CILAMCE 2017
Florianópolis - Santa Catarina - Brazil

A Symmetry and Structure Preserving State Derivative Feedback Control Strategy for Second-order Linear Systems

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Abstract. *Second-order Linear Control Systems are usually modeled by systems of differential equations in the form $M\ddot{x} + C\dot{x} + Kx = f$, where M, C and K are the mass, damping and stiffness matrices, which are symmetric and sometimes positive definite; x, $\dot{x}$ and f are the time-dependent state and external force vectors. In order to drive the system to achieve better performance, control strategies can be accomplished by feeding back into the system the information on its state vectors x and $\dot{x}$, using Sensors, estimators, and actuators. In this work we propose a state derivative control strategy that is capable of preserving the eigenstructure of the system, meaning that the second-order poles and modes that are not intended to change are assured to not change, while the poles that are intended to change are assigned to the controlled system through the solution of an algebraic matrix equation. Since the proposed strategy also feeds back into the system the information on the state vector $\ddot{x}$, it must be considered as derivative state feedback. Mathematical framework of the associated Model Updating problem and real-life numerical examples are presented.*

Keywords: *Structure Preserving, State, Derivative, Feedback, Second-order*

1 INTRODUCTION

The quadratic eigenvalue problem arises from second-order dynamic equations of the type

$$M\frac{d^2x}{dt^2} + C\frac{dx}{dt} + Kx = f(t) \tag{1}$$

where $M \in \mathbb{R}^{n\times n}$ is a symmetric positive definite matrix; $C \in \mathbb{R}^{n\times n}$ and $K \in \mathbb{R}^{n\times n}$ are, respectively, positive semi definite damping and stiffness matrices. Also $x \in \mathbb{R}^n$ is the displacement vector, while $f \in \mathbb{R}^n$ is the input vector.

The homogeneous solutions of the system in Eq (1) are given by $x(t) = xe^{\lambda t}$ where λ and nonzero vector x, which are called eigenvalue and eigenvector, are solutions of the quadratic eigenvalue problem (QEP)

$$Q(\lambda)x = 0 \tag{2}$$

where the matrix (also called λ-matrix, or pencil matrix)

$$Q(\lambda) = \lambda^2 M + \lambda C + K \tag{3}$$

is singular for every eigenvalue λ. In mechanical systems' applications, under the hypotheses above, there is a set of $2n$ of such pairs (λ_i, x_i) commonly referred to as frequencies and mode shape vectors (or modes), of the vibrating system in Eq (1).

Aiming to drive the system to achieve better performance, a control strategy is proposed. It is accomplished by feeding back the information on the states $x(t)$ and $\dot{x}(t)$, and derivative $\ddot{x}(t)$, using estimators, sensors and actuators, into the system through $f(t)$, which is then assumed to have the following form

$$f(t) = v(t) + B_1F_1\ddot{x}(t) - (B_1F_3 + B_2F_2)\dot{x}(t) + B_2F_4x(t) \tag{4}$$

for some design matrices $B_1, B_2 \in \mathbb{R}^{n\times m}$, $F_1, F_2, F_3, F_4 \in \mathbb{R}^{m\times n}$, for some positive integer $m \leq n$. The closed-loop control system is therefore

$$(M - B_1F_1)\frac{d^2x}{dt^2} + (C + B_1F_3 + B_2F_2)\frac{dx}{dt} + (K - B_2F_4)x = v(t) \tag{5}$$

where it must be emphasized that (4) implies

$$f(t) = v(t) + Bu(t) \tag{6}$$

since

$$f(t) - v(t) = \begin{bmatrix} B_1F_1 & -(B_1F_3 + B_2F_2) & B_2F_4 \end{bmatrix} \begin{bmatrix} \ddot{x}(t) \\ \dot{x}(t) \\ x(t) \end{bmatrix} \tag{7}$$

and therefore, through QR factorization, becomes

$$f(t) - v(t) = B\bar{F}_1\ddot{x}(t) + B\bar{F}_2\dot{x}(t) + B\bar{F}_3x(t) = Bu(t) \tag{8}$$

which gives the more usual form (6).

Several structure preserving feedback control strategies have been proposed for the undamped case ($C = 0$) in last two decades. They frequently take advantage of known results for the so called Model Updating Problem (MUP) (Datta et al, 1997; Carvalho et al, 2007; Mao et al, 2012; Zhigang et al., 2015) for undamped second order vibrating systems with structure preserving hypothesis. However, the solution of the MUP itself, which has been addressed since long ago (Baruch et al, 1982; Berman et al, 1983; Caesar et al, 1987; Inman et al, 1990) takes advantage of natural and well-known results on the decomposition of the modes of linear undamped vibrating systems.

The MUP intends to update a set of eigenvalues and eigenvectors of a model (M, C, K) in a new model $(\overline{M}, \overline{C}, \overline{K})$, usually to fit measurements done in the original system. Sometimes, MUP has been called FEMUP in literature, in allusion to the fact that very often the system matrices come from the Finite Element method.

As the strategy rapidly spread through literature, a deep account on the mathematical foundations for the solvability of the MUP with structure preserving hypothesis, that is, which preserves both the symmetry and the rest (unknown) set of eigenvalues and eigenvectors of the original model, was presented by Chu et al (2007; 2008; 2009). As one of their results states that the structure preserving MUP is only solvable if the new set of eigenvectors lies in the same vector space of the original eigenvectors, assigning any set of measured modes would not in general preserve the symmetry of the original system, implying that the symmetry preserving MUP in reality is unpractical.

However, a particular case of the MUP, called the Eigenvalue Embedding Problem (EEP) only aims to change a subset of the second-order eigenvalues, without changing the rest of eigenvalues and eigenvectors and keeping symmetry of the system matrices. This problem was addressed in (Carvalho, 2006) for linear second order damped vibrating systems.

This paper address the task of clarifying the claim that, under the light of the results in Chu et al (2007, 2008, 2009), the EEP is solvable in some practical situations. A particular state derivative feedback control strategy is derived for linear second order damped vibrating systems.

In Section 2, a set of results from (Datta, 1997) and (Carvalho, 2006) are re-stated in the light of the results in (Chu, 2009). We present application to EEP and to symmetry preserving feedback control.

In Section 3, we present an algorithm and numerical examples.

2 METHODS

The Eigenvalue Embedding Problem (EEP) aims to assign part of the second-order eigenvalues without changing the rest of eigenvalues and eigenvectors.

Definition of EEP: Let $\lambda_1, \ldots, \lambda_{2n}$ be second-order eigenvalues of the model (M, C, K) and let $x_1, \ldots, x_{2n}$ be the associated second-order eigenvectors. Let $\mu_1, \ldots, \mu_p$ be given values. We also make the following assumptions:

(H1) $\{\lambda_1, \ldots, \lambda_p\} \bigcap \{\lambda_{p+1}, \ldots, \lambda_{2n}\} = \emptyset$; $\{\lambda_1, \ldots, \lambda_p\} = \overline{\{\lambda_1, \ldots, \lambda_p\}}$

(H2) $\{\mu_1, \ldots, \mu_p\} \bigcap \{\lambda_{p+1}, \ldots, \lambda_{2n}\} = \emptyset$; $\{\mu_1, \ldots, \mu_p\} = \overline{\{\mu_1, \ldots, \mu_p\}}$

The Eigenvalue Embedding Problem aims to construct a new updated model $(\overline{M}, \overline{C}, \overline{K})$ having second-order eigenvalues $\mu_1, \ldots, \mu_p, \lambda_{p+1}, \ldots, \lambda_{2n}$ as well as second-order eigenvectors $x_{p+1}, \ldots, x_{2n}$ corresponding to $\lambda_{p+1}, \ldots, \lambda_{2n}$, with the hypothesis that system matrices' symmetry and positiveness do not change.

□

If $M = M^T > 0$, $C = C^T$, $K = K^T \geq 0$, then the system in Eq 1 has a set of $2n$ frequencies λ_i, and associated modes $x_i \neq 0$. Using these quantities, we can assemble the Spectral Decomposition

$$MX\Lambda^2 + CX\Lambda + KX = 0 \tag{9}$$

where $X \in \mathbb{R}^{n\times 2n}, \Lambda \in \mathbb{R}^{2n\times 2n}$, for (M, C, K). In this assemblage, real quantities λ_i, x_i are placed over the diagonal of Λ and over the respective column of X. However, complex valued quantities $\lambda_i = \alpha_i + i\beta_i$, where $\alpha_i, \beta_i \in \mathbb{R}$, which are associated to complex quantities $x_i = x_{ir} + ix_{ii}$, where $x_{ir}, x_{ii} \in \mathbb{R}^n$ are linearly independent, are placed in blocks $\begin{bmatrix} \alpha_i & \beta_i \\ -\beta_i & \alpha_i \end{bmatrix}$ over the diagonal of Λ, while $\begin{bmatrix} x_{ir} & x_{ii} \end{bmatrix}$ are placed over the respective columns of X.

In (Datta, 1997; Datta, 2010), a set of matrix equations is presented as a consequence of symmetry and positiveness of a vibrating system given by Eq (1). They are called the Orthogonality Relations for the QEP in Eq (2) and are presented in the next Theorem.

Theorem 2.1 *If $M = M^T > 0$, $C = C^T$, $K = K^T \geq 0$, then a real spectral decomposition as in Eq (9) can be found, and, furthermore, there exist diagonal block matrices D_1, D_2, D_3 such that*

$$(X\Lambda)^T MX\Lambda - X^T KX = D_1 \tag{10}$$
$$(X\Lambda)^T CX\Lambda + (X\Lambda)^T KX + X^T KX\Lambda = D_2 \tag{11}$$
$$(X\Lambda)^T MX + X^T MX\Lambda + X^T CX = D_3 \tag{12}$$

Proof: details on this demonstration can be found in (Datta, 2010).

□

Corollary 2.1 *Under the hypothesis (H1), which means that those two sets are self-conjugated and do not overlap, for any block (subspace) decomposition*

$$X = \begin{bmatrix} X_1 & X_2 \end{bmatrix}, \; \Lambda = \begin{bmatrix} \Lambda_1 & \\ & \Lambda_2 \end{bmatrix} \tag{13}$$

the following equations hold

$$(X_2\Lambda_2)^T MX_1\Lambda_1 - X_2^T KX_1 = 0 \tag{14}$$
$$(X_2\Lambda_2)^T CX_1\Lambda_1 + (X_2\Lambda_2)^T KX_1 + X_2^T KX_1\Lambda_1 = 0 \tag{15}$$
$$(X_2\Lambda_2)^T MX_1 + X_2^T MX_1\Lambda_1 + X_2^T CX_1 = 0 \tag{16}$$

Proof: Can be found in (Datta, 2010), but is also given in our Appendix section.

□

Corollary 2.2 *Given matrices Λ_1 and X_1 such that $MX_1\Lambda_1^2 + CX_1\Lambda_1 + KX_1 = 0$ and under* *the hypothesis (H1), then for any matrix Φ, the update matrices*

$$\triangle M = -MX_1\Lambda_1\Phi\Lambda_1^T X_1^T M \tag{17}$$

$$\triangle C = MX_1\Lambda_1\Phi X_1^T K + KX_1\Phi\Lambda_1^T X_1^T M \tag{18}$$

$$\triangle K = -KX_1\Phi X_1^T K \tag{19}$$

are such that

$$(M + \triangle M)X_2\Lambda_2^2 + (C + \triangle C)X_1\Lambda_2 + (K + \triangle K)X_2 = 0 \tag{20}$$

that is, none of those updates changes the possibly unknown spectrum (Λ_2, X_2). In particular, all these update matrices are symmetric if and only if such matrix Φ is symmetric.

Proof: In our Appendix section.

□

The next theorem follows the results in (Chu, 2009).

Theorem 2.2 *Given matrices Λ_1, Σ_1 and X_1 such that*

$$MX_1\Lambda_1^2 + CX_1\Lambda_1 + KX_1 = 0, \tag{21}$$

and under the hypothesis (H1) and (H2), and given a matrix G_1 for which we define

$$Y_1 = X_1G_1 \tag{22}$$

the problem of computing a matrix Φ for which

$$(M + \triangle M)Y_1\Sigma_1^2 + (C + \triangle C)Y_1\Sigma_1 + (K + \triangle K)Y_1 = 0 \tag{23}$$

can be solved by defining

$$W = (X_1\Lambda_1)^T MX_1G_1\Sigma_1 - X_1^T KX_1G_1 \tag{24}$$

and, if W is nonsingular, by defining

$$\Phi = \Lambda_1^{-1}(G_1\Sigma_1 - \Lambda_1G_1)W^{-1} \tag{25}$$

Proof: can be found in (Chu et al, 2009), but is also given in our Appendix section.

□

Theorem 2.3 *Under the hypotheses of Theorem 2.2, the matrix Φ verifying Eq (25) is symmetric if and only if the matrix*

$$H = G_1^T D_{11}\Lambda_1^{-1}G_1\Sigma_1 \tag{26}$$

is also symmetric, where

$$D_{11} = (X_1\Lambda_1)^T MX_1\Lambda_1 - X_1^T KX_1 \tag{27}$$

is a symmetric block diagonal matrix which also satisfies

$$D_{11}\Lambda_1 = \Lambda_1^T D_{11}. \tag{28}$$

Proof: can be found in (Chu et al, 2009) but is also presented in our Appendix section.

Lemma 2.1 *Under the hypotheses of Theorem 2.2, if the matrices Λ_1 and Σ_1 have the same block structure, meaning that we update real valued λ_i to real valued μ_i and complex valued λ_i*

to complex valued μ_i, then it follows

$$\Lambda_1\Sigma_1 = \Sigma_1\Lambda_1 \tag{29}$$
$$D_{11}\Sigma_1 = \Sigma_1^T D_{11} \tag{30}$$

Proof of Lemma 2.1: presented in our Appendix section.

Theorem 2.4 *Under the hypotheses of Theorem 2.2, if the matrices Λ_1 and Σ_1 are both diagonal and we choose $G_1 = I$, or else, if matrices Λ_1 and Σ_1 do have the same block structure, and we choose $G_1 = \Lambda_1$, then the matrix*

$$H = G_1^T D_{11}\Lambda_1^{-1} G_1\Sigma_1 \tag{31}$$

is symmetric. Furthermore, the closed-loop control system in Eq (5), for the following choices

$$B_1 = MX_1\ ,\ B_2 = KX_1 \tag{32}$$
$$F_1 = \Lambda_1\Phi\Lambda_1^T X_1^T M\ ,\ F_2 = \Phi\Lambda_1^T X_1^T M \tag{33}$$
$$F_3 = \Lambda_1\Phi X_1^T K\ ,\ F_4 = \Phi X_1^T K \tag{34}$$
$$\tag{35}$$

has second-order eigenvalues $\{\mu_1, \ldots, \mu_p, \lambda_{p+1}, \ldots, \lambda_{2n}\}$ as well as associated eigenvectors $\{y_1, y_2, \ldots, y_p, x_{p+1}, \ldots, x_{2n}\}$, for some $\{y_1, y_2, \ldots, y_p\}$.

Proof:

For the first case, if both Λ_1 and Σ_1 are diagonal, then, for $G_1 = I$, we have

$$H = G_1^T D_{11}\Lambda_1^{-1} G_1\Sigma_1 = D_{11}\Lambda_1^{-1}\Sigma_1 \tag{36}$$

which is the product of three diagonal matrices, therefore it is also diagonal and therefore symmetric.

For the second case, if Λ_1 and Σ_1 are not diagonal but have the same block structure, by Lemma 2.1, then Eq (29) and Eq (30) hold, and the choice $G_1 = \Lambda_1$ implies

$$H = G_1^T D_{11}\Lambda_1^{-1} G_1\Sigma_1 = \Lambda_1^T D_{11}\Sigma_1 \tag{37}$$

and now we obtain

$$H^T = \Sigma_1^T D_{11}\Lambda_1 = D_{11}\Sigma_1\Lambda_1 = D_{11}\Lambda_1\Sigma_1 = \Lambda_1^T D_{11}\Sigma_1 = H \tag{38}$$

after applying Eq (29) and Eq (30). Therefore the matrix H is symmetric.

Now, substituting the definitions in Eq (17)-(19) gives

$$\triangle M = -MX_1\Lambda_1\Phi\Lambda_1^T X_1^T M = -B_1F_1 \tag{39}$$
$$\triangle C = MX_1\Lambda_1\Phi X_1^T K + KX_1\Phi\Lambda_1^T X_1^T M = B_1F_3 + B_2F_2 \tag{40}$$
$$\triangle K = -KX_1\Phi X_1^T K = -B_2F_4 \tag{41}$$

and therefore the closed-loop system defined by the feedback law in Eq (4), and which corresponds to the closed-loop control system given in Eq (5), is equivalent to the new model

$$(\overline{M}, \overline{C}, \overline{K}) = (M - B_1F_1, C + B_1F_3 + B_2F_2, K - B_2F_4), \tag{42}$$

which verifies the claimed properties.

□

3 ALGORITHM AND NUMERICAL EXAMPLES

Algorithm : Structure Preserving State Derivative Feedback Control

Input: Matrices $M, C, K \in \mathbb{R}^{n\times n}$, $\Lambda_1, \Sigma_1 \in \mathbb{R}^{p\times p}$, $X_1 \in \mathbb{R}^{n\times p}$.

Output: Control matrices B_1 and B_2 and feedback gain matrices F_1, F_2, F_3 and F_4.

Requirements: real matrices Λ_1 and Σ_1 either are both diagonal or have the same block structure.

Step 1: Compute the matrix $D_{11} = (X_1\Lambda_1)^T M X_1\Lambda_1 - X_1^T K X_1$.

Step 2: If Λ_1 and Σ_1 are real and diagonal, then choose $G_1 = I$; otherwise, choose $G_1 = \Lambda_1$.

Step 3: Compute $W = (X_1\Lambda_1)^T M X_1 G_1 \Sigma_1 - X_1^T K X_1 G_1$ and verify that it is nonsingular.

Step 4: Compute $\Phi = \Lambda_1^{-1}(G_1\Sigma_1 - \Lambda_1 G_1)W^{-1}$

Step 5: Compute matrices $B_1 = MX_1$ and $B_2 = KX_1$

Step 6: Compute matrices

$$F_1 = \Lambda_1\Phi\Lambda_1^T X_1^T M \ , \ F_2 = \Phi\Lambda_1^T X_1^T M \ , \ F_3 = \Lambda_1\Phi X_1^T K \ , \ F_4 = \Phi X_1^T K$$

Return

3.1 Small System

Consider the dynamical system governed by Eq (1) with matrices

$$M = \begin{bmatrix} 6 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 3 \end{bmatrix}, \ C = \begin{bmatrix} 12 & 0 & 0 \\ 0 & 10 & 0 \\ 0 & 0 & 4 \end{bmatrix}, \ K = \begin{bmatrix} 2 & -0.5 & 0 \\ -0.5 & 2 & 0.5 \\ 0 & 0.5 & 2 \end{bmatrix}$$

the system has open-loop poles in

$$\Omega = \{\ -2.2894 \quad -1.8081 \quad -0.1373 \quad -0.2459 \quad -0.6764 + 0.4883i \quad -0.6764 - 0.4883i\ \}$$

while the respective modes are in

$$\left\{ \begin{matrix} 0.0833 & 0.9867 & 0.7137 & -0.6176 & -0.0009 - 0.0108i & -0.0009 + 0.0108i \\ 0.9948 & -0.1613 & 0.6648 & 0.7258 & 0.0491 + 0.1001i & 0.0491 - 0.1001i \\ -0.0581 & 0.0176 & -0.2205 & -0.3029 & 0.8293 + 0.5475i & 0.8293 - 0.5475i \end{matrix} \right\}$$

We choose

$$X_1 = \begin{bmatrix} 0.0833 & 0.9867 \\ 0.9948 & -0.1613 \\ -0.0581 & 0.0176 \end{bmatrix}, \ \Lambda_1 = \begin{bmatrix} -2.2894 & 0 \\ 0 & -1.8081 \end{bmatrix},$$

$$\Sigma_1 = \begin{bmatrix} -1.9000 & 0 \\ 0 & -1.5000 \end{bmatrix}$$

and since Λ_1 and Σ_1 are diagonal, choosing $G_1 = I$ clearly guarantees that matrix H in Eq (26) is symmetric, and we use our formulas to compute

$$W = \begin{bmatrix} 15.5853 & 0.1072 \\ 0.1070 & 13.9730 \end{bmatrix}, \; \Phi = \begin{bmatrix} -0.0109 & 0.0001 \\ 0.0001 & -0.0122 \end{bmatrix} \tag{43}$$

and since W is nonsingular, we can further compute

$$B_1 = \begin{bmatrix} 0.4995 & 5.9205 \\ 3.9793 & -0.6452 \\ -0.1742 & 0.0529 \end{bmatrix}, \; B_2 = \begin{bmatrix} -0.3309 & 2.0541 \\ 1.9190 & -0.8072 \\ 0.3813 & -0.0454 \end{bmatrix}$$

$$F_1 = \begin{bmatrix} -0.0265 & -0.2278 & 0.0100 \\ -0.2359 & 0.0271 & -0.0022 \end{bmatrix}, \; F_2 = \begin{bmatrix} 0.0116 & 0.0995 & -0.0044 \\ 0.1305 & -0.0150 & 0.0012 \end{bmatrix}$$

$$F_3 = \begin{bmatrix} -0.0087 & 0.0481 & 0.0095 \\ 0.0453 & -0.0181 & -0.0011 \end{bmatrix}, \; F_4 = \begin{bmatrix} 0.0038 & -0.0210 & -0.0042 \\ -0.0251 & 0.0100 & 0.0006 \end{bmatrix}$$

Finally, it can be shown that the closed-loop system has poles in

$$\overline{\Omega} = \{ \; -1.9000 \quad -1.5000 \quad -0.1373 \quad -0.2459 \quad -0.6764 + 0.4883i \quad -0.6764 - 0.4883i \; \}$$

while the eigenvectors of the closed-loop system are

$$\left\{ \begin{array}{cccccc} 0.0833 & 0.9867 & 0.7137 & 0.6176 & -0.0009 - 0.0108i & -0.0009 + 0.0108i \\ 0.9948 & -0.1613 & 0.6648 & -0.7258 & 0.0495 + 0.0999i & 0.0495 - 0.0999i \\ -0.0581 & 0.0176 & -0.2205 & 0.3029 & 0.8314 + 0.5443i & 0.8314 - 0.5443i \end{array} \right\}$$

and that shows that the eigenvalue embedding was successful, and therefore that structure preserving state derivative feedback control is possible.

3.2 Vibrating Rod

This example was considered in (Datta et al, 1997). It regards the finite difference model of an axially vibrating non-conservative rod. The model is parametrized by the number of nodes n, and the matrices are given by $M = 2(I + SS^T) + S + S^T$, $C = F^TGF$ and $K = 1000F^TF$, where $S = [\delta_{i+1,j}]$ and $F = I_n - S$, $G = 0.01\text{diag}\{\sin \frac{i\pi}{2n}\}$, for $i = 1, \ldots, n$. Taking $n = 4$ gives

$$M = \begin{bmatrix} 4 & 1 & 0 & 0 \\ 1 & 4 & 1 & 0 \\ 0 & 1 & 4 & 1 \\ 0 & 0 & 1 & 2 \end{bmatrix}, \; C = 0.01 \begin{bmatrix} 1.090 & -0.7071 & 0 & 0 \\ -0.7071 & 1.631 & -0.9239 & 0 \\ 0 & -0.9239 & 1.924 & -1 \\ 0 & 0 & -1 & 1 \end{bmatrix}$$

$$K = 1000 \begin{bmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 1 \end{bmatrix}$$

The system has open-loop poles in
$\Omega = \{-0.0082 \pm 42.2823i, -0.0033 \pm 29.2391i, -0.0010 \pm 16.0961i - 0.0001 \pm 5.1024i\}$

and the respective complex modes are in

$$\left\{ \begin{matrix} -0.0001 - 0.2420i & -0.0001 + 0.2420i & 0.0001 + 0.5843i & 0.0001 - 0.5843i \\ 0.0001 + 0.4472i & 0.0001 - 0.4472i & -0.0000 - 0.4472i & -0.0000 + 0.4472i \\ -0.0001 - 0.5843i & -0.0001 + 0.5843i & -0.0001 - 0.2420i & -0.0001 + 0.2420i \\ 0.0001 + 0.6325i & 0.0001 - 0.6325i & 0.0001 + 0.6325i & 0.0001 - 0.6325i \end{matrix} \right.$$

$$\left. \begin{matrix} -0.0000 - 0.5843i & -0.0000 + 0.5843i & -0.0000 - 0.2420i & -0.0000 + 0.2420i \\ -0.0000 - 0.4472i & -0.0000 + 0.4472i & -0.0000 - 0.4472i & -0.0000 + 0.4472i \\ 0.0000 + 0.2420i & 0.0000 - 0.2420i & -0.0000 - 0.5843i & -0.0000 + 0.5843i \\ 0.0000 + 0.6325i & 0.0000 - 0.6325i & -0.0000 - 0.6325i & -0.0000 + 0.6325i \end{matrix} \right\}$$

We choose

$$X_1 = \begin{bmatrix} -0.0001 & -0.2420 & 0.0001 & 0.5843 & -0.0000 & -0.5843 & -0.0000 & -0.2420 \\ 0.0001 & 0.4472 & -0.0000 & -0.4472 & -0.0000 & -0.4472 & -0.0000 & -0.4472 \\ -0.0001 & -0.5843 & -0.0001 & -0.2420 & 0.0000 & 0.2420 & -0.0000 & -0.5843 \\ 0.0001 & 0.6325 & 0.0001 & 0.6325 & 0.0000 & 0.6325 & -0.0000 & -0.6325 \end{bmatrix}$$

$$\Lambda_1 = \begin{bmatrix} -0.0082 & 42.2823 & 0 & 0 & 0 & 0 & 0 & 0 \\ -42.2823 & -0.0082 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -0.0033 & 29.2391 & 0 & 0 & 0 & 0 \\ 0 & 0 & -29.2391 & -0.0033 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -0.0010 & 16.0961 & 0 & 0 \\ 0 & 0 & 0 & 0 & -16.0961 & -0.0010 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -0.0001 & 5.1024 \\ 0 & 0 & 0 & 0 & 0 & 0 & -5.1024 & -0.0001 \end{bmatrix}$$

and also $\Sigma_1 = \Lambda_1 - \text{ diag } \begin{pmatrix} 2 & 2 & 1.6 & 1.6 & 1.2 & 1.2 & 1 & 1 \end{pmatrix}$, which gives

$$\Sigma_1 =$$
$$\begin{bmatrix} -2.0082 & 42.2823 & 0 & 0 & 0 & 0 & 0 & 0 \\ -42.2823 & -2.0082 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1.6033 & 29.2391 & 0 & 0 & 0 & 0 \\ 0 & 0 & -29.2391 & -1.6033 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1.2010 & 16.0961 & 0 & 0 \\ 0 & 0 & 0 & 0 & -16.0961 & -1.2010 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1.0001 & 5.1024 \\ 0 & 0 & 0 & 0 & 0 & 0 & -5.1024 & -1.0001 \end{bmatrix}$$

and since Λ_1 and Σ_1 share the same block structure, we choose $G_1 = \Lambda_1$, and obtain

$$Y_1 =$$
$$\begin{bmatrix} 10.2336 & -0.0016 & -17.0848 & -0.0000 & 9.4052 & 0.0003 & 1.2349 & 0.0000 \\ -18.9092 & 0.0011 & 13.0761 & 0.0013 & 7.1984 & -0.0002 & 2.2818 & 0.0000 \\ 24.7061 & 0.0003 & 7.0767 & -0.0011 & -3.8957 & -0.0002 & 2.9814 & 0.0000 \\ -26.7417 & -0.0011 & -18.4924 & 0.0003 & -10.1801 & 0.0001 & 3.2270 & -0.0000 \end{bmatrix}$$

$$W =$$
$$10^5 \begin{bmatrix} -0.0615 & 1.3015 & -0.0000 & 0.0000 & 0.0000 & -0.0000 & 0.0000 & -0.0000 \\ 1.3015 & -0.0000 & -0.0000 & -0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 \\ -0.0000 & -0.0000 & -0.0354 & 0.6469 & -0.0000 & 0.0000 & -0.0000 & 0.0000 \\ 0.0000 & -0.0000 & 0.6469 & 0.0000 & -0.0000 & 0.0000 & -0.0000 & -0.0000 \\ 0.0000 & 0.0000 & -0.0000 & -0.0000 & -0.0119 & 0.1590 & 0.0000 & -0.0000 \\ -0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.1590 & 0.0000 & 0.0000 & 0.0000 \\ 0.0000 & 0.0000 & -0.0000 & -0.0000 & 0.0000 & 0.0000 & -0.0012 & 0.0062 \\ -0.0000 & 0.0000 & 0.0000 & -0.0000 & -0.0000 & 0.0000 & 0.0062 & 0.0000 \end{bmatrix}$$

$$\Phi =$$

$$10^{-2}\begin{bmatrix} -0.0000 & -0.0015 & 0.0000 & -0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 \\ -0.0015 & -0.0001 & 0.0000 & -0.0000 & -0.0000 & 0.0000 & -0.0000 & 0.0000 \\ 0.0000 & 0.0000 & 0.0000 & -0.0025 & 0.0000 & -0.0000 & -0.0000 & -0.0000 \\ -0.0000 & -0.0000 & -0.0025 & -0.0001 & 0.0000 & -0.0000 & 0.0000 & -0.0000 \\ 0.0000 & -0.0000 & 0.0000 & 0.0000 & 0.0000 & -0.0075 & -0.0000 & 0.0000 \\ 0.0000 & 0.0000 & -0.0000 & -0.0000 & -0.0075 & -0.0006 & -0.0000 & 0.0000 \\ 0.0000 & -0.0000 & -0.0000 & 0.0000 & -0.0000 & -0.0000 & 0.0000 & -0.1609 \\ 0.0000 & 0.0000 & -0.0000 & -0.0000 & 0.0000 & 0.0000 & -0.1609 & -0.0315 \end{bmatrix}$$

and then

$$B_1 =$$
$$\begin{bmatrix} -0.0002 & -0.5209 & 0.0003 & 1.8900 & -0.0001 & -2.7845 & -0.0000 & -1.4153 \\ 0.0003 & 0.9625 & -0.0000 & -1.4466 & -0.0002 & -2.1311 & -0.0000 & -2.6152 \\ -0.0002 & -1.2576 & -0.0002 & -0.7829 & 0.0000 & 1.1534 & -0.0000 & -3.4169 \\ 0.0001 & 0.6806 & 0.0001 & 1.0229 & 0.0001 & 1.5069 & -0.0000 & -1.8492 \end{bmatrix}$$

$$B_2 =$$
$$10^{3}\begin{bmatrix} -0.0003 & -0.9313 & 0.0001 & 1.6158 & 0.0000 & -0.7214 & 0.0000 & -0.0368 \\ 0.0004 & 1.7208 & -0.0000 & -1.2367 & -0.0001 & -0.5521 & 0.0000 & -0.0681 \\ -0.0004 & -2.2483 & -0.0002 & -0.6693 & 0.0000 & 0.2988 & -0.0000 & -0.0890 \\ 0.0002 & 1.2168 & 0.0001 & 0.8745 & 0.0000 & 0.3904 & -0.0000 & -0.0481 \end{bmatrix}$$

and also

$$F_1 = \begin{bmatrix} -0.0143 & 0.0264 & -0.0345 & 0.0187 \\ -0.0000 & -0.0000 & 0.0000 & -0.0000 \\ 0.0400 & -0.0306 & -0.0166 & 0.0216 \\ -0.0000 & 0.0000 & -0.0000 & -0.0000 \\ -0.0545 & -0.0417 & 0.0226 & 0.0295 \\ 0.0000 & 0.0000 & -0.0000 & -0.0000 \\ -0.0593 & -0.1096 & -0.1431 & -0.0775 \\ 0.0000 & 0.0000 & 0.0000 & 0.0000 \end{bmatrix}$$

$$F_2 = \begin{bmatrix} 0.0000 & -0.0000 & -0.0000 & 0.0000 \\ -0.0003 & 0.0006 & -0.0008 & 0.0004 \\ -0.0000 & -0.0000 & 0.0000 & -0.0000 \\ 0.0014 & -0.0010 & -0.0006 & 0.0007 \\ -0.0000 & 0.0000 & 0.0000 & -0.0000 \\ -0.0034 & -0.0026 & 0.0014 & 0.0018 \\ -0.0000 & 0.0000 & 0.0000 & 0.0000 \\ -0.0116 & -0.0215 & -0.0281 & -0.0152 \end{bmatrix}$$

$$F_3 = \begin{bmatrix} 0.0287 & -0.0529 & 0.0691 & -0.0374 \\ -0.6051 & 1.1180 & -1.4608 & 0.7906 \\ -0.0639 & 0.0489 & 0.0266 & -0.0347 \\ 1.1686 & -0.8944 & -0.4841 & 0.6325 \\ 0.0653 & 0.0501 & -0.0271 & -0.0354 \\ -0.8765 & -0.6708 & 0.3630 & 0.4743 \\ 0.0593 & 0.1096 & 0.1432 & 0.0775 \\ -0.3025 & -0.5590 & -0.7304 & -0.3953 \end{bmatrix}$$

$$F_4 = \begin{bmatrix} 0.0143 & -0.0264 & 0.0345 & -0.0187 \\ 0.0007 & -0.0013 & 0.0016 & -0.0009 \\ -0.0400 & 0.0306 & 0.0166 & -0.0216 \\ -0.0022 & 0.0017 & 0.0009 & -0.0012 \\ 0.0545 & 0.0417 & -0.0226 & -0.0295 \\ 0.0041 & 0.0031 & -0.0017 & -0.0022 \\ 0.0593 & 0.1096 & 0.1431 & 0.0775 \\ 0.0116 & 0.0215 & 0.0281 & 0.0152 \end{bmatrix}$$

and it be be shown that the closed-loop poles are in

$$\overline{\Omega} = \{ \; -2.0082 \pm 42.2823i \quad -1.6033 \pm 29.2391i \quad -1.2010 \pm 16.0961i \quad -1.0001 \pm 5.1024i \; \}$$

and that the complex modes of the closed-loop system are

$$\left\{\begin{array}{cccc} -0.0115-0.2418i & -0.0115+0.2418i & 0.0320+0.5834i & 0.0320-0.5834i \\ 0.0212+0.4467i & 0.0212-0.4467i & -0.0244-0.4465i & -0.0244+0.4465i \\ -0.0277-0.5837i & -0.0277+0.5837i & -0.0133-0.2417i & -0.0133+0.2417i \\ 0.0300+0.6317i & 0.0300-0.6317i & 0.0346+0.6315i & 0.0346-0.6315i \end{array}\right.$$

$$\left.\begin{array}{cccc} 0.0435-0.5827i & 0.0465+0.2375i & 0.0465-0.2375i & 0.0435+0.5827i \\ 0.0333-0.4460i & 0.0860+0.4389i & 0.0860-0.4389i & 0.0333+0.4460i \\ -0.0180+0.2414i & 0.1124+0.5734i & 0.1124-0.5734i & -0.0180-0.2414i \\ -0.0471+0.6307i & 0.1216+0.6206i & 0.1216-0.6206i & -0.0471-0.6307i \end{array}\right\}$$

and that shows that the eigenvalue embedding was successful, and therefore that structure preserving state derivative feedback control is possible.

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4 APPENDIX

Proof of Corollary 2.1:

Substituting Eq (13) into Eq (10) gives

$$\begin{bmatrix} \Lambda_1^T & \\ & \Lambda_2^T \end{bmatrix} \begin{bmatrix} X_1^T \\ X_2^T \end{bmatrix} M \begin{bmatrix} X_1 & X_2 \end{bmatrix} \begin{bmatrix} \Lambda_1 & \\ & \Lambda_2 \end{bmatrix} - \begin{bmatrix} X_1^T \\ X_2^T \end{bmatrix} K \begin{bmatrix} X_1 & X_2 \end{bmatrix} = \begin{bmatrix} D_{11} & 0 \\ 0 & D_{22} \end{bmatrix} \tag{44}$$

for the some block diagonal matrices D_{11} e D_{22}. Now considering the off-diagonal terms if the equation immediately gives the claimed result.

Proof of Corollary 2.2:

Let (Λ_2, X_2) be the rest of the spectrum in regard to (Λ_1, X_1). Then

$$\begin{aligned}
&(M + \triangle M)X_2\Lambda_2^2 + (C + \triangle C)X_2\Lambda_2 + (K + \triangle K)X_2 \\
&\quad = (M - MX_1\Lambda_1\Phi\Lambda_1^T X_1^T M)X_2\Lambda_2^2 \\
&\qquad + (C + MX_1\Lambda_1\Phi X_1^T K + KX_1\Phi\Lambda_1^T X_1^T M)X_2\Lambda_2 + (K - KX_1\Phi X_1^T K)X_2 \\
&\quad = 0 - MX_1\Lambda_1\Phi\Lambda_1^T X_1^T MX_2\Lambda_2^2 + MX_1\Lambda_1\Phi X_1^T KX_2\Lambda_2 \\
&\qquad + KX_1\Phi\Lambda_1^T X_1^T MX_2\Lambda_2 - KX_1\Phi X_1^T KX_2 \\
&\quad = MX_1\Lambda_1\Phi(-\Lambda_1^T X_1^T MX_2\Lambda_2 + X_1^T KX_2)\Lambda_2 - KX_1\Phi(\Lambda_1^T X_1^T MX_2\Lambda_2 - X_1^T KX_2) \\
&\quad = 0
\end{aligned}$$

since both terms between parentheses are null by the first orthogonality relation in Eq (14). Clearly, the update matrices in Eq (17)-(19) are symmetric if and only if the matrix Φ is also symmetric, completing the proof.

Proof of Theorem 2.2:

Substituting the expressions for $\triangle M$, $\triangle C$ and $\triangle K$ into Eq (23) gives

$$(M - MX_1\Lambda_1\Phi\Lambda_1^T X_1^T M)X_1G_1\Sigma_1^2 + (C + MX_1\Lambda_1\Phi X_1^T K + KX_1\Phi\Lambda_1^T X_1^T M)X_1G_1\Sigma_1 + (K - KX_1\Phi X_1^T K)X_1G_1 = 0 \tag{45}$$

and then

$$MX_1\Lambda_1\Phi(\Lambda_1^T X_1^T MX_1G_1\Sigma_1 - X_1^T KX_1G_1)\Sigma_1 - KX_1\Phi(\Lambda_1^T X_1^T MX_1G_1\Sigma_1 - X_1^T KX_1G_1) = MX_1G_1\Sigma_1^2 + CX_1G_1\Sigma_1 + KX_1G_1 \tag{46}$$

and now, the left hand side can be written as

$$MX_1\Lambda_1\Phi W\Sigma_1 - KX_1\Phi W \tag{47}$$

for matrix W defined as in Eq (24), while the right hand side gives

$$MX_1G_1\Sigma_1^2 + CX_1G_1\Sigma_1 + KX_1G_1 = MX_1G_1\Sigma_1^2 + CX_1G_1\Lambda_1\Lambda_1^{-1}\Sigma_1 + KX_1G_1 = \tag{48}$$

$$MX_1G_1\Sigma_1^2 + (-MX_1\Lambda_1^2 - KX_1)\Lambda_1^{-1}G_1\Sigma_1 + KX_1G_1 = \tag{49}$$

$$MX_1(G_1\Sigma_1 - \Lambda_1G_1)\Sigma_1 - KX_1\Lambda_1^{-1}(G_1\Sigma_1 - \Lambda_1G_1) \tag{50}$$

and therefore we have

$$MX_1\Lambda_1\Phi W\Sigma_1 - KX_1\Phi W = MX_1(G_1\Sigma_1 - \Lambda_1G_1)\Sigma_1 - KX_1\Lambda_1^{-1}(G_1\Sigma_1 - \Lambda_1G_1) \tag{51}$$

and we observe that a simultaneous solution of

$$\begin{cases} \Lambda_1\Phi W &= G_1\Sigma_1 - \Lambda_1G_1 \\ \Phi W &= \Lambda_1^{-1}(G_1\Sigma_1 - \Lambda_1G_1) \end{cases} \tag{52}$$

which fortunately exists if W is nonsingular, and satisfies the expression we claimed.

Proof of Theorem 2.3:

Clearly, using Eq (25), $\Phi^T = \Phi$ is equivalent to

$$W^{-T}(\Sigma_1^T G_1^T - G_1^T\Lambda_1^T)\Lambda_1^{-T} = \Lambda_1^{-1}(G_1\Sigma_1 - \Lambda_1G_1)W^{-1} \tag{53}$$

which is equivalent to

$$(\Sigma_1^T G_1^T - G_1^T\Lambda_1^T)\Lambda_1^{-T}W = W^T\Lambda_1^{-1}(G_1\Sigma_1 - \Lambda_1G_1) \tag{54}$$

Now, substituting the expression for W in Eq (24) gives

$$(\Sigma_1^T G_1^T - G_1^T\Lambda_1^T)(\Lambda_1^T X_1^T MX_1G_1\Sigma_1 - X_1^T KX_1G_1) = (\Sigma_1^T G_1^T X_1^T MX_1\Lambda_1 - G_1^T X_1^T KX_1)\Lambda_1^{-1}(G_1\Sigma_1 - \Lambda_1G_1) \tag{55}$$

which can be expanded to

$$\Sigma_1^T G_1^T X_1^T MX_1G_1\Sigma_1 - G_1^T\Lambda_1^T X_1^T MX_1G_1\Sigma_1 - \Sigma_1^T G_1^T\Lambda_1^{-T}X_1^T KX_1G_1 + G_1^T X_1^T KX_1G_1 = \Sigma_1^T G_1^T X_1^T MX_1G_1\Sigma_1 - \Sigma_1^T G_1^T X_1^T MX_1\Lambda_1\Sigma_1 - G_1^T X_1^T KX_1\Lambda_1^{-1}G_1\Sigma_1 + G_1^T X_1^T KX_1G_1 \tag{56}$$

and, after cancellations, simplified to

$$-G_1^T(\Lambda_1^T X_1^T MX_1 - X_1^T KX_1\Lambda_1^{-1})G_1\Sigma_1 = \Sigma_1^T G_1^T(\Lambda_1^{-T}X_1^T KX_1 - X_1^T MX_1\Lambda_1)G_1 \tag{57}$$

which can also be written

$$-G_1^T(\Lambda_1^T X_1^T M X_1 \Lambda_1 - X_1^T K X_1)\Lambda_1^{-1} G_1 \Sigma_1 = -\Sigma_1^T G_1^T \Lambda_1^{-T}(\Lambda_1^T X_1^T M X_1 \Lambda_1) G_1 - X_1^T K X_1) G_1 \tag{58}$$

which is clearly equivalent to

$$G_1^T D_{11} \Lambda_1^{-1} G_1 \Sigma_1 = (G_1^T D_{11} \Lambda_1^{-1} G_1 \Sigma_1)^T \tag{59}$$

after using Eq (27). This is equivalent to $H = H^T$ for H defined in Eq (26).

Now, to show that Eq (28) holds, we start from Eq (21) to obtain

$$-C X_1 \Lambda_1 = M X_1 \Lambda_1^2 + K X_1 \tag{60}$$

$$-(X_1\Lambda_1)^T C X_1 \Lambda_1 = (X_1\Lambda_1)^T M X_1 \Lambda_1^2 + (X_1\Lambda_1)^T K X_1 \tag{61}$$

$$\tag{62}$$

and now we transpose Eq (21) to obtain

$$-(X_1\Lambda_1)^T C = \Lambda_1^T (X_1\Lambda_1)^T M + X_1^T K \tag{63}$$

$$-(X_1\Lambda_1)^T C X_1 \Lambda_1 = \Lambda_1^T (X_1\Lambda_1)^T M X_1 \Lambda_1 + X_1^T K X_1 \Lambda_1. \tag{64}$$

and now we can combine Eq (61) with Eq (64) to get

$$(X_1\Lambda_1)^T M X_1 \Lambda_1^2 + (X_1\Lambda_1)^T K X_1 = \Lambda_1^T (X_1\Lambda_1)^T M X_1 \Lambda_1 + X_1^T K X_1 \Lambda_1. \tag{65}$$

Re-arranging, it follows

$$\left[(X_1\Lambda_1)^T M X_1 \Lambda_1 - X_1^T K X_1\right] \Lambda_1 = \Lambda_1^T \left[(X_1\Lambda_1)^T M X_1 \Lambda_1 - X_1^T K X_1\right] \tag{66}$$

which imples Eq 28, completing the proof. □

Proof of Lemma 2.1: we consider the partition of Λ_1 and Σ_1 in diagonal blocks that have sizes either 1 x 1 or 2 x 2. Since the properties of Eq (29) and Eq (30) are easily verified for groups of 1 x 1 blocks along the diagonal, as we are dealing with diagonal matrices having distinct diagonal entries, it remains to show those properties only for the 2 x 2 blocks on the diagonal. Without loss of generality, suppose

$$\Lambda_1 = \begin{bmatrix} \alpha_1 & \beta_1 \\ -\beta_1 & \alpha_1 \end{bmatrix}, \ \Sigma_1 = \begin{bmatrix} \gamma_1 & \delta_1 \\ -\delta_1 & \gamma_1 \end{bmatrix}, \ D_{11} = \begin{bmatrix} a_{11} & a_{12} \\ a_{12} & a_{22} \end{bmatrix} \tag{67}$$

where $\beta_1 \neq 0$ and $\delta_1 \neq 0$.

Define

$$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \ J_1 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \ J_2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \ J_3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \tag{68}$$

$$\mathcal{Z} = \{Z : Z = \alpha I + \beta J_1, \alpha, \beta \in \mathbb{R}\} \tag{69}$$

$$\mathcal{V} = \{V : V = \gamma J_2 + \delta J_3, \gamma, \delta \in \mathbb{R}\} \tag{70}$$

Since $J_1^2 = -I$, it follows that product of elements of $\mathcal{Z}$ also belongs to $\mathcal{Z}$. Besides, since I and J commute, $\mathcal{Z}$ is a commutative space of matrices. Since both Λ_1 and Σ_1 belong to $\mathcal{Z}$, it follows that they commute, and Eq (29) holds.

Since J_2 and J_3 are symmetric matrices, then all matrices in $\mathcal{V}$ are symmetric. Besides, when we impose $D_{11}\Lambda_1 = \Lambda_1^T D_{11}$, since $\beta \neq 0$, it follows

$$D_{11} = \begin{bmatrix} a_{11} & a_{12} \\ a_{12} & -a_{11} \end{bmatrix} = a_{12}J_2 + a_{11}J_3 \tag{71}$$

and therefore $D_{11} \in \mathcal{V}$. Since it is straightforward to show that

$$J_2J_1 = -J_3\ ,\ J_3J_1 = J_2\ ,\ J_1J_2 = J_3\ ,\ J_1J_3 = -J_2 \tag{72}$$

and also $J_1^T = -J_1$, then

$$D_{11}\Sigma_1 = (a_{12}J_2 + a_{11}J_3)(\gamma I + \delta J_1) = (\gamma a_{12} + \delta a_{11})J_2 + (\gamma a_{11} - \delta a_{12})J_3 \tag{73}$$

while

$$\Sigma_1^T D_{11} = (\gamma I - \delta J_1)(a_{12}J_2 + a_{11}J_3) = (\gamma a_{12} + \delta a_{11})J_2 + (\gamma a_{11} - \delta a_{12})J_3 \tag{74}$$

and the combination of the last two equations imply Eq (30), completing the proof.