E}_h$. The interface condition $[\![\kappa \nabla p]\!] = [\![\mathbf{u}]\!] = 0$ and $[\![p]\!] = 0$ on $e \in \mathcal{E}_h^0$ are weakly imposed on $e \in \mathcal{E}_h$. Furthermore, the boundary conditions are imposed on the space for the multiplier only, leaving the space for the pressure free on every edge e.

Based on the alternative definition (35) of our model problem (20), a well established hybrid discontinuous Galerkin method, see Egger (2009); Arruda et al. (2013), may be presented as: Find $p_h \in V_h^k$ and $\hat{p}_h \in M_h^l$ such that for all $q_h \in V_h^k$ and $\hat{q}_h \in M_h^l$

$$a_{SHDG}([p_h, \hat{p_h}], [q_h, \hat{q_h}]) = f_{SHDG}([q_h, \hat{q_h}]), \tag{36}$$

where

$$
\begin{aligned}
a_{SHDG}([p_h,\hat{p_h}],[q_h,\hat{q_h}]) = & \sum_{K\in\mathcal{T}_h}\int_K (\kappa\nabla p_h\cdot\nabla q_h + \lambda p_h\, q_h\,)\, d\mathcal{X} \\
& - \sum_{K\in\mathcal{T}_h}\int_{\partial K} \kappa\nabla p_h\cdot\mathbf{n}_e(q_h-\hat{q}_h)\, ds \\
& + s\sum_{K\in\mathcal{T}_h}\int_{\partial K} \kappa\nabla q_h\cdot\mathbf{n}_e(p_h-\hat{p}_h)\, ds \\
& + \beta\sum_{K\in\mathcal{T}_h}\int_{\partial K} (p_h-\hat{p}_h)\,(q_h-\hat{q}_h)\, ds,
\end{aligned}
\tag{37}
$$

and

$$
f_{SHDG}([q_h,\hat{q_h}]) = \sum_{K\in\mathcal{T}_h}\int_K f\, q_h\, d\mathcal{X}, \tag{38}
$$

for s and β defined as before. The spaces V_h^k and M_h^l are defined as

$$
\begin{aligned}
V_h^k &= \left\{q_h \in L^2(\Omega);\; q_h|_K \in \mathbb{S}_k(K),\, \forall K\in\mathcal{T}_h\right\}, \\
M_h^l &= \left\{\hat{q}_h \in M(\mathcal{E}_h);\; \hat{q}_h|_e \in p_l(e),\, \forall e\in\mathcal{E}_h^0;\; \hat{q}_h|_e = 0,\, \forall e\in\mathcal{E}_h^\partial\right\},
\end{aligned}
$$

where $M(\mathcal{E}_h)\subset L^2(\mathcal{E}_h)$ for a globally discontinuous Lagrange multiplier, or $M(\mathcal{E}_h)\subset C^0(\mathcal{E}_h)$ for a globally continuous Lagrange multiplier, see Arruda et al. (2013), and $p_l(e)$ is the space of polynomials of degree at most l defined on e.

The Eq. (36) is usually split into two equations by separating the terms multiplied by q_h from the ones multiplied by $\hat{q}_h$. The resulting two different type of equations give rise to a *local problem* and a *global problem*. The local problem is defined on each element K and it may be solved for the local solution p_h if the Lagrange multiplier $\hat{p}_h$ is provided. The global problem couples the local solution p_h to the Lagrange multiplier $\hat{p}_h$, and even though it depends on both p_h and $\hat{p}_h$, we may use the local problem to derive a linear system whose variables are only those related to $\hat{p}_h$. This solver strategy is usually known as static condensation. Explicitly, those problems are written as:

Local Problem: for each $K\in\mathcal{T}_h$, find $p_h|_K\in V_h^k$ such that for all $q_h|_K\in V_h^k$

$$
a_K(p_h,q_h) = \int_K f\, q_h\, d\mathcal{X} - b_K(\hat{p}_h,q_h). \tag{39}
$$

Global Problem: find $\hat{p}_h\in M_h^l$ satisfying, for all $\hat{q}_h\in M_h^l$

$$
\sum_{K\in\mathcal{T}_h} c_K(p_h,\hat{q}_h) - \sum_{K\in\mathcal{T}_h} d_K(\hat{p}_h,\hat{q}_h) = 0, \tag{40}
$$

where the bilinear forms $a_K(\cdot,\cdot)$, $b_k(\cdot,\cdot)$, $c_k(\cdot,\cdot)$ and $d_k(\cdot,\cdot)$ are given by

$$
\begin{aligned}
a_k(p_h,q_h) = & \int_K (\kappa\nabla p_h\cdot\nabla q_h + \lambda p_h\, q_h\,)\, d\mathcal{X} \\
& - \int_{\partial K}\kappa\nabla p_h\cdot\mathbf{n}_e q_h\, ds + s\int_{\partial K}\kappa\nabla q_h\cdot\mathbf{n}_e p_h\, ds + \beta\int_{\partial K} p_h\, q_h\, ds,
\end{aligned}
\tag{41}
$$

$$b_K(\hat{p}_h, q_h) = s \int_{\partial K} \kappa \nabla q_h \cdot \mathbf{n}_e \hat{p}_h \, ds - \beta \int_{\partial K} \hat{p}_h \, q_h \, ds, \tag{42}$$

$$c_K(p_h, \hat{q}_h) = \int_{\partial K} \kappa \nabla p_h \cdot \mathbf{n}_e \hat{q}_h \, ds - \beta \int_{\partial K} p_h \, \hat{q}_h \, ds \tag{43}$$

$$d_K(\hat{p}_h, \hat{q}_h) = \beta \int_{\partial K} \hat{p}_h \, \hat{q}_h \, ds. \tag{44}$$

Remark 3.3. *To obtain a numerical local solution p_h using the above mentioned approach, one needs to be able to solve the local problems. Given the form of $a_K(\cdot,\cdot)$, we observe that both λ and β_0 are not allowed to be zero, simultaneously. If those two parameters are zero, then the local problems can be viewed as a second order problem for which only Neumann boundary conditions are imposed. However, if β_0 is non-zero, then the local problem can be viewed as a second order problem with Robin boundary conditions. Similarly, one cannot use the above approach to find p_h if both s and β_0 are null, since the matrix generated by $b_K(\cdot,\cdot)$ would not couple the local problems to the global problem.*

The analysis of hybrid DG methods is similar to the analysis of DG methods. This means that we must convert the integrals on ∂K for all K in $\mathcal{T}_h$ in terms of integrals on e for all e in $\mathcal{E}_h$. That is, we must obtain a formulation written in terms of averages and jumps, as defined by Eqs. (1)-(3). To do that we apply identities (4)-(5) to the bilinear form $a_{SDG}([\cdot,\cdot],[\cdot,\cdot])$ in (36), yielding

$$\begin{aligned} a_{SHDG}([p_h,\hat{p_h}],[q_h,\hat{q_h}]) &= a_{SDG}(p_h,q_h) \\ &+ \sum_{e\in\mathcal{E}_h} \int_e (s[\![\kappa\nabla q_h]\!]\,(\langle p_h\rangle - \hat{p}_h) - [\![\kappa\nabla p_h]\!]\,(\langle q_h\rangle - \hat{q}_h))\, ds \\ &+ 2\beta \sum_{e\in\mathcal{E}_h} \int_e (\langle p_h\rangle - \hat{p}_h)\,(\langle q_h\rangle - \hat{q}_h)\, ds. \end{aligned} \tag{45}$$

We first notice that in the DG formulation (25) we have $[\![\kappa\nabla p]\!] = 0$ as natural interface condition, while in the hybrid DG method (36), $[\![\kappa\nabla p]\!] = 0$ is weakly imposed through the Lagrange multiplier. Furthermore, if we assume that the Lagrange multiplier $\hat{p}_h$ is globally discontinuous, then the hybrid DG method can be viewed as a modification of the DG method (25) as follows. We isolate the terms multiplying $\hat{q}_h$, leading to

$$\int_e [\![\kappa\nabla p_h]\!]\,\hat{q}_h\, ds - 2\beta \sum_{e\in\mathcal{E}_h} \int_e (\langle p_h\rangle - \hat{p}_h)\,\hat{q}_h\, ds = 0. \tag{46}$$

Assuming that $l \geq k$, we can solve the above equation for $\hat{p}_h$, showing that

$$\hat{p}_h = \langle p_h\rangle - \frac{1}{2\beta}[\![\kappa\nabla p_h]\!], \quad \text{on} \quad e \in \mathcal{E}_h. \tag{47}$$

Replacing the above on the expression for $a_{SHDG}([p_h,\hat{p_h}],[q_h,\hat{q_h}])$ and taking $\hat{q}_h = 0$ shows that

$$a_{SHDG}([p_h,\hat{p_h}],[q_h,\hat{q_h}]) = a_{SDG}(p_h,q_h) + \sum_{e\in\mathcal{E}_h} \frac{s}{2\beta} \int_e [\![\kappa\nabla q_h]\!]\,[\![\kappa\nabla p_h]\!]\, ds. \tag{48}$$

The extra term above clearly does not affect consistency, and since $s/2\beta$ can always be bounded, the proof of stability and coercivity of the bilinear form $a_{SHDG}([\cdot,\cdot],[\cdot,\cdot])$ above follows basically the same steps applied in the previous section.

4 NUMERICAL EXPERIMENTS

This section exposes results of some numerical experiments to identify whether or not the formulation proposed by Bevilacqua et al. is stable, even though it does not have the extra stabilization term involving β_0.

4.1 The Bevilacqua-Rojas-Feijóo Experiment

Bevilacqua et al. studied a variation of our model problem (20) for a mesh $\mathcal{T}_h$ with only one edge on $\mathcal{E}_h^0$. Their formulation corresponds to a DG formulation (25) with $s = -1$ and $\beta = 0$. Even though the theory predicts that β has to be different than zero for $s \neq 1$, their numerical experiments showed that the DG method is stable for their case test.

In this subsection we make a p-convergence study using the case test exposed in Bevilacqua et al. (1974). That is, we take $\mathcal{T}_h = \{K_1, K_2\}$ such that κ is constant on each K_i, and we define the problems:

$$\begin{cases} -\nabla \cdot (\kappa \nabla p) = 1, & \text{in } \Omega \\ p = 0, & \text{on } \Gamma \end{cases}, \tag{49}$$

and

$$\begin{cases} -\nabla \cdot (\kappa \nabla p) = \kappa \cos(\pi x)\, \cos(\pi y) \frac{\pi^2}{2} & \text{in } \Omega \\ p = 0, & \text{on } \Gamma \end{cases}, \tag{50}$$

where

$$\kappa = \begin{cases} 2\,\mathbf{I}, & \text{in } K_1 \\ 0.2\,\mathbf{I}, & \text{in } K_2 \end{cases}.$$

Here K_1 is given by the rectangle $[a, 0] \times [d, c]$ and K_2 is $[0, b] \times [d, c]$. Therefore, the gradient ∇p has a discontinuity at the segment connecting the points $(0, d)$ and $(0, c)$. Following Bevilacqua et al. (1974), we define the trigonometric basis as

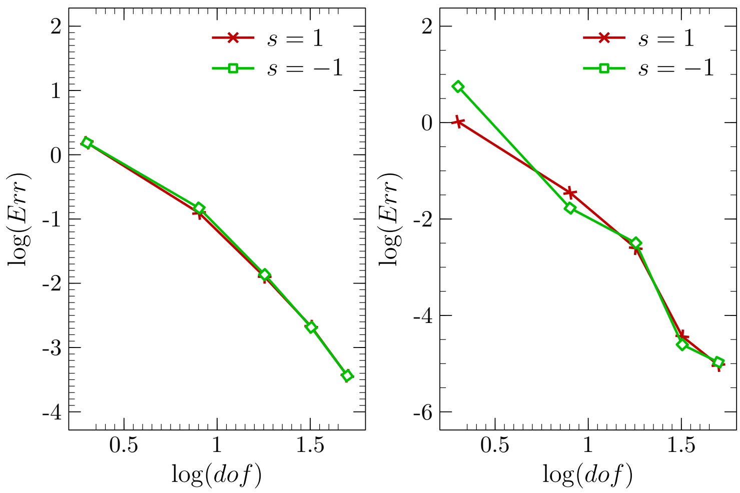
$$p_{i,j} = \begin{cases} \sin(i\pi \dfrac{x-a}{b-a}) \sin(j\pi \dfrac{y-c}{c}), & \text{in } K_1 \\ \\ \sin(i\pi \dfrac{b-x}{b-a}) \sin(j\pi \dfrac{y-c}{c}), & \text{in } K_2 \end{cases}. \tag{51}$$

We also use the polynomial approximation

$$p_{i,j} = \begin{cases} \left(\dfrac{x-a}{b-a}\right)^i \left(\dfrac{y-d}{c-d}\right)^j (y-c), & \text{in } K_1 \\ \\ \left(\dfrac{b-x}{b-a}\right)^i \left(\dfrac{y-d}{c-d}\right)^j (y-c), & \text{in } K_2 \end{cases}, \tag{52}$$

for $i = 1, \dots, M_x$ and $j = 1, \dots, M_y$. Notice that both set of basis satisfy the boundary conditions.

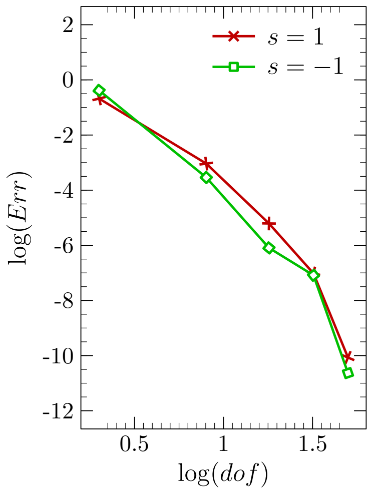
In Fig. 1 we show the results of our first test. In this test we increase M_x and M_y from 1 to 6, and use both trigonometric and polynomial approximations on problem (49). In the figure, $Err = ||p_{i,i} - p_{6,6}||$, for $i = 1, \dots, 5$. We observe that even with $\beta = 0$ on formulation (25), we obtain good results for $s = 1$ or $s = -1$. For the case of trigonometric approximation, the convergence is the expect one. For the case of polynomial approximation, there seems to be a different convergence rate for even and odd polynomial degrees when $s = -1$. For the case $s = 0$ we did not observe any convergence.



(a) Trigonometric approximations (b) Polynomial approximations

Figure 1: p-convergence test for DG formulation applied to problem (49) with $\beta = 0$

In Fig. 2 we show the results for problem (50). For this case the solution is known: $p = \cos(\pi x)\cos(\pi y)$, and therefore we have $Err = ||p_{i,i} - p||$, for $i = 1, \dots, 5$. In this case we only obtained convergence for the polynomial approximations and $s \neq 0$. We can see that the formulation is also stable for this case, with the nonsymmetric case providing more accurate results.



(a) Polynomial approximations

Figure 2: p-convergence test for DG formulation applied to problem (50) **with** $\beta = 0$

Remark 4.1. *The formulation introduced in Bevilacqua et al. (1974) is very unlikely to be extended for a general number of elements, since the space of trigonometric approximations would have to satisfy the boundary conditions of the problem. When we compare their formulation to the SDG formulation* (25)*, it is easy to tell why it does not need stabilization on the boundary edges: the space of trigonometric functions already satisfy the boundary conditions. The reason for which stabilization is not needed in the interior edge may be related to the fact that that edge is connected to the boundary on both ends. We are still working on a proof for this affirmation.*

4.2 The SHDG formulation

As we observed in subsection 3.4, the SHDG formulation can be viewed as a SDG formulation in which an extra consistent term is added, if the spaces V_h^k and Q_h^l are chosen correctly and we adopt a globally discontinuous Lagrange multiplier. Since hybrid DG methods are considerably less complex than DG methods, it is easier to study the influence of the stabilization term using formulation (36).

In order to keep our SHDG formulation similar to the DG formulation in Bevilacqua et al. (1974), we adopted in our tests the following edge-dependent stabilization parameter β_0:

$$\beta_0(e) = \begin{cases} \delta_{k,h}, & e \in \mathcal{E}_h^\partial \\ 0, & e \in \mathcal{E}_h^0 \end{cases},$$

that is, we have stabilization turned on for boundary edges, and turned off for interior edges. The positive value $\delta_{k,h}$ varies according to the approximation space V_h^k and element length h.

The problem we solve in this section is the following:

$$\begin{cases} -\nabla \cdot (\kappa \nabla p) + \lambda p = (\kappa \frac{\pi^2}{2} + \lambda) \sin(\pi x)\, \sin(\pi y) & \text{in } \Omega \\ p = \sin(\pi x)\, \sin(\pi y), & \text{on } \Gamma \end{cases}, \tag{53}$$

where $\Omega = [0,1] \times [0.25, 0.75]$,

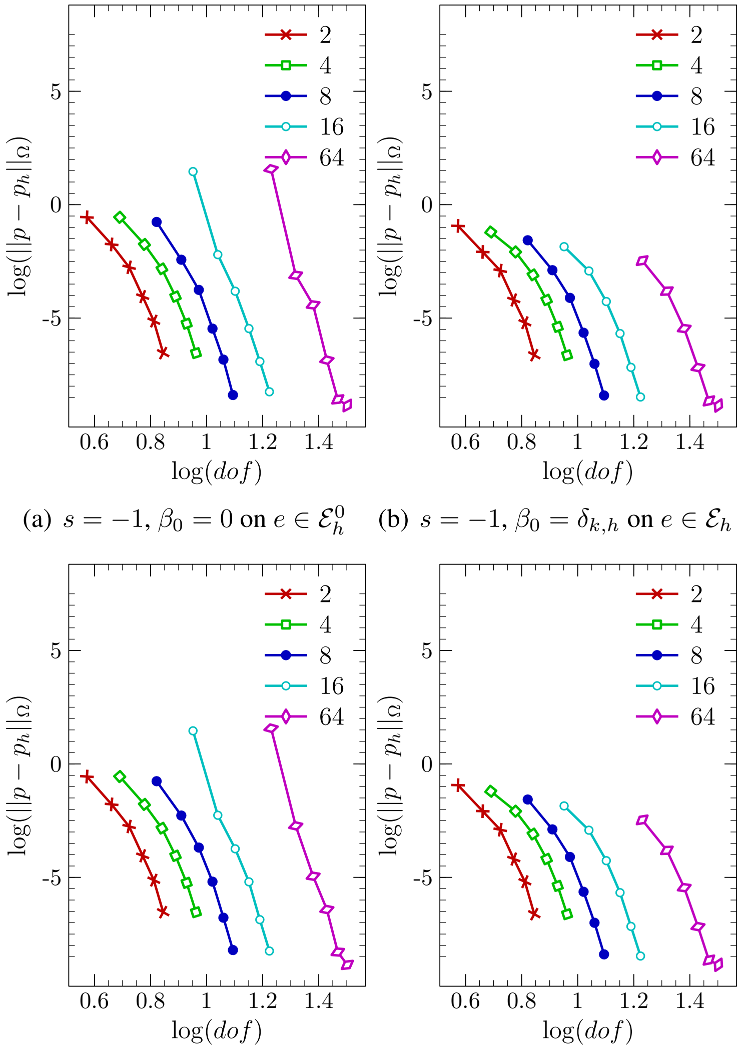
$$\kappa(x,y) = \begin{cases} 2\,\mathbf{I}, & (x,y) \in [0,0.5] \times [0.25,0.75] \\ 0.2\,\mathbf{I}, & (x,y) \in [0.5,1] \times [0.25,0.75] \end{cases},$$

and

$$\lambda(x,y) = \begin{cases} 0.3, & (x,y) \in [0,0.5] \times [0.25,0.75] \\ 3, & (x,y) \in [0.5,1] \times [0.25,0.75] \end{cases}.$$

We notice that the solution is given by $p = \sin(\pi x)\,\sin(\pi y)$ despite the fact that κ and λ are discontinuous on the line connecting $(0.5, 0.25)$ to $(0.5, 0.75)$.

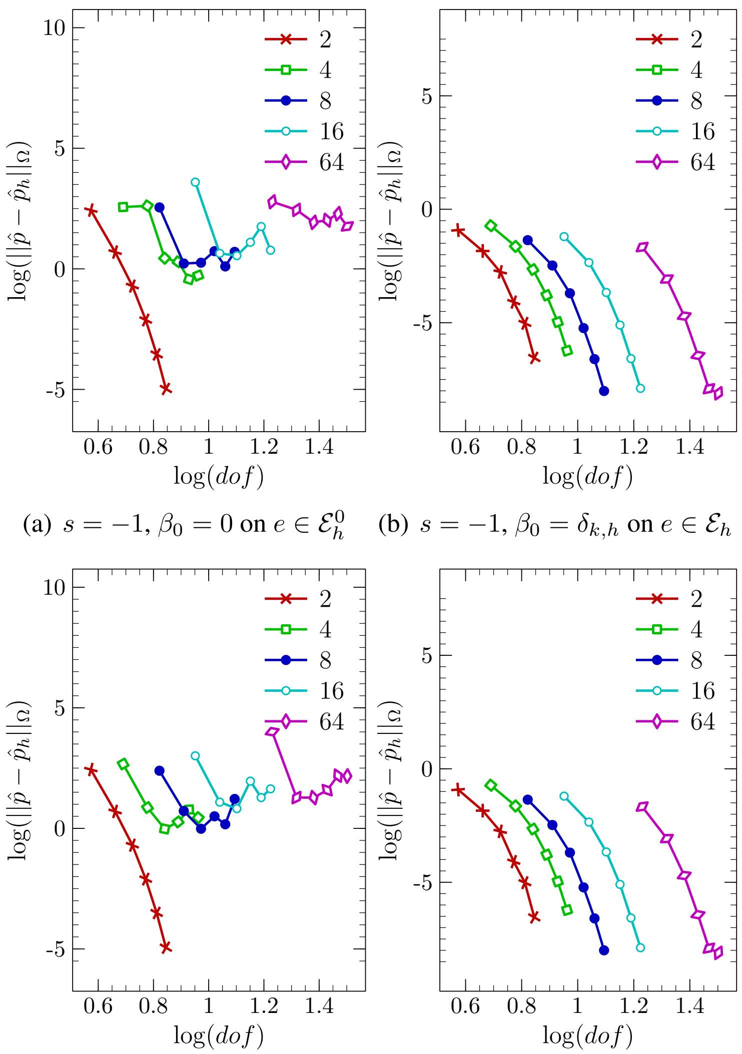
We have already verified that the DG formulation exposed in Bevilacqua et al. (1974) is stable for two elements. Therefore, a natural question that arises is whether formulation (36) is stable for more than 2 elements. In order to answer this question, we made a p-convergence test for $\mathcal{T}_h$ being composed of 2, 4, 8, 16 and 64 elements. In Fig. 3 we show a plot of $\log(dof)$ by $\log(Err)$ with dof being the number of degrees of freedom, $Err = ||p - p_h||_{\mathcal{T}_h}$ and p_h being the numerical solution for the spaces $\mathbb{Q}_i \times p_i$, for $i = 1, \cdots, 6$. By looking at the plots, we check that both the symmetric and non-symmetric formulations provide the expected results for $i > 1$ for all number of elements. Furthermore, for a general number of elements the case where the space is $\mathbb{Q}_1 \times p_1$ is not stable in general. In Fig. 4 we show similar plots in which now $Err = ||\hat{p} - \hat{p}_h||_{\mathcal{E}_h}$. In this case it is clear that the Lagrange multiplier convergence is affected by the lack of the extra stability term. This fact is observed by comparing the figures in the left (no interior stabilization) to the figures on the right (stabilization everywhere). Our conclusion is that even though the SHDG formulation is stable for symmetric and nonsymmetric cases with no interior stabilization and spaces $\mathbb{Q}_i \times p_i$ for $i > 1$, the Lagrange multiplier is unstable if no stabilization is imposed on the edges where it is defined. We also observe that the only case for which the Lagrange multiplier seems to be stable numerically is when there are only two elements. These facts make it evident that without the extra stabilization term, the Lagrange multiplier cannot be stable if we choose $k = l$ for spaces V_h^k and Q_h^l.



(a) $s = -1$, $\beta_0 = 0$ on $e \in \mathcal{E}_h^0$ (b) $s = -1$, $\beta_0 = \delta_{k,h}$ on $e \in \mathcal{E}_h$

(c) $s = 1$, $\beta_0 = 0$ on $e \in \mathcal{E}_h^0$ (d) $s = 1$, $\beta_0 = \delta_{k,h}$ on $e \in \mathcal{E}_h$

Figure 3: p-convergence test for the primal variable p_h on the SHDG formulation applied to problem (53).

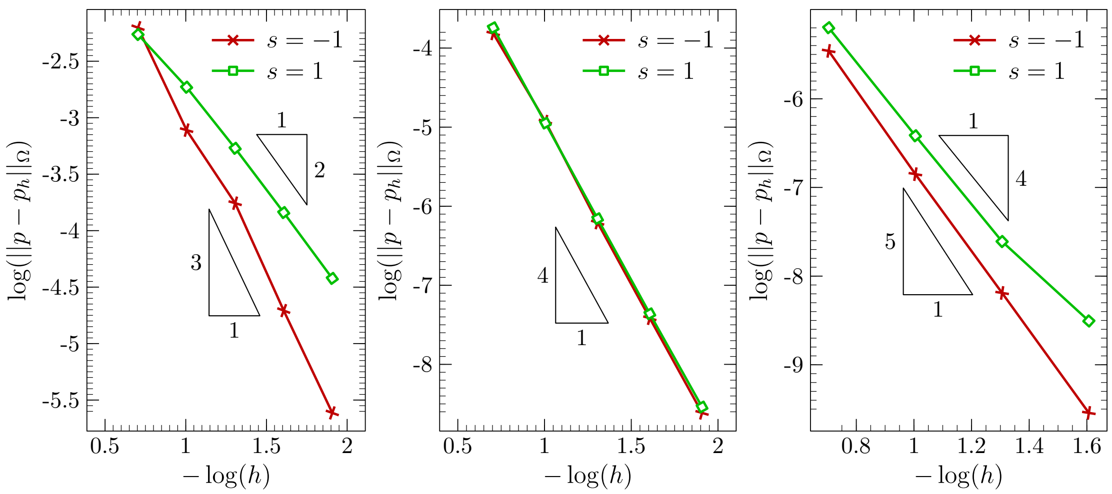


(a) $s = -1$, $\beta_0 = 0$ on $e \in \mathcal{E}_h^0$ (b) $s = -1$, $\beta_0 = \delta_{k,h}$ on $e \in \mathcal{E}_h$

(c) $s = 1$, $\beta_0 = 0$ on $e \in \mathcal{E}_h^0$ (d) $s = 1$, $\beta_0 = \delta_{k,h}$ on $e \in \mathcal{E}_h$

Figure 4: p-convergence test for the Lagrange multiplier $\hat{p}_h$ on the SHDG formulation applied to problem (53).

Since we have already identified the cases for which both p_h and $\hat{p}_h$ seem to be stable, we can now check whether or not the convergence rates are optimal. In Fig 5 we show the convergence rates for cases $\mathbb{Q}_i \times p_i$ with $i = 2, 3, 4$. Since the expected optimal rate is $i + 1$ for $\mathbb{Q}_i \times p_i$, the simulations show that the main variable p_h converges with the desired rate for the symmetric formulation ($s = -1$), despite of the lack of the extra stability term on formulation (36). On the other hand, the nonsymmetric formulation ($s = 1$) achieves the optimal rates only if $i > 1$ is odd. When i is even, then the convergence rate for the main variable is of order i, which is not optimal.



(a) $\mathbb{Q}_2 \times p_2$ (b) $\mathbb{Q}_3 \times p_3$ (c) $\mathbb{Q}_4 \times p_4$

Figure 5: h-convergence test for SHDG formulation applied to problem (53) **using** $\mathbb{Q}_k \times p_k$ **spaces,** $\beta_0 = 0$ **on** $e \in \mathcal{E}_h^0$ **and** $\beta_0 = \delta_{k,h}$ **on** $e \in \mathcal{E}_h^\partial$

5 CONCLUSIONS

We have presented numerical evidence that formulation (36) with $s = -1$ (symmetric) and $s = 1$ (nonsymmetric) is stable for the main variable even if no stabilization term is added ($\beta_0 = 0$) and the space of admissible functions/variations is, for each element K and edge e, $\mathbb{Q}_i(K) \times p_i(e)$ with $i > 1$. We have also verified that the convergence rates for the main variable are the optimal ones for the symmetric case. In the nonsymmetric case, the method achieves the optimal rate $i+1$ for i odd, and sub-optimal rate i for i even. However, the Lagrange multiplier was showed unstable numerically in the same setting, despite the fact that the main variable is stable.

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