# NONLINEAR VIBRATIONS OF COLUMNS RESTING ON PARTIAL ELASTIC BASE

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***Abstract.*** *In this work the nonlinear vibrations of columns resting on partial elastic base and subjected to axial and lateral loads are studied. For this, the nonlinear Euler-Bernoulli theory is used to describe the column and the Winkler theory is applied to describe the foundation. To obtain the nonlinear dynamic equations, the Rayleigh-Ritz method together with the Hamilton principle are used. Several boundary conditions are considered. Obtained results show the strong effect of boundary conditions, foundation and loads on the nonlinear vibrations of columns resting on partial elastic base.*

***Keywords:*** *Columns, Elastic base, Nonlinear vibrations*

# 1 INTRODUCTION

The study of the dynamic behavior of columns is a research subject in several engineering areas such as civil, mechanical, naval, etc. then, its study continues to be an important research subject due to the large applications mainly in if dynamics loads are considered.

In literature it is possible to find some works related to the behavior of columns but, only a few works are related to the analysis of columns resting on partial elastic base. Here only the most important works are mentioned.

Reis et al (2016) studied free and forced vibrations of both beams and columns resting on elastic foundations using the finite element method. Both Winkler and Pasternak models were considered to describe the foundation. It was observed that axial critical load and natural frequencies are increased as base stiffness is increased, results were compared with SAP2000 and good agreement was observed.

Amabili and Garziera (1999) used several admissible functions to obtain the potential energy and to obtain the natural frequencies of beams with Winkler elastic foundations with variable boundary conditions.

Nayfeh e Emam (2008) presentem na exact beam solution for the post critical paths of beam with several boundary conditions. The geometric non linearity was considered to sudy the dynamics instability of each pots critical solution.

Serebrenick (2004) studied the non linear behavior of columns resting on partial elastic foundation. The Ritz method was applied to obtain the set of non linear equations and both linear and non linear models were used to describe the elastic foundation.

Sampaio (2004) studied the non linear dynamic behavior and obtained exact solutions for the natural frequencies and vibration modes of partially supported columns. The Galerkin-Urabe and Harminic Balance methods were applied to solve the governing equations of the problem.

In this paper the nonlinear vibrations of columns resting on partial elastic base subjected to axial and transversal time varying loads are studied. To model the columns, the nonlinear Eulher-Bernoulli theory is used and the elastic base is considered as a Winkler type. The Rayleigh Ritz method is applied to obtain a set of ordinary nonlinear differential equations of motions which are solved using the Newton-Raphson method, for he static analysis, and he Runge-Kutta method for the dynamic case. Obtained results show the large influence of elastic base and boundary conditions on the nonlinear vibrations of columns.

# 2 MATHEMATICAL FORMULATION

Consider a column of length $L$, density $\rho$, transversal area $A$, inertia modulus $I$, Young modulus $E$ and resting on a partial elastic base with stiffness $K_s$. It is assumed that at both ends of the column there are two rotational springs with stiffness $kt_1$ and $kt_2$ and the column is subjected to an axial load P and a transversal load F as shown in Fig. 01. The lateral displacement field is $w(x,t)$.



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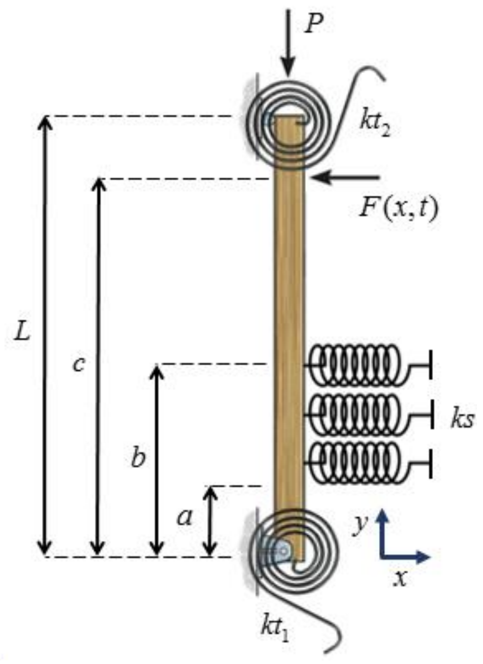



Figure 01 – Column resting on partial elastic base

The strain energy of the column is given by:

$$U = \frac{1}{2} E\, I \int_0^L \left(\frac{\partial^2 w(x,t)}{\partial x^2}\right)^2 + \left(\frac{\partial^2 w(x,t)}{\partial x^2}\right)^2 \left(\frac{\partial w(x,t)}{\partial x}\right)^2 +$$
$$\frac{1}{4}\left(\frac{\partial^2 w(x,t)}{\partial x^2}\right)^2 \left(\frac{\partial w(x,t)}{\partial x}\right)^4 dx \qquad (1)$$
$$+\frac{1}{2} K_s \int_a^b w(x,t)^2 dx + \frac{1}{2} K_{t1} w(x,t)^2 \delta(x-0) + \frac{1}{2} K_{t2} w(x,t)^2 \delta(x-L)$$

Where $\delta$ is the delta of Dirac function, *a* and *b* are the beginning and end of the elastic base.

The kinetic energy, considering translational inertia only is given by:

$$T = \frac{1}{2} \rho\, A \int_0^L \left(\frac{\partial w(x,t)}{\partial t}\right)^2 dx \qquad (2)$$

It is considered that the column is subjected to an axial load $P = P_e + P_d \cos(\Omega_1 t)$, where $P_e$ is the amplitude of the static load, $P_d$ is the amplitude of the dynamic load and $\Omega_1$ is the frequency of the dynamic load. Also, it is assumed the transversal harmonic load $F = F_L \cos(\Omega_2 t)$ is acting at the point c of the column, where $F_L$ is the amplitude of the dynamic transversal load and $\Omega_2$ is the frequency of the transversal load. Then, the work done by the external loads can be written as:

$$W = \int_0^L \left(P_e + P_d \cos(\varpi_{ex1} t)\right)\left[\frac{1}{2}\left(\frac{\partial w(x,t)}{\partial x}\right)^2 + \frac{1}{8}\left(\frac{\partial w(x,t)}{\partial x}\right)^4\right]dx \\ + F_L \cos(\omega_{ex2} t)\delta(x-c)w(x,t) \tag{3}$$

The work done by the non-conservative forces can be written as:

$$F_{NC} = \frac{1}{2} C \int_0^L \left(\frac{\partial w(x,t)}{\partial t}\right)^2 dx \tag{4}$$

Where $C = 2\xi\rho\omega_o$, $\xi$ is the damping coefficient and $\omega_o$ is he natural frequency of the column.

To describe the lateral displacements of the column, the following expansion is assumed:

$$w(x,t) = \sum_{i=1}^{n} C(t)_i \sin\left(n\frac{\pi x}{L}\right) \tag{5}$$

This expansion satisfies the simply supported boundary conditions but, the assumption of rotational springs at both ends of the column, together with a large number of terms in Eq. (5) allows to consider clamped-clamped and clamped-supported boundary conditions.

To obtain the set of ordinary nonlinear differential equations of dynamic equilibrium, the Rayleirh-Ritz method and the Hamilton principle are used as follows:

$$\bar{L} = T - U + W \tag{6}$$

$$\frac{d}{dt}\left(\frac{\partial \bar{L}}{\partial \dot{C}_i}\right) - \frac{\partial \bar{L}}{\partial C_i} = \frac{\partial F_{NC}}{\partial \dot{C}_i} \tag{7}$$

Where $\bar{L}$ is the Lagrangean functional of the system.

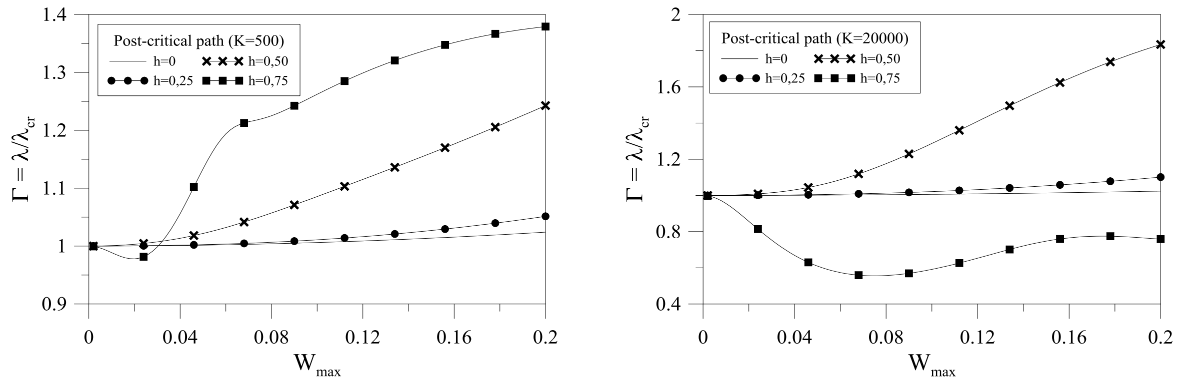
In analysis the following non-dimentional parameters where used:

$$\lambda^2 = \frac{P_d L^2}{\pi^2 EI} \qquad \mu^2 = \frac{FL^2}{\pi^2 EI} \qquad \Omega^4 = \frac{\rho A \omega o_o^2 L^4}{\pi^4 EI} \qquad K = \frac{k_s L^4}{\pi^4 EI}$$
$$\varphi_1 = \frac{\omega_{ex1}}{\omega_o} \qquad \varphi_2 = \frac{\omega_{ex2}}{\omega_o} \qquad \tau = t\omega_o \qquad h = \frac{b}{L} \qquad H = \frac{c}{L} \tag{8}$$

# 3 NUMERICAL RESULTS

Consider a simply supported steel square column with $b = 0.10$ m, Young modulus $E = 10e6$ N/m$^2$, density $\rho = 7850.0$ kg/m$^3$, length 3.0 m and damping coefficient $\xi$ =- 0.01.

First, the variation of the post critical paths due to axial load as the length of the elastic based is varied will be analyzed. Fig. 02 displays the variation of the post critical paths as the length of the elastic base is varied considering two values of the elastic base stiffness parameter. As can be observed in Fig. 02a for K =500, if no elastic base is considered, the column after the critical load, displays stable post critical behavior now; if the elastic base is considered, for h =0.25 and h = 0.50 displays also post critical stable paths but, for h = 0.75 the column after the critical load displays initial unstable post-critical path and after a certain point stable path. This behavior shows the strong influence of elastic base on the static non linear behavior. In Fig. 02b, when k = 2000, the column displays stable post critical paths for a column without elastic base and for h = 0.25 and h = 0.25 but, for h = 0.75 the column displays unstable post critical path with small gain of stiffness.




(a) (b)

Figure 02 – Variation of post critical paths. a) K = 500 and b) K 20000

Figure 03 displays the variation of the critical load parameter as the length of the elastic base is varied. As can be observed as the length of the elastic base is incremented, the value of the axial critical load parameter is also increased. For value of h lower than 0.6, the critical load increase in a similar rate but, after this value the critical load are strongly affected and the critical load are higher for large values of elastic base parameter *K* than for lower values of *K*.

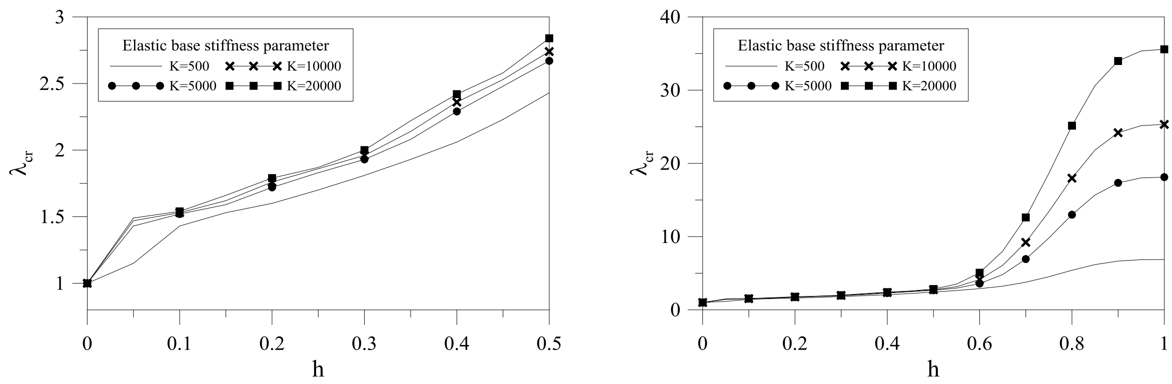



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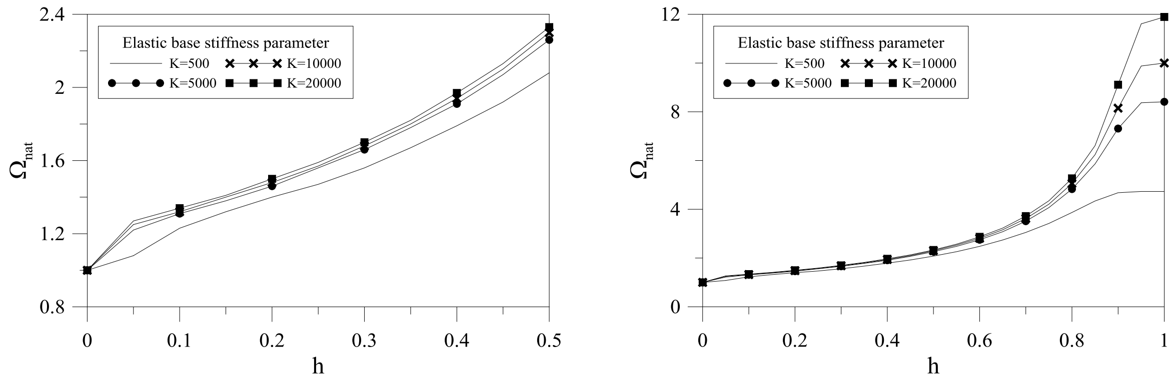
Figure 03 – Variation of critical load parameter.

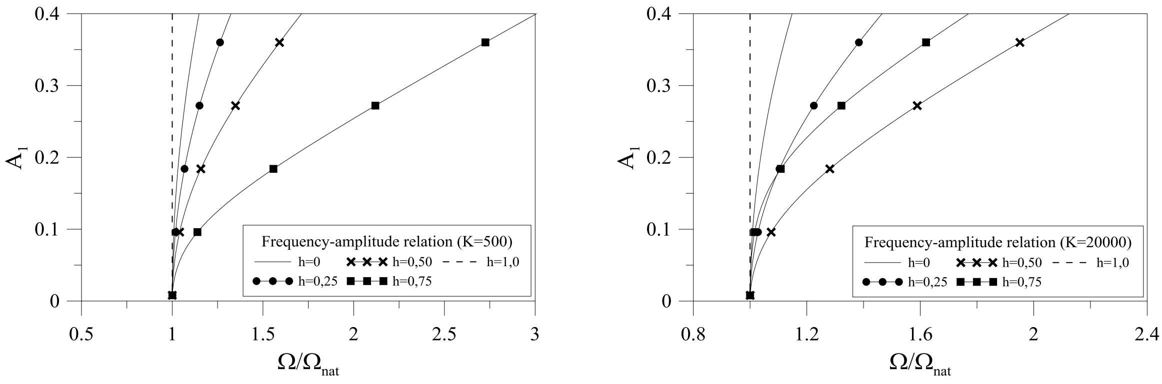
Figure 04 displays the variation of the natural frequency parameter as the length of the elastic base is varied. As can be observed as the length of the elastic base is incremented, the value of the frequency parameter is also increased showing that the column is stiffer. The rate if the natural frequency parameter is similar for all values of h, but for a values higher than 0.6, the amount os frequency parameter is stronger for large values of *K* then for low values of *K*.




(a) (b)

Figure 043 – Variation of natural frequency parameter.

Now, the frequency amplitude relation will be studied. Figure 05 displays the frequency amplitude relations for two different values of the elastic base parameter *K*. As can be observed, for higher values of the length of the elastic base, the column stiffness is higher than for low values of *h*. Is it possible to observe that for both large and low values of the elastic base parameter, the nonlinear behavior of the column is quite similar.




(a) (b)

Figure 05 – Frequency-amplitude relation.

Figure 06 shows the parameric instability boundaries of the column for increasing values of the elastic base parameter considering $h$ = 0.5. As can be observed, as the values of the axial load frequency is increased, it is possible to observe the principal ($\phi_1$ = 2.0) and secondary ($\phi_2$ = 1.0) regions of parametric instablity. In the secondary region, the ascending and descending boundaries are similar for all values of $K$ but, in the descending boundary of the principal region, the boundary related to the column without elastic base is lower than the bundary of the column with elastic base. Now, for the ascending part, all boundaries are similar for any values of the elastic base.

This means that the escape region of the column is affected mainly in the descending part of the boundary and is not affect in other boundaries.

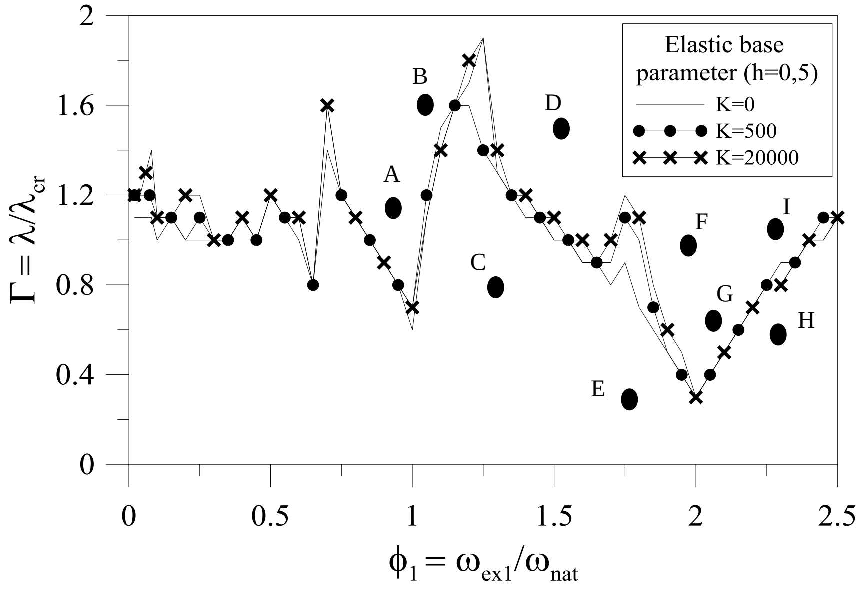



Figure 06 – Parametric instability boundaries.



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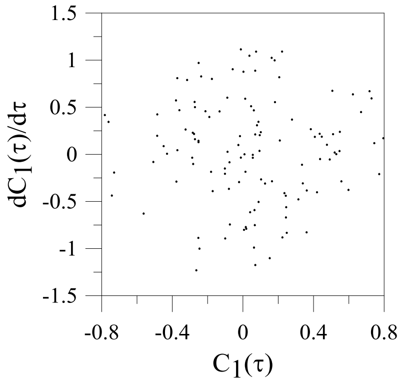
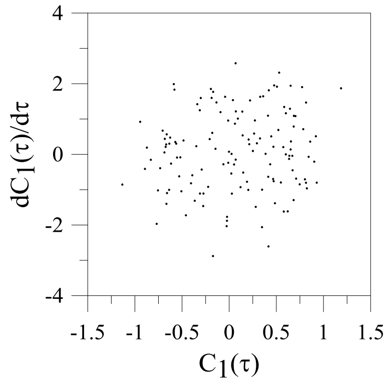
Finally, Fig. 07 show the phase portraits and time response of the points showed in Fig. 06. This figure aims to show the kind of vibrations that the column displays up and below the instability boundary. As n be observed, for point below the instability boundary the column displays trivial solutions but, for point with initial conditions above the boundary the column displays chaotic and periodic solutions.




(a)

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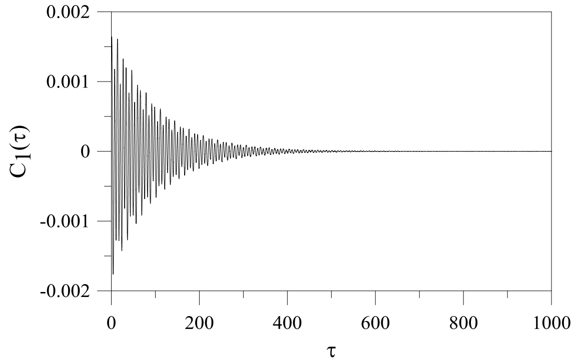


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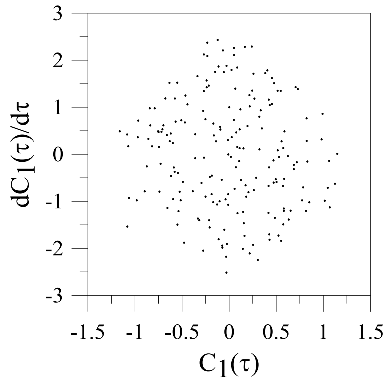


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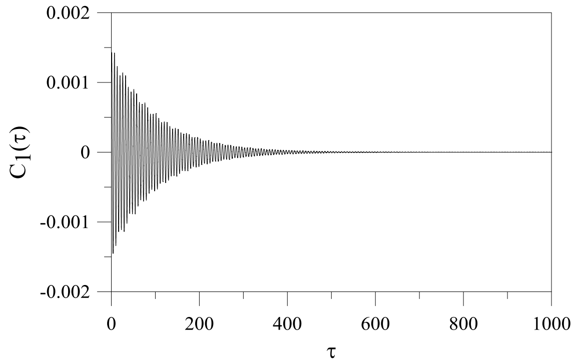


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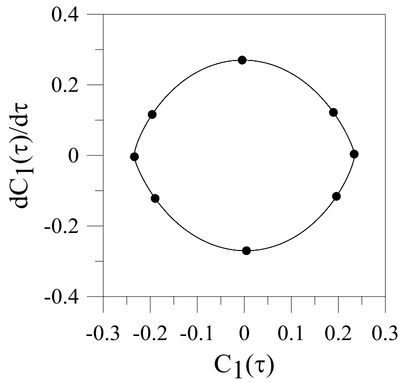


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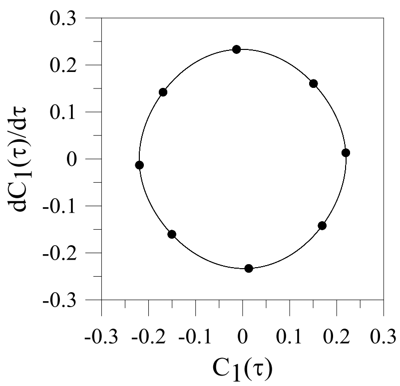


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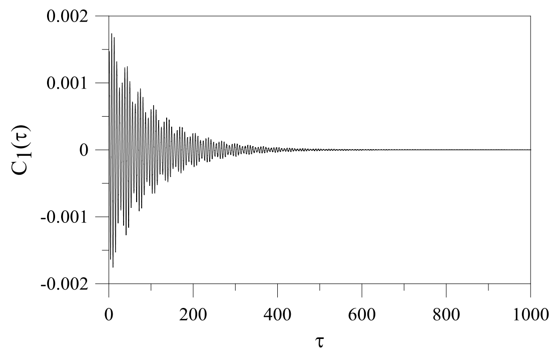
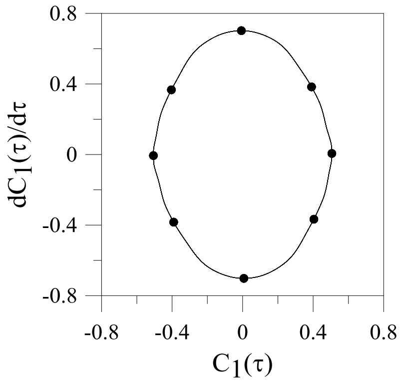



(h)




(i)

Figure 07 – Phase portrais and ttime responses.

# 4 CONCLUDING REMARKS

In this work the nonlinear oscillations of columns resting on partial elastic base was studied. It was possible to observe in the analysis that partial elastic base affects strongly the static and dynamic nonlinear behavior of the column.

In the static case, as the length or stiffness of the elastic base is varied, the column will gain or lose stiffness affecting the whole of the system. When axial harmonic loads are applied to the column, is no elastic base is considered, the column will display stable solutions but, if elastic base is considered, it is possible to obtain even chaotic large amplitude oscillations.

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