



GENERALIZED-STRAIN MESH-FREE METHOD (GSMF) FOR TWO-DIMENSIONAL ELASTICITY PROBLEMS

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Abstract. Recently meshless methods have been more used due to their accuracy and performance in numerical analysis. Some of them are derived from a weak - form formulation on global domain and others from a local sub - domains. The weighted - residual method is the basis for the meshless formulation. The Generalized - Strain Mesh - Free (GSMF) local method, is derived through a weighted-residual formulation that leads to the work theorem of theory of structures. In a local region, the work theorem establishes an energy relationship between a statically - admissible stress field and an independent kinematically - admissible strain field. In the GSMF formulation, the local form of work theorem is simply an integration - free formula. The Moving Least Squares (MLS) approximation of the elastic field is used to construct the trial function in this meshless formulation. GSMF has a high computational efficiency leading to accurate numerical results in two-dimensional elasticity problems. This paper is concerned with the numerical comparison of the energy and displacement error for different regular nodal distribution, for the Timoshenko cantilever beam and the infinite plate with circular hole. The results are compared with the exact solution and optimal parameters have been determined.

Keywords: Generalized-Strain Mesh-free (GSMF), Elasticity problems, Moving Least Squares (MLS), Work theorem, Energy error.

1 INTRODUCTION

There are different methodologies to solve mechanical problems. The methodology generally used is to idealize the problem according to certain conditions and to build a physical model governed by mathematical equations. These equations can be solved by analytical methods with exact solutions or by numerical methods with approximate solutions. The meshless is a new numerical method that has been used more frequently due to their accuracy and performance in numerical analysis for complex problem where the finite element method (FEM) has not shown much efficiency in cases where a great refinement mesh is needed.

In these last decades, many different meshless methods have been proposed with their formulation based in the weighted – residual method, Finalyson (1972). Element-free Galerkin (EFG) method, presented by Belytscho et al. (1994). The Meshless Local Petrov – Galerkin (MLPG) method presented by Atluri and Shen (2002). The meshless Finite Volume (FVM) method presented Atluri et al. (2004), Atluri et al. (2006) presented the Meshless Local Petrov-Galerkin (MLPG) for elastic problem. Liu (2009) presented the Meshfree methods moving beyond the finite element method and Oliveira and Portela (2016) presented the Generalized-Strain (GSMF) where the work theorem generates a weak form that is completely integration free.

The GSMF formulation has two important parameters associated to support and quadrature/collocation domain, this parameters have a greater influence in the problem solution obtained by Meshfree methods. The size of the local support domain (α_s) and the local quadrature/collocation domain (α_q) are very important meshless parameters, both related to accuracy and computational efficiency. A comparative study of the effect of these size parameters using the Meshless Local Petrov – Galerkin method (MLPG) for the solution of a cantilever beam was presented by Moussaoui and Bouziane (2013) and, using the same method, a similar study for the cantilever beam and the infinite plate with circular hole was carried out by Erdayi (2014).

This paper presents a numerical comparison using the GSMF formulation to identify the size effect of the configuration parameters of the local support domain and the local quadrature/collocation domain for different regular nodal distribution with the same number nodes for the Timoshenko cantilever beam and the plate with centralized circular hole. The results obtained in this work, both for the energy and displacement, are compared with the exact solution presented for plane stress case problems-2D and important results are presented in the end.

2 MLS APPROXIMATION

Let Ω be the domain of a body with boundary Γ and let $N = \{X_1, X_2, \dots, X_N\} \in \Omega$ be a set of scattered nodal points that represents a meshless discretization. Some of them are located on the boundary Γ where Ω_s , represented as Ω_P , Ω_Q and Ω_R , is the local compact support of a node X_t , represented as X_P , X_Q and X_R . Ω_x is the domain of definition of a sampling point X and Ω_q is the local weak-form domain or quadrature domain of node X_i , as represent in Fig. 1. Circular or rectangular local supports centered at each nodal point can be used. In the region of a

sampling point x , the domain of definition of MLS approximation is the subdomain Ω_x , where the approximation is defined.

2.1 Shape Functions

Let Ω_x be the domain of definition of the MLS approximation, in a neighborhood of a sampling point x . To approximate the displacement $u(x) \in \Omega_x$, over a number of scattered nodes $X_i \in \Omega$, $i = 1, 2, \dots, n$, where the nodal parameters $\hat{u}_i$ are defined, the MLS approximation is given by

$$u^h(x) = \mathbf{p}^T(x) \mathbf{a}(x), \quad (1)$$

for $x \in \Omega_x$, in which

$$\mathbf{p}^T(x) = [p_1(x), p_2(x), \dots, p_m(x)], \quad (2)$$

is a vector of the complete monomial basis of order m and $\mathbf{a}(x)$ is the vector of unknown coefficients $a_j(x)$, $j=1, 2, \dots, m$ that are functions of the space coordinates $x=[x_1, x_2]^T$, for 2-D problems.

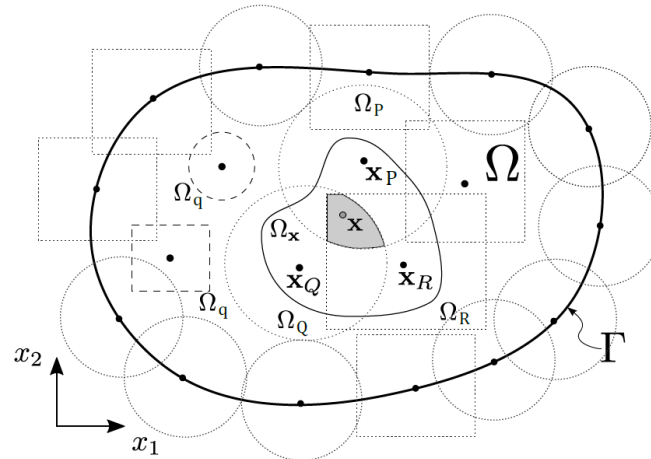


Figure 1. Representation of a global domain Ω and boundary Γ in a meshless discretization, with X_i nodes distributed within the body.

The coefficient vector $\mathbf{a}(x)$ is determined by minimizing the weighted discrete L_2 norm

$$J(x) = \frac{1}{2} \sum_{i=1}^n w_i(x) [u^h(x_i) - \hat{u}_i]^2 = \frac{1}{2} \sum_{i=1}^n w_i(x) [\mathbf{p}^T(x_i) \mathbf{a}(x) - \hat{u}_i]^2, \quad (3)$$

with respect to each term of $\mathbf{a}(x)$, in which $w_i(x)$ is the weight function associated with the node x_i , with the compact support that is $w_i(x) > 0$, for all x in the support of $w_i(x)$. In the Fig. 1 is represented the compact support of the MLS weight functions associated with a few nodes. Finding the extremum of $J(x)$ with respect to each term of $\mathbf{a}(x)$, leads to

$$\mathbf{A}(x) \mathbf{a}(x) = \mathbf{B}(x) \hat{\mathbf{u}}, \quad (4)$$

in which

$$\mathbf{A}(x) = \sum_{i=1}^n w_i(x) \mathbf{p}(x_i) \mathbf{p}^T(x_i), \quad (5)$$

$$\mathbf{B}(x) = [w_1(x) \mathbf{p}(x_1), w_2(x) \mathbf{p}(x_2), \dots, w_n(x) \mathbf{p}(x_n)], \quad (6)$$

$$\hat{\mathbf{u}} = [\hat{u}_1, \hat{u}_2, \dots, \hat{u}_n]. \quad (7)$$

Solving equation (4) for $a(x)$ yields

$$a(x) = \mathbf{A}^{-1}(x) \mathbf{B}(x) \hat{\mathbf{u}}, \quad (8)$$

provided $n \geq m$, for each some point x , condition defined in the MLS approximation.

$$u^h(x) = \sum_{i=1}^n \phi_i(x) \hat{u}_i, \quad (9)$$

in which

$$\phi_i(x) = \sum_{j=1}^n p_j(x) [\mathbf{A}^{-1}(x) \mathbf{B}(x)]_{ji}. \quad (10)$$

The shape function of the MLS approximation for the node x_i is represented Figure 2.

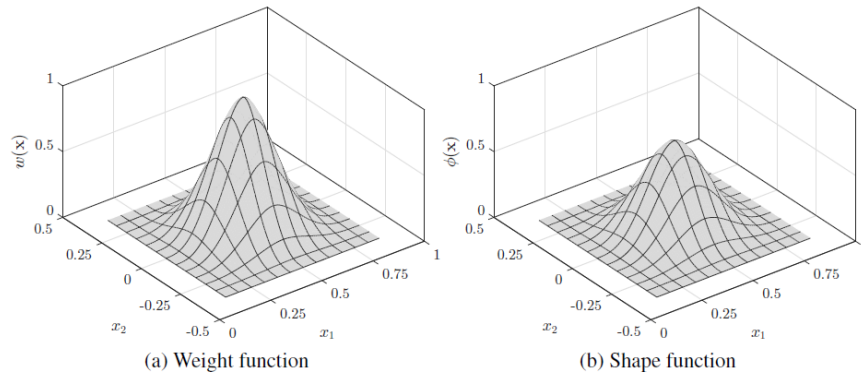


Figure 2. Weight function and shape function - MLS approximation

The MLS shape functions are not nodal interpolants which means mathematically that $\phi_i(x_j) \neq \delta_{ij}$. Since $\phi_i(x)$ vanishes for x not in the local domain of the node x_i , is preserved the local character of the MLS approximation. The nodal shape function is complete up to the order of the basis. The smoothness of the nodal shape function is determined by the smoothness of the basis and of the weight function. The spatial derivatives of the shape function $\phi_i(x)$ are giving by

$$\phi_{i,k}(x) = \sum_{j=1}^m \left[p_{j,k} (\mathbf{A}^{-1} \mathbf{B})_{ji} + p_j (\mathbf{A}^{-1} \mathbf{B}_{,k} - \mathbf{A}^{-1} \mathbf{A}_{,k} \mathbf{A}^{-1} \mathbf{B})_{ji} \right], \quad (11)$$

in which $\mathbf{0}_{,k} = \partial \mathbf{0} / \partial x_k$.

2.2 Weight Functions

In the Figure 2 the weight functions are represented, in Eq. (3) are introduced for each node x_i , have a compact support for all x , which defines the subdomain where $w_i(x) > 0$. Rectangular compact supports with weight functions defined as

$$w_i(x) = w_{i_x}(x)w_{i_y}(x), \quad (12)$$

the weight function is given by the quartic spline function

$$w_{i_x}(x) = \begin{cases} 1 - 6\left(\frac{d_{i_x}}{r_{i_x}}\right)^2 + 8\left(\frac{d_{i_x}}{r_{i_x}}\right)^3 - 3\left(\frac{d_{i_x}}{r_{i_x}}\right)^4 & \text{for } 0 \leq d_{i_x} \leq r_{i_x}, \\ 0 & \text{for } d_{i_x} > r_{i_x} \end{cases}, \quad (13)$$

$$w_{i_y}(x) = \begin{cases} 1 - 6\left(\frac{d_{i_y}}{r_{i_y}}\right)^2 + 8\left(\frac{d_{i_y}}{r_{i_y}}\right)^3 - 3\left(\frac{d_{i_y}}{r_{i_y}}\right)^4 & \text{for } 0 \leq d_{i_y} \leq r_{i_y}, \\ 0 & \text{for } d_{i_y} > r_{i_y} \end{cases}, \quad (14)$$

in which $d_{i_x} = \|x - x_i\|$ and $d_{i_y} = \|y - y_i\|$. The parameters r_{i_x} and r_{i_y} represent the size of the support for the node i , respectively in the x and y directions.

2.3 Elastic Field

The elastic field is approximated at a sampling point x . Considering Eq. (9) the displacement and strain components are

$$\mathbf{u} = \begin{bmatrix} u^h(x) \\ v^h(x) \end{bmatrix} = \begin{bmatrix} \phi_1(x) & 0 & \dots & \phi_n(x) & 0 \\ 0 & \phi_1(x) & \dots & 0 & \phi_n(x) \end{bmatrix} \begin{bmatrix} \hat{u}_1 \\ \hat{v}_1 \\ \vdots \\ \hat{u}_n \\ \hat{v}_n \end{bmatrix} = \mathbf{\Phi} \hat{\mathbf{u}}, \quad (15)$$

$$\boldsymbol{\varepsilon} = \mathbf{L}\mathbf{u} = \mathbf{L}\mathbf{\Phi}\hat{\mathbf{u}} = \mathbf{B}\hat{\mathbf{u}}, \quad (16)$$

in which geometrical linearity is assumed in the differential operator $\mathbf{L}$ and thus,

$$\mathbf{B} = \begin{bmatrix} \phi_{1,1} & 0 & \dots & \phi_{n,1} & 0 \\ 0 & \phi_{1,2} & \dots & 0 & \phi_{n,2} \\ \phi_{2,1} & \phi_{1,1} & \dots & \phi_{n,2} & \phi_{n,1} \end{bmatrix}, \quad (17)$$

Stress and traction components are

$$\boldsymbol{\sigma} = \mathbf{D}\boldsymbol{\varepsilon} = \mathbf{D}\mathbf{B}\hat{\mathbf{u}}, \quad (18)$$

$$\mathbf{t} = \mathbf{n}\boldsymbol{\sigma} = \mathbf{n}\mathbf{D}\mathbf{B}\hat{\mathbf{u}}, \quad (19)$$

in which $\mathbf{D}$ is the matrix of the elastic constants and $\mathbf{n}$ is the matrix of the components of the unit outward normal, defined as

$$\mathbf{n} = \begin{bmatrix} n_1 & 0 & n_2 \\ 0 & n_2 & n_1 \end{bmatrix}. \quad (20)$$

Eq. (15) to (19) show that the variables of the elastic field are defined in terms of the nodal unknowns $\hat{u}$, for all point $x \in \Omega_x$.

3 LOCAL FORM OF THE WORK THEOREM

The development for the local form of the work theorem introduced by Oliveira and Portela (2016) is presented in this section. Let Ω be the domain of a body and Γ its boundary, subdivided in Γ_u and Γ_t that is $\Gamma = \Gamma_u \cup \Gamma_t$; in the Fig. 3 show the nodal points P, Q and R have corresponding local domains Ω_P , Ω_Q , and Ω_R .

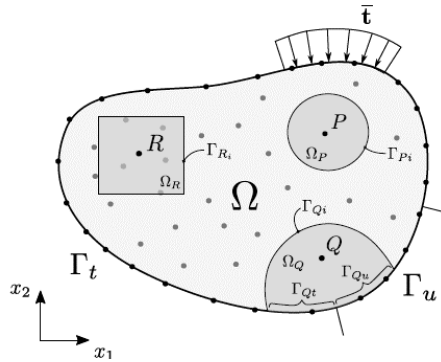


Figure 3. Global domain Ω and the local domains Ω_P , Ω_Q and Ω_R , with boundary $\Gamma = \Gamma_u \cup \Gamma_t$ represented.

The mixed fundamental boundary value problem of linear electrostatics aims to determine the distribution of stress σ , strains ε and displacements u throughout the body, when it has constrained displacements $\bar{u}$ defined on Γ_u and is loaded by an external system of distributed surface and body forces with densities denoted by $\bar{t}$ on Γ_t and b in Ω , respectively. The entire admissible elastic field is the solution of the posed problem that simultaneously satisfies the kinematic admissibility and the static admissibility. If this solution exists, it can be shown that it is unique, provided linearity and stability of the material are admitted, see Fredholm (1906) and Fichera (2006).

The general work theorem establishes an energy relationship between any statically – admissible stress field and any kinematically – admissible strain field that can be defined in the body. Derived as a weighted residual statement, the work theorem serves as a unifying basis for the formulation of numerical models Continuum Mechanics presented by Brebbia and Tottenham (1985). In the domain of the body, consider a statically-admissible stress field that is

$$\mathbf{L}^T \sigma + b = 0, \quad (21)$$

in the domain Ω , with boundary conditions

$$t = n\sigma = \bar{t}, \quad (22)$$

on the static boundary Γ_t , σ : Stress components, L : Matrix differential operator, t : represent the traction components, $\bar{t}$: Prescribed tractions values, n : Unit normal components to the boundary.

In the global domain Ω , consider an arbitrary local subdomain Ω_Q , centered at the point Q , with boundary $\Gamma = \Gamma_{Qi} \cup \Gamma_{Qt} \cup \Gamma_{Qu}$, Γ_{Qi} : Interior local boundary and Γ_{Qt} and Γ_{Qu} : Local boundaries that respectively share a global boundary. Due to its arbitrariness, this local domain can be overlapping with other similar subdomains. For the local domain Ω_Q the strong form of the weighted-residual equation is written as

$$\int_{\Omega_Q} (\mathbf{L}^T \boldsymbol{\sigma} + \mathbf{b})^T \mathbf{W}_\Omega d\Omega + \int_{\Gamma_{Qt}} (\mathbf{t} - \bar{\mathbf{t}})^T \mathbf{W}_\Gamma d\Gamma = 0, \quad (23)$$

in which $\mathbf{W}_\Omega$ and $\mathbf{W}_\Gamma$ are arbitrary weighting functions defined, respectively in Ω and on Γ . When the domain term of Eq. (23) is integrated by parts, the following local weak form of the weighted residual equation is obtained

$$\int_{\Gamma_Q} (\mathbf{n}\boldsymbol{\sigma})^T \mathbf{W}_\Omega d\Gamma - \int_{\Omega_Q} (\boldsymbol{\sigma}^T \mathbf{L} \mathbf{W}_\Omega - \mathbf{b}^T \mathbf{W}_\Omega) d\Omega + \int_{\Gamma_{Qt}} (\mathbf{t} - \bar{\mathbf{t}})^T \mathbf{W}_\Gamma d\Gamma = 0, \quad (24)$$

which now requires continuity of $\mathbf{W}_\Omega$, as an admissibility condition for integrability. For the sake of convenience, the arbitrary weighting function $\mathbf{W}_\Gamma$ is chosen as

$$\mathbf{W}_\Gamma = -\mathbf{W}_\Omega, \quad (25)$$

on the boundary Γ_{Qt} . Thus, Eq. (24) leads to

$$\int_{\Gamma_Q - \Gamma_{Qt}} \mathbf{t}^T \mathbf{W}_\Omega d\Gamma + \int_{\Gamma_{Qt}} \bar{\mathbf{t}}^T \mathbf{W}_\Omega d\Gamma - \int_{\Omega_Q} (\boldsymbol{\sigma}^T \mathbf{L} \mathbf{W}_\Omega - \mathbf{b}^T \mathbf{W}_\Omega) d\Omega = 0. \quad (26)$$

Consider further an arbitrary kinematically-admissible strain field $\boldsymbol{\varepsilon}^*$, with continuous displacements $\mathbf{u}^*$ and small derivatives, in order to assume geometrical linearity, defined in the global domain that is

$$\boldsymbol{\varepsilon}^* = \mathbf{L} \mathbf{u}^*, \quad (27)$$

in the domain Ω , with boundary conditions

$$\mathbf{u}^* = \bar{\mathbf{u}}, \quad (28)$$

on the kinematic boundary Γ_u . When the continuous arbitrary weighting function $\mathbf{W}_\Omega$, is defined as

$$\mathbf{W}_\Omega = \mathbf{u}^*, \quad (29)$$

the weak form (26), of the weighted residual equation, becomes

$$\int_{\Gamma_Q - \Gamma_{Qt} - \Gamma_{Qu}} \mathbf{t}^T \mathbf{u}^* d\Gamma + \int_{\Gamma_{Qu}} \mathbf{t}^T \bar{\mathbf{u}}^* d\Gamma + \int_{\Gamma_{Qt}} \bar{\mathbf{t}}^T \mathbf{u}^* d\Gamma - \int_{\Omega_Q} (\boldsymbol{\sigma}^T \mathbf{L} \mathbf{u}^* - \mathbf{b}^T \mathbf{u}^*) d\Omega = 0, \quad (30)$$

which can be written in a compact form as

$$\int_{\Gamma_Q} \mathbf{t}^T \mathbf{u}^* d\Gamma + \int_{\Omega_Q} \mathbf{b}^T \mathbf{u}^* d\Omega = \int_{\Omega_Q} \boldsymbol{\sigma}^T \boldsymbol{\varepsilon}^* d\Omega. \quad (31)$$

This equation is the starting point of kinematically admissible formulations of the local mesh-free method presented in this paper. Equation (31) which expresses the static-kinematic duality, is the local form of the well-known work theorem, the fundamental identity of solid mechanics, Timoshenko and Goodier (1970).

It is important to notice that the stress field σ , is any one that satisfies equilibrium with the applied external forces b and t , which is not necessarily the stress field that actually settles in the body. Also, the strain field ε^* , is any one that is compatible with the constraints $u^* = \bar{u}$ which is not necessarily the strain field that actually settles in the body. These two fields are not connected function W_Ω they are completely independent. For that reason, Eq. (31) can be used under the only assumption of geometrical linearity. It is the independence of the two admissible fields of the Eq. (31) that allows generation of different meshfree methods, when the strain field is locally defined through different options, as carried out in this paper. A final important remark, worth of mentioning, is that the local domain Ω_Q , is any arbitrary subdomain Ω , of the body.

4 MODELING STRATEGY

In the local meshfree methods different formulations can be derived when the arbitrary kinematically – admissible field ε^* , is locally defined in the work theorem, Eq. (31). In the following section, simple kinematically – admissible local fields will be used to derive the meshless formulation presented in this paper, the Generalized-Strain Mesh-Free (GSMF) formulation. On the other hand, the statically – admissible local field σ , will be always assumed as the elastic field that actually settles in the body. Not only satisfying static admissibility, through Eq. (21) and (22), but also satisfying kinematic admissibility in the elastic field defined as

$$\varepsilon = \mathbf{L}u. \quad (32)$$

In the domain Ω , with boundary conditions

$$u = \bar{u}, \quad (33)$$

on the kinematic boundary Γ_u ; in the which the displacements u , are assumed continuous with small derivatives, in order to allow for geometrical linearity of the strain field ε . Therefore, Eq. (33) must be enforced in the numerical model, in order to provide a unique solution of the posed problem. For a meshless discretization of the body, the local weak-form domain or quadrature domain Ω_Q , centered at the node Q , can be defined in this paper as a rectangular or circular subdomain, as represented in Fig. 3.

5 GENERALIZED-STRAIN FORMULATION

The Equation (31), the kinematically – admissible displacement field u^* , was assumed as a continuous function leading to a regular integral function that is the kinematically-admissible strain field ε^* . For more information about the method, see Oliveira and Portela (2016). However, this continuity assumption on u^* , enforced in the local form of the work theorem, is not required but can be relaxed by convenience, provided ε^* can be useful as a generalized function, in the sense of the theory of distributions, Gedalf and Shilov (1964). Hence, this formulation considers that the kinematically-admissible displacement field is a piece-wise continuous function, defined in terms of the Heaviside step function and therefore the corresponding kinematically-admissible strain field is a generalized function, defined in terms of Dirac delta function. For the sake of the simplicity, in dealing with Heaviside and Dirac delta functions in a two-dimensional coordinate space, consider a scalar function d , defined as

$$d = \|x - x_Q\| \quad \text{that is} \quad \begin{cases} d = 0 & \text{if } x \equiv x_Q \\ d > 0 & \text{if } x \neq x_Q \end{cases}, \quad (34)$$

which represents the absolute-value function of the distance between a field point x and a particular reference point x_Q , in the local domain $\Omega_Q \cup \Gamma_Q$ assigned to the field node Q . Therefore, this definition always assumes $d = d(x, x_Q) \geq 0$, as a positive or null value, in this case when-ever x and x_Q are coincident points. It is important to remark that, in Eq. (34), neither the field point x nor the reference point x_Q is necessarily a nodal point of the local domain. For a scalar coordinate, $d \supset d(x, x_Q)$, the Heaviside step function can be defined as

$$H(d) = \begin{cases} 1 & \text{if } d \leq 0 \quad (d = 0 \text{ for } x \equiv x_Q) \\ 0 & \text{if } d > 0 \quad \text{that is } x \neq x_Q \end{cases}, \quad (35)$$

in which the discontinuity is assumed at x_Q and consequently, the Dirac delta function is defined with the following properties

$$\delta(d) = H'(d) = \begin{cases} \infty & \text{if } d = 0 \quad \text{that is } x \equiv x_Q \\ 0 & \text{if } d \neq 0 \quad (d > 0 \text{ for } x \neq x_Q) \end{cases} \quad \text{and} \quad \int_{-\infty}^{+\infty} \delta(d) dd = 1, \quad (36)$$

in which $H'(d)$ represents the distributional derivative of $H(d)$. Note that the derivative of $H(d)$, with respect to the coordinate x_i , can be defined as

$$H(d)_{,i} = H'(d)d_{,i} = d(d)d_{,i} = d(d)n_i. \quad (37)$$

Since the result of this equation is not affected by any particular value of the constant n_i , this constant will be conveniently redefined later on. Kronecker delta function can be defined through Heaviside step function as

$$\Delta(d) = H(d) + H(-d) - 1 = \begin{cases} 1 & \text{if } d = 0 \quad \text{that is } x \equiv x_Q \\ 0 & \text{if } d > 0 \quad \text{that is } x \neq x_Q \end{cases}, \quad (38)$$

Which has the distributional derivative always null that is

$$\Delta'(d) = \delta(d) - \delta(-d) = \delta(d) - \delta(d) = 0, \quad (39)$$

as a consequence of the symmetry of Dirac delta function.

Now considerer that d_i , d_j and d_k represent the distance function d , defined in Eq. (34), for corresponding field points x_i , x_j and x_k . Then, the kinematically – admissible displacement field can be defined as a linear combination of Kronecker delta function evaluations at an arbitrary number of collocation points, conveniently arranged in the local domain $\Omega_Q \cup \Gamma_Q$ of the field node Q , that is

$$u^*(x) = \left[\frac{L_i}{n_i} \sum_{l=1}^{n_i} \Delta(d_l) + \frac{L_t}{n_t} \sum_{j=1}^{n_t} \Delta(d_j) + \sum_{k=1}^{n_\Omega} \Delta(d_k) \right] e, \quad (40)$$

in which $e = [1 \quad 1]^T$ represents the metric of the orthogonal directions; n_i , n_t and n_Ω represent the number of collocation points, respectively on the local interior boundary $\Gamma_{Qi} = \Gamma_Q - \Gamma_{Qt} - \Gamma_{Qu}$ with length L_i , on the local static boundary Γ_Q , with length L_t and in the local domain Ω_Q

with area S . This assumed displacement field $u^*(x)$, a discrete rigid-body unit displacement defined at collocation points, schematically represent in Fig. 4, conveniently leads to a null

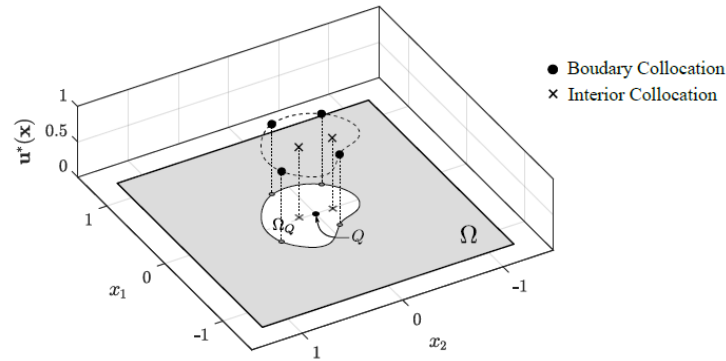


Figure 4. Schematic representation of the displacement $u^*(x)$ of Eq. (38), a discrete rigid-body unit displacement defined at collocation points, for a local domain associated with a field node Q .

generalized strain field that is

$$\varepsilon^*(x) = 0, \quad (41)$$

as a consequence of Eq. (39). The local work theorem, Eq. (31), can be written as

$$\int_{\Gamma_Q - \Gamma_{Qt}} \mathbf{t}^T \mathbf{u}^* d\Gamma + \int_{\Gamma_{Qt}} \bar{\mathbf{t}}^T \mathbf{u}^* d\Gamma + \int_{\Omega_Q} \mathbf{b}^T \mathbf{u}^* d\Omega = \int_{\Omega_Q} \boldsymbol{\sigma}^T \varepsilon^* d\Omega, \quad (42)$$

which, after considering the assumed displacement and the strain components of the kinematically-admissible field, respectively Eq. (40) and (41), leads to

$$\frac{L_i}{n_i} \sum_{l=1}^{n_i} \int_{\Gamma_Q - \Gamma_{Qt}} \mathbf{t}^T \Delta(d_l) d\Gamma + \frac{L_t}{n_t} \sum_{j=1}^{n_t} \int_{\Gamma_{Qt}} \bar{\mathbf{t}}^T \Delta(d_j) d\Gamma + \frac{S}{n_\Omega} \sum_{k=1}^{n_\Omega} \int_{\Omega_Q} \mathbf{b}^T \Delta(d_k) d\Omega = 0, \quad (43)$$

Now considering the properties of Kronecker delta function, defined in Eq. (38), the Eq. (43) simply leads to

$$\mathbf{e}^T \left[\frac{L_i}{n_i} \sum_{l=1}^{n_i} \mathbf{t}_{x_l} + \frac{L_t}{n_t} \sum_{j=1}^{n_t} \bar{\mathbf{t}}_{x_j} + \frac{S}{n_\Omega} \sum_{k=1}^{n_\Omega} \mathbf{b}_{x_k} \right] = 0, \quad (44)$$

and finally to

$$\frac{L_i}{n_i} \sum_{l=1}^{n_i} \mathbf{t}_{x_l} = -\frac{L_t}{n_t} \sum_{j=1}^{n_t} \bar{\mathbf{t}}_{x_j} - \frac{S}{n_\Omega} \sum_{k=1}^{n_\Omega} \mathbf{b}_{x_k}. \quad (45)$$

Equation (45) states the equilibrium of tractions and body forces, pointwisely defined at collocation points, as schematically represented in Fig. 5; obviously, the pointwise version of the Euler – Cauchy stress principle.

This is the equation used in the GSMF formulation which, therefore, is free of integration. Since the work theorem is a weighted – residual weak form, it can be easily seen that this integration – free formulation is nothing else other than a weighted – residual weak – form collocation. Equations (45), of the Generalized – Strain Mesh – free formulation, can be derived from another kinematically – admissible displacement field, directly defined in terms of Heaviside step function, see Finalyson (1972).

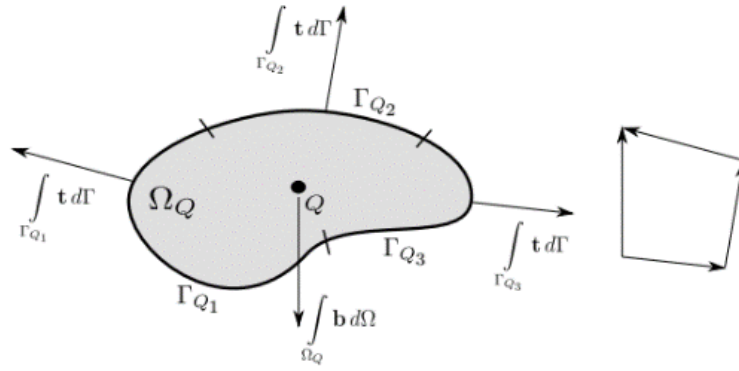


Figure 5. Schematic representation of the equilibrium of tractions and body forces of Eq. (44), pointwisely defined at collocation points of a local domain associated with a field node Q.

Discretization of Eq. (45) is carried out with the MLS approximation, Eq. (15) to (19), for the local domain Ω_Q , in terms of the nodal unknowns $\hat{\mathbf{u}}$, thus leading to the system of two linear algebraic equations

$$\frac{L_t}{n_l} \sum_{i=1}^{n_l} \mathbf{n}_{x_i} \mathbf{D} \mathbf{B}_{x_i} \hat{\mathbf{u}} = -\frac{L_t}{n_t} \sum_{j=1}^{n_t} \bar{\mathbf{t}}_{x_j} - \frac{S}{n_\Omega} \sum_{k=1}^{n_\Omega} \mathbf{b}_{x_k}, \quad (46)$$

that can be written as

$$\mathbf{K}_Q \hat{\mathbf{u}} = \mathbf{F}_Q, \quad (47)$$

in which $\mathbf{K}_Q$, the nodal stiffness matrix associated with the local Ω_Q , is a $2 \times 2n$ matrix

$$\mathbf{K}_Q = \frac{L_t}{n_l} \sum_{i=1}^{n_l} \mathbf{n}_{x_i} \mathbf{D} \mathbf{B}_{x_i}, \quad (48)$$

and $\mathbf{F}_Q$ is the respective force vector given by

$$\mathbf{F}_Q = -\frac{L_t}{n_t} \sum_{j=1}^{n_t} \bar{\mathbf{t}}_{x_j} - \frac{S}{n_\Omega} \sum_{k=1}^{n_\Omega} \mathbf{b}_{x_k}, \quad (49)$$

Consider that the problem has a total of N field nodes Q , each one associated with the respective local region Ω_Q . Assembling Eq. (47), for all M interior and static – boundary field nodes leads to the global system of $2M \times 2N$ equations

$$\mathbf{K} \hat{\mathbf{u}} = \mathbf{F}, \quad (50)$$

Finally, the remaining equations are obtained from the $N - M$ boundary field nodes on the kinematic boundary. For a field node on the kinematic boundary, a direct interpolation method is used to impose the Kinematic boundary condition as

$$u_k^h(\mathbf{x}_j) = \sum_{i=1}^n \phi_i(\mathbf{x}_j) \hat{u}_{ik} = \bar{\mathbf{u}}_k, \quad (51)$$

or, in matrix form as

$$\mathbf{u}_k = \Phi_k \hat{\mathbf{u}} = \bar{\mathbf{u}}_k, \quad (52)$$

with $k = 1, 2$, where $\bar{\mathbf{u}}_k$ is specified nodal displacement component. Equations (51) are directly assembled into the global system of equations (50).

6 NUMERICAL EXAMPLES

This section presents some numerical results for Cantilever beam and the Plate with a centralized circular hole for different nodal configurations. The effects of the size of local support and quadrature/collocation domain are analyzed and compared with the exact solution. For a generic node i , the size of the local support Ω_s and the local domain of integration Ω_q are respectively given by

$$r\Omega_s = \alpha_s c_i, \quad (53)$$

$$r\Omega_q = \alpha_q c_i, \quad (54)$$

in which c_i represents the distance of the node i to the nearest neighboring node. Displacement and energy norms can be used for error estimation. These norms can be computed, respectively as

$$\|u\| = \left[\int_{\Omega} u^T u d\Omega \right]^{1/2}, \quad (55)$$

$$\|\varepsilon\| = \left[\frac{1}{2} \int_{\Omega} \varepsilon^T D \varepsilon d\Omega \right]^{1/2}, \quad (56)$$

The relative error for $\|u\|$ and $\|\varepsilon\|$ is given, respectively by

$$r_u = \frac{\|u_{num} - u_{exact}\|}{\|u_{exact}\|}, \quad (57)$$

$$r_{\varepsilon} = \frac{\|\varepsilon_{num} - \varepsilon_{exact}\|}{\|\varepsilon_{exact}\|}. \quad (58)$$

The local support domain (α_s) must be greater than 1.0, the algorithm of MLS approximation may be singular and the shape function cannot be constructed for the small values. The local quadrature/collocation domain (α_q) must be less than 1.0, the principal reason is to ensure that the local sub – domains of the internal nodes are entirely within the solution domain, without being intersected by the global boundary.

6.1 Cantilever Beam

A Cantilever beam showed in Fig. 6, is subjected to a parabolic traction at the free end. The principal properties are presented in Table 1 and the problem is solved for plane stress case.

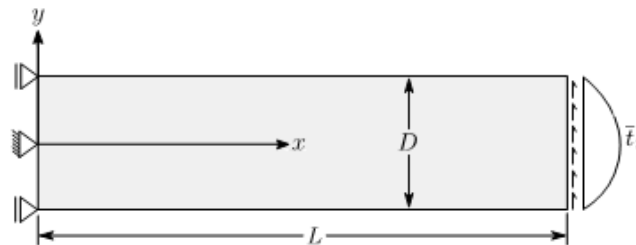


Figure 6. Timoshenko Cantilever beam

Table 1. Properties of Cantilever beam

Parameter	Nomenclature	Value
Height	D	12 m
Length	L	48 m
Thickness	t	1 m
Load	P	1000 N
Modulus of Elasticity	E	30 MPa
Poisson's Ratio	ν	0.3

The parabolic traction and the moment of inertia is given by

$$\bar{t}_2(x_2) = -\frac{P}{2I} \left(\frac{D^2}{4} - x_2^2 \right), \quad (59)$$

$$I = \frac{D^3}{12}, \quad (60)$$

The equations for the exact displacement are:

$$u_1(x, y) = -\frac{Py}{6EI} \left[(6L - 3x) + (2 + \nu) \left(y^2 - \frac{D^2}{4} \right) \right], \quad (61)$$

$$u_2(x, y) = \frac{P}{6EI} \left[3\nu y^2 (L - x) + (4 + 5\nu) \frac{D^2 x}{4} + (3L - x) y^2 \right]. \quad (62)$$

And the exact stress components are given by

$$\sigma_{11}(x, y) = -\frac{P(L - x)y}{I}, \quad \sigma_{12}(x, y) = -\frac{P}{2I} \left(\frac{D^2}{4} - y^2 \right) \text{ and } \sigma_{22}(x, y) = 0. \quad (63)$$

The GSMF was used for solving this problem, two regular nodal distributions were considered with a discretization of $4 \times 60 = 240$ nodes, $5 \times 48 = 240$ nodes. The second discretization is showed in the Fig. 7.

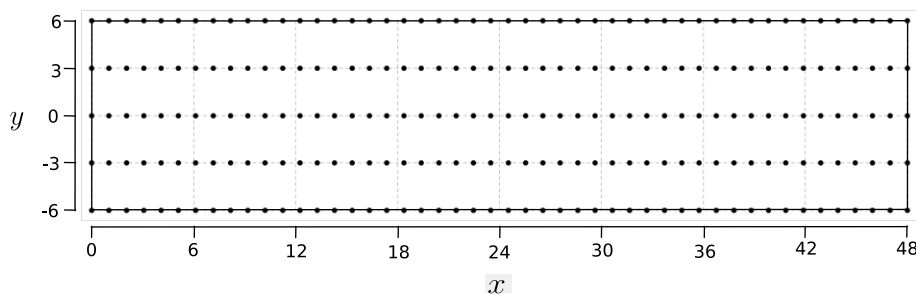


Figure 7. The regular nodal distribution of $5 \times 48 = 240$ nodes.

The Figure 8 shows the variation of the energy relative error as function of the size local support domain (α_s), with 15 ratios, which vary from 3.0 to 10.0 with 0.5 increments for fixed value of $\alpha_q = 0.5$.

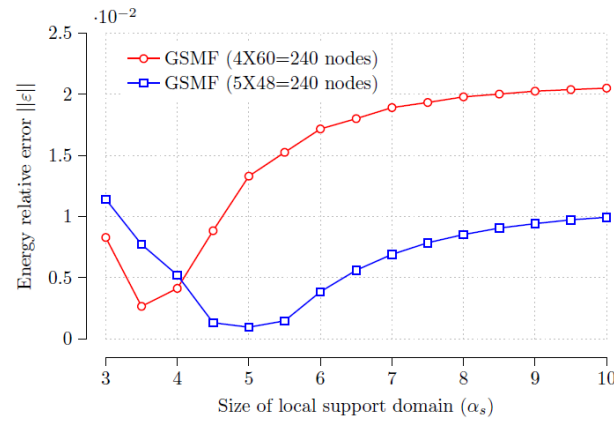


Figure 8. Influence of the local support domain (α_s) for different nodal distributions with $\alpha_q = 0.5$ – Timoshenko cantilever beam.

In the Figure 8 it is possible to observe that there are two region. First region showed that better results are obtained for the configuration nodal (4x60). After $\alpha_s = 4.0$ the best results are obtained for the other nodal configuration (5x48) which is the second region, specifically for α_s values between 4.5 and 5.5.

The Figure 9 shows the variation of the energy relative error as function of the size local quadrature/collocation domain (α_q), with 7 ratios, which vary from 0.4 to 0.7 with 0.05 increments for fixed value of $\alpha_q = 4.0$.

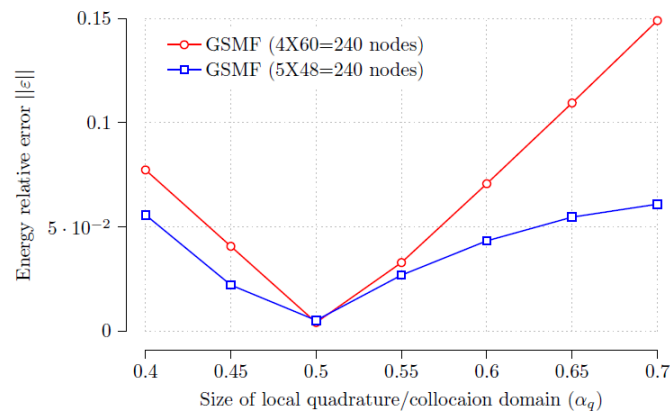


Figure 9. Influence of the local quadrature/collocation domain size (α_q) for nodal distributions with $\alpha_s = 4.0$ - Timoshenko cantilever beam.

The Figure 9 shows that the minimum error is obtained when $\alpha_q = 0.5$ for both nodal configurations and independent of the value used for α_s .

The Figure 10 shows the variation of the energy and displacement relative error as function of the size local support domain (α_s). The axis α_s has 15 ratios varying from 3.0 to 10.0 with 0.5 increments and a fixed value of $\alpha_q = 0.5$, this value was chosen because presents the best results independently of the nodal configuration. The results are obtained for the nodal configuration showed in the Fig. 7.

The Figure 10 displays that the minimum error for energy and displacement is obtained when $\alpha_s = 5$ for both nodal configurations. It is important to mention that the region determined for $\alpha_s = 4.5$ and $\alpha_s = 5.5$ presents the fewest errors.

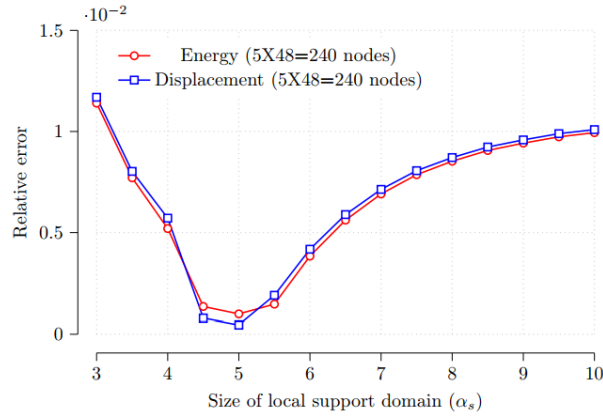


Figure 10. Influence of the local support domain (α_s) in energy and displacement error with $\alpha_q=0.5$, carried out by the regular discretization with $5 \times 48 = 240$ nodes Timoshenko cantilever beam.

6.2 Plate with centralized circular hole

Consider an infinite plate with a centered circular hole under unidirectional unit tension along the x direction, as represented in Fig. 11. Due to the symmetry of the problem about the horizontal and vertical axes, only a portion of the upper right quadrant of the plate is considered. The modeled section of the plate has dimensions $b \times b$ and the center circle has a radius $a = 1$, with $b = 5a$. The principal properties are presented in Table 2 and the problem is solved for plane stress case. In order to investigate the influence domain size, three sets of run are performed. Results of exact solution is used to calculate the relative error.

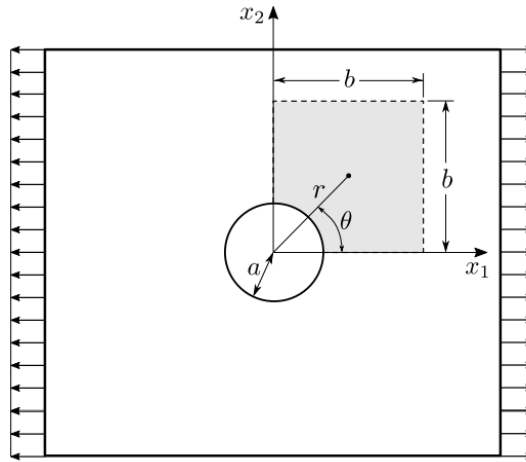


Figure 11. Plate with centralized circular hole.

The exact stress distribution in the plate is given by

$$\sigma_{11}(r, \theta) = 1 - \frac{a^2}{r^2} \left(\frac{3}{2} \cos 2\theta + \cos 4\theta \right) + \frac{3}{2} \frac{a^4}{r^4} \cos 4\theta \quad (63)$$

$$\sigma_{22}(r, \theta) = -\frac{a^2}{r^2} \left(\frac{1}{2} \cos 2\theta - \cos 4\theta \right) - \frac{3}{2} \frac{a^4}{r^4} \cos 4\theta \quad (64)$$

$$\sigma_{12}(r, \theta) = -\frac{a^2}{r^2} \left(\frac{1}{2} \sin 2\theta - \sin 4\theta \right) - \frac{3}{2} \frac{a^4}{r^4} \sin 4\theta \quad (65)$$

Where r and θ are the usual polar coordinates, centered at the center of the hole. A plane-stress state is considered which leads to the following displacements

$$u_1(r, \theta) = -\frac{\cos \theta}{2r^3 E} \left[4a^4 \cos^2 \theta (1+\nu)(1-r^2) - 3a^4(1+\nu) + (ar)^2(1-3\nu) - 2r^4 \right] \quad (66)$$

$$u_2(r, \theta) = -\frac{\sin \theta}{2r^3 E} \left[4a^4 \cos^2 \theta (1+\nu)(1-r^2) - a^4(1+\nu) + (ar)^2(\nu-3) + 2r^4 \nu \right] \quad (67)$$

Table 2. Properties of Plate with Circular Hole

Parameter	Nomenclature	Value
Hole radius	a	1 m
Height	D	5 m
Length	L	5 m
Thickness	t	1 m
Load	P	1000 N
Modulus of Elasticity	E	100000 Pa
Poisson's Ratio	ν	0.25

The bottom and left edges of the plate are assumed as kinematic boundaries, with displacements specified on the bottom u_2 ($x, y = 0$) and left edges (u_1 ($x = 0, y = L, y = 0$)). The rigid and top edges are assumed as static boundaries, loaded by tractions computed from the stresses of the exact solution (63, 64 e 65) as $t_j = \sigma_{ij}$, in wich n_i represents the components of the unit outward normal to the edge of the plate.

To solve this problem, the plate was discretized by Alturi and Zhu (2000), with 9 nodes in the tangential direction and 11 nodes in the radial direction, see Fig. 12a, with circular local supports. In this work, another configuration is proposed with 11 nodes in the tangential direction and 9 nodes in the radial direction, see 12b. GSMF was applied with 1 boundary collocation point per quadrant of the circular local domain.

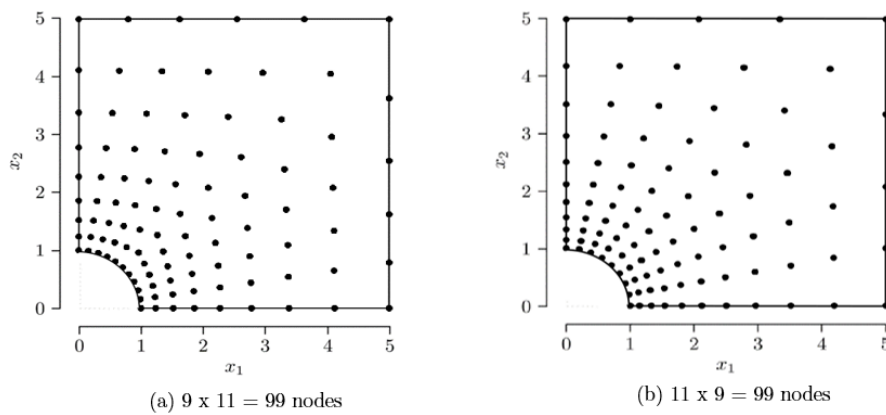


Figure 12. Nodal distribution for the infinite plate of 9x11 = 99 nodes (a) and 11x9 = 99 nodes (b).

The Figure 13 shows the first set, the variation of the energy relative error as function of the size local support domain (α_s), with 5 ratios, which vary from 3.0 to 5.0 with 0.5 increments for fixed value of $\alpha_q = 0.3$.

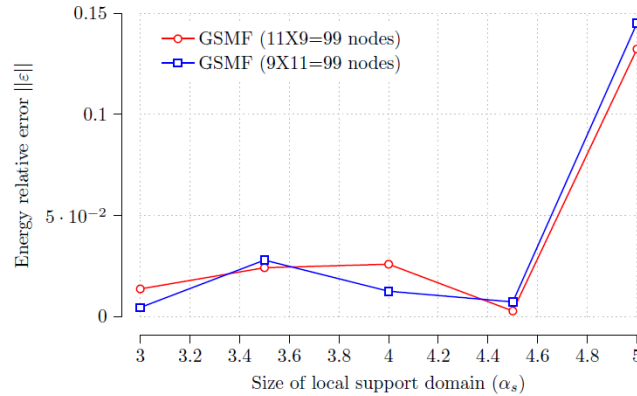


Figure 13. Influence of the local support domain (α_s) for different nodal distributions with $\alpha_q = 0.3$ - Plate with centralized circular hole.

The Figure 13 shows that the minimum error is obtained when $\alpha_s = 4.5$ for both nodal configurations, especially for the nodal configuration presented in the Fig. 12.a. Other good values are obtained for $\alpha_s = 3.0$.

In the second set, the variation of the energy relative error as function of the size local quadrature/collocation domain (α_q), with 10 ratios varying from 0.1 to 0.55 with 0.05 increments for fixed value of $\alpha_s = 4.5$. Results are shown in the Fig. 14.

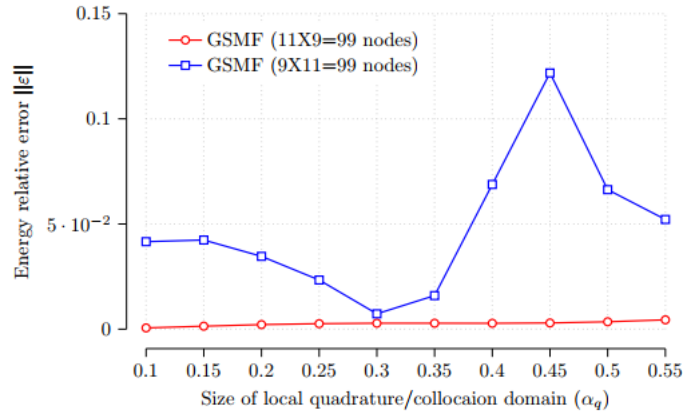


Figure 14. Influence of the local quadrature/collocation domain size (α_q) for nodal distributions with $\alpha_s = 4.0$ - Plate with centralized circular hole.

In the Figure 14 it is possible to observe that the configuration with 9 nodes in the tangential direction and 11 nodes in the radial direction always have the minimum error. When $\alpha_q=0.3$, both nodal configuration presented similar values.

In third set, the Fig. 15 shows the variation of the energy and displacement relative error as function of the size local quadrature/collocation domain (α_s), with 10 ratios, which vary from 0.1 to 0.55 with 0.05 increments and fixed value of $\alpha_s = 4.5$. The results are presented for the nodal configuration showed in the Fig. 12a, because this configuration presents the smallest errors according to the results obtained in Fig. 14.

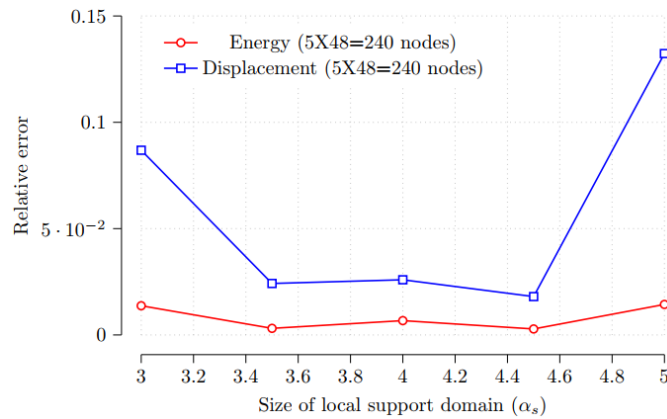


Figure 15. Influence of the local support domain (α_s) in energy and displacement error with $\alpha_q=0.5$, carried out by the regular discretization with $5 \times 48 = 240$ nodes Plate with centralized circular hole.

The Figure 15 displays that the minimum error for energy and displacement is obtained when $\alpha_s = 4.5$ for both nodal configurations. It is important to mention that the region determined for $\alpha_s = 4.5$ and $\alpha_s = 5.5$ presents the fewest errors and the less distance between the curves.

CONCLUSIONS

The size of local support domain (α_s), local quadrature/collocation domain (α_q) and the configuration nodal affect the accuracy and performance of GSMF local method. However, for certain values of α_s and α_q and specificity nodal configurations it is possible to obtain the minimum relative errors.

According to the results obtained for the Timoshenko cantilever beam and the plate with centralized circular hole, it can be observed that the nodal configuration influences the results obtained for energy and displacement. In the two cases studied, was observed that the best results were obtained for the configurations that presented the greatest number of nodes in the transversal direction, similar to the one obtained in the finite element method (FEM).

The study realized for the Timoshenko cantilever beam, the parameter local quadrature/collocation domain $\alpha_q = 0.5$, presented the lowest relative errors independently of values of local support domain (α_s) and the configuration nodal. For the nodal discretization with 5 nodes in the transversal direction and 48 nodes in the longitudinal direction (240 nodes), the errors for energy and displacement were the minimums.

For the Plate with centralized circular hole, the parameter local support domain $\alpha_s = 4.5$, presented the lowest relative errors independently of values of quadrature/collocation support domain (α_q). The nodal discretization with 9 nodes in the tangential direction and 11 nodes in the radial direction (99 nodes) were obtained the minimum errors for energy and displacement.

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