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ANALYSIS OF QUATERNIONS COMPONENTS IN QUEST ALGORITHM - THE DUALITY PROBLEM

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Abstract. *The attitude estimation of aerospace vehicles is an important step in many contexts. Different algorithms were proposed in literature addressing this estimation, and problems related with computational implementations of them. In this context, some algorithms are based in quaternion algebra to represent attitudes. In this work, the problem of distinct quaternions representing the same attitude, presented by QUEST algorithm, is addressed. In fact, it is analyzed each quaternion component, and it is proposed a solution to this duality problem. The solution presented not provides loss of information in the attitude estimation.*

Keywords: *QUEST algorithm, quaternion, attitude.*

1 INTRODUCTION

The description of an object in space is a important issue in many contexts. For instance, in robotics applications the control of an end-effectuator element is dependent of its position in space, which is generally related to some referential system (Craig, 2005). This description can be translated into the measurement of torques and moments necessary to align the object with respect a tridimensional frame (Hu, 2015) or, in other words, a measurement of the attitude of the object. In the same way, for aerospace vehicles the attitude estimation is essential to implement controls laws for guidance (Oursland, 2010).

A well-know algorithm used to estimate the attitude is the QUEST (QUaternion ESTimate) algorithm (Shuster, 1981). In this algorithm rotations in space is represented by an equivalent angle/axis described by the quaternion form. A recursive form of QUEST algorithm, called REQUEST, was presented by (Bar-Itzhack, 1996). In (Quan, 2013) the REQUEST is combined with Unscented Kalman Filter (Julier, 1997).

The use of quaternion can lead to a lower computational cost, compared to others representations, e.g., Euler angles or attitude matrix. In fact, using quaternions, a complete rotation can be represented by four parameters, whereas for attitude matrix, nine parameters are needed. However, the rotations described by quaternions presents the problem of duality, that is, quaternions with opposite signals can represent the same attitude.

In this work is proposed an method to detect and prevent this duality in the process of attitude estimation. This method is based on the variation of real component of quaternion, combined with a routine of point correction. The proposed method is applied to the output of the QUEST algorithm, which has as input a signal obtained from sensors (accelerometer and magnetometer) collected during $60s$, with a $200\mu s$ sampling time. It is also investigated the performance of the proposed method in data with different Signal-to-Noise Ratios *(SNR)*.

The article is organized as follow. Section 2 presents a brief review of quaternion representation. The QUEST algorithm and the q-method, on which QUEST is based, is detailed in section 3. In section 4 the proposed method for the correction of duality in the QUEST results is presented. The numerical results are discussed in section 5. Finally, in section 6 is pointed out the main remarks of this work.

2 QUATERNIONS

The set of quaternions is represented by $\mathbb{H}$, in tribute to creator, the Irish mathematician William Rowan Hamilton. The quaternions are represented by a real scalar quantity and three orthogonally arranged imaginary numbers that establish the relation between the canonical bases of space $\mathbb{R}^3$:

$$
\begin{aligned}
\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = \mathbf{ijk} &= -1, \\
\mathbf{ij} = \mathbf{k} &= -\mathbf{ji}, \\
\mathbf{jk} = \mathbf{i} &= -\mathbf{kj}, \\
\mathbf{ki} = \mathbf{j} &= -\mathbf{ik}.
\end{aligned}
$$

The quaternion algebra is more complicated in relation with the algebra of the complex numbers, since it has some peculiarities. In fact, the quaternion has other two imaginary components, which become analytical solutions more larger and complex. The quaternions can be represented in rectangular notation as:

$$q = q_0 + \mathbf{i}q_1 + \mathbf{j}q_2 + \mathbf{k}q_3 \equiv q_0 + \mathbf{q}^{\mathrm{T}}.$$

Complex numbers can be used to represent attitudes or describe rotations on a single axis. On the other hand, quaternions can represents a three axis rotational operator. Thus, the quaternions are an alternative form for attitude representation in three-dimensional space. In (Großekatthöfer, 2012) are presented a brief introduction of quaternion algebra, and examples of operations with quaternions to represent the attitude of a rigid body.

The normalized quaternion can be introduced as an angle/axis spatial representation,

$$q = \left\{ cos\left(\frac{\theta}{2}\right) + \vec{k}^{\mathrm{T}} \left[sin\left(\frac{\theta}{2}\right)\right] \right\}, \tag{1}$$

where $\vec{k}$ is the unitary axis of rotate and θ is the angle of rotation on that axis.

Assuming that attitude of an object in time t_0 is $\vec{r}$, and the rotation of θ degrees occurs on the k axis. Then, the quaternion operation sequence to find the attitude in the next time instant, can be represented by:

1. Represent $\vec{r}$ by quaternion $p = \left(0, \vec{r}\right)$;
2. Represent the rotation by quaternion $q = \left[cos\frac{\theta}{2}, \vec{k} \cdot \left(sin\frac{\theta}{2}\right)\right]$;
3. Compute $R_q\left(p\right) = qpq^{-1}$;
4. The attitude in the next instant of time is the imaginary component of R_q.

It is important to note that, if an attitude is represented by a quaternion with unit magnitude, the inverse of this quaternion will be equal to its conjugate (Valenti, 2015). In fact, if q represents the attitude of a vehicle in the reference system $\{B\}$ relative to the reference system $\{A\}$, then $\overline{q}$ represents the attitude of this vehicle in the reference system $\{A\}$ compared to the reference system $\{B\}$, or

$${}^{A}_{B}q \;=\; {}^{B}_{A}\overline{q} \;=\; \left[q_0 \;\; -q_1 \;\; -q_2 \;\; -q_3\right]^{\mathrm{T}}.$$

The inverse of Equation (1), which reports the attitude conversion using the angle/axis representation for the quaternion representation, is given by:

$$\begin{cases} k_x = \frac{q_1}{\sqrt{q_1^2+q_2^2+q_3^2}}, \\ k_y = \frac{q_2}{\sqrt{q_1^2+q_2^2+q_3^2}}, \\ k_z = \frac{q_3}{\sqrt{q_1^2+q_2^2+q_3^2}}, \\ \theta = 2cos^{-1}\left(q_0\right). \end{cases} \tag{2}$$

3 ALGORITHMS OF ATTITUDE

In (Wahba, 1965), the problem of finding the orthogonal matrix A, with positive unit determinant, that minimizes the *Loss Function* is defined by:

$$L(A) = \frac{1}{2}\sum_{i=1}^{N} a_i \|b_i^* - Ab_i\|^2, \tag{3}$$

where b^* are the direction cosines of the observed sensors, b are the direction cosines of the reference sensors, and a are the non-negative weights related to the sensor accuracy through their error variances. In addition, the total variance can by represented by:

$$\left(\sigma_{total}^2\right)^{-1} = \sum_{i=1}^{N} \left(\sigma_i^2\right)^{-1}, \qquad a_i = \frac{\sigma_{total}^2}{\sigma_i^2}, \qquad \sum_{i=1}^{N} a_i = 1.$$

The *Wahba's Problem* has a close relationship with the *Procrustes's Problem*, which is equivalent to finding an orthogonal matrix A that is nearest to B in the sense of *Frobenius's Norm*. The difference remains in the fact that in *Wahba's Problem* the determinant of an attitude matrix A must be $+1$.

From Eq. (3) it is possible to represent the solution of *Wahba's Problem* as the *Gain Function* given by:

$$G(A) = tr\left(AB^{\mathrm{T}}\right). \tag{4}$$

The matrix B is known as attitude profile matrix, and can be expressed by:

$$B = \sum_{i=1}^{\mathrm{N}} a_i b_i^* b_i^{\mathrm{T}}.$$

The *Wahba's Problem* has been subject of research in past decades (Markley, 1988), (Markley, 1993) and (Mortari, 1997). However, the most known work is the QUEST algorithm. In fact, (Markley, 2014) presents the main estimators used to minimize the *Loss Function* (3), and analysis its covariance matrices.

Keat (1977) presented a minimum square estimation of the attitude, from the solution of *Wahba's Problem*. This procedure requires at least a pair of directional sensors attached on the vehicle, and another pair assigned as the referential system. In this work, it was used information obtained from an accelerometer and a magnetometer, that are orthogonally fixed on three Cartesian axis.

Indeed, the objective is still to find the nine elements of attitude matrix A that maximize Eq. (4). In this context, such equation can be rewritten in function of quaternions instead of A. This is the main contribution of the *q-method*. However, is more suitable write the real term in the last position of the quaternion q,

$$q = \begin{bmatrix} \mathbf{q} \\ q_0 \end{bmatrix}. \tag{5}$$

The attitude matrix A, from the representation angle/axis has the form:

$$A = \left[cos^2\left(\frac{\theta}{2}\right) - sin^2\left(\frac{\theta}{2}\right)\right] I + 2sin^2\left(\frac{\theta}{2}\right)\vec{k}\vec{k}^{\mathrm{T}} - 2sin\left(\frac{\theta}{2}\right) cos\left(\frac{\theta}{2}\right)\underset{\sim}{\vec{k}}, \tag{6}$$

where I is an identity matrix of order 3 and $\underset{\sim}{\vec{k}}$ is an skew-symmetric matrix defined by:

$$\underset{\sim}{\vec{k}} = \begin{bmatrix} 0 & -k_z & k_y \\ k_z & 0 & -k_x \\ -k_y & k_x & 0 \end{bmatrix}. \tag{7}$$

Thus, the relation between the equations (2) and (6), using the same formalism of (5), results in the attitude matrix A in terms of the quaternium q takes form:

$$A = \left(q_0^2 - \mathbf{q}^{\mathrm{T}}\mathbf{q}\right) I + 2\mathbf{q}\mathbf{q}^{\mathrm{T}} - 2q_0\underset{\sim}{Q}, \tag{8}$$

where $\underset{\sim}{Q}$ is a skew-symmetric matrix as a function of $\mathbf{q}$, defined from (1) and (7):

$$\underset{\sim}{Q} = sin\left(\frac{\theta}{2}\right)\underset{\sim}{\vec{k}}.$$

From equations (4) and (8), the *Gain Function* of *Wahba's Problem* has only a quaternion variable,

$$G(q) = tr\left[\left(q_0^2 - \mathbf{q}^{\mathrm{T}}\mathbf{q}\right) B^{\mathrm{T}}\right] + 2tr\left(\mathbf{q}\mathbf{q}^{\mathrm{T}}B^{\mathrm{T}}\right) - 2q_0 tr\left(\underset{\sim}{Q}B^{\mathrm{T}}\right). \tag{9}$$

The Eq. (9) can also be given as:

$$G(q) = \left(q_0^2 - \mathbf{q}^{\mathrm{T}}\mathbf{q}\right)\epsilon + \mathbf{q}^{\mathrm{T}}S\mathbf{q} + 2q_0\left(\mathbf{q}^{\mathrm{T}}Z\right), \tag{10}$$

where,

$$\epsilon = tr(B), \qquad S = B + B^{\mathrm{T}}, \qquad Z = \sum_{i=1}^{N} a_i\left(b_i^* \times b_i\right). \tag{11}$$

The Eq. (10) can be expressed in the matrix form, which makes it more suitable to develop an algorithm for attitude estimation:

$$G(q) = \begin{bmatrix}\mathbf{q}^{\mathrm{T}} & q_0\end{bmatrix}\left[\begin{array}{c|c} S - \epsilon I & Z \\ \hline Z^{\mathrm{T}} & \epsilon \end{array}\right]\begin{bmatrix}\mathbf{q} \\ q_0\end{bmatrix} = q^{\mathrm{T}}Kq.$$

As pointed out before, the quaternion used in the function of gain must be unit, which means $q^{\mathrm{T}}q = 1$. This requirement is necessary to determine the maximum eigenvalue of matrix K. From *Method of Lagrange Multipliers* it is possible to write:

$$G_L(q) = q^{\mathrm{T}}Kq - \lambda\left(q^{\mathrm{T}}q - 1\right). \tag{12}$$

Taking the first derivative of Eq. (12) with respect to q^{T}, and set this derivative equal to zero it is possible to obtain the optimal quaternion, or the quaternion that maximize the *Gain Function*:

$$\lambda q = Kq. \tag{13}$$

Indeed, the maximum eigenvector of K, related with the maximum eigenvalue of K, will represents the optimum quaternium, which is used to describe the attitude of the aerospace vehicle.

3.1 QUEST Algorithm

The QUEST algorithm is based on q-method and was first presented by Shuster (1981), with a republication in (Shuster, 2012). The main advantage of this algorithm is the reduction of computation cost to estimate the attitude. Representing the quaternion as a *Gibbs's Vector* (Wertz, 2012), it is possible to write:

$$q = \frac{1}{\sqrt{1 + ||\vec{\mathbf{Y}}||^2}} \begin{bmatrix} \vec{\mathbf{Y}} \\ 1 \end{bmatrix},$$

where $\vec{\mathbf{Y}}$ denotes the Gibbs's vector, which is defined by the ratio between the real and the complex part of q:

$$\vec{\mathbf{Y}} = \frac{\mathbf{q}}{q_0} = \vec{k} \cdot tan\left(\frac{\theta}{2}\right). \tag{14}$$

From (14), the vector $\vec{k}$ and θ denotes, respectively, the rotation axis of quaternion q, and the rotation angle on that axis.

The Eq. (13) in the matrix form is given by:

$$\lambda \begin{bmatrix} \mathbf{q} \\ q_0 \end{bmatrix} = \begin{bmatrix} S - \epsilon I & Z \\ Z^{\mathrm{T}} & \epsilon \end{bmatrix} \begin{bmatrix} \mathbf{q} \\ q_0 \end{bmatrix}. \tag{15}$$

If λ is equal to λ_{max}, then $\vec{\mathbf{Y}}$ and q represents the optimum attitude. From the solution of Eq. (15), the eigenvalue of matrix K can be obtained by:

$$\lambda = \epsilon + Z^{\mathrm{T}} \frac{1}{(\lambda + \epsilon) I - S} Z. \tag{16}$$

Eq. (16) is equivalent to the characteristic equation of the eigenvalues of matrix K. Note that the maximum eigenvalue of K becomes close to unity as the *Gain Function* is maximized. However, there is a singularity in attitude estimation when the θ in Eq. (14) is close to π. In this case, the estimate of the Gibbs vector does not converge. To overcome this problem, the *Cayley-Hamilton Theorem* can be used.

To avoid high value of *Gibbs's Vector*, it is necessary define a new variable ζ given by:

$$\vec{\mathbf{Y}} = \zeta^{-1}\vec{\mathbf{X}}.$$

Thus, the optimum quaternion can be estimated by QUEST algorithm from:

$$q_{opt} = \frac{1}{\sqrt{\zeta^2 + \|\vec{\mathbf{X}}\|^2}} \begin{bmatrix} \vec{\mathbf{X}} \\ \zeta \end{bmatrix},$$

where:

$$\vec{\mathbf{X}} = \left(\kappa_4 I + \kappa_5 S + S^2\right) Z.$$

The *Cayley-Hamilton Theorem* applied to the characteristic equation of the matrix S can be expressed by:

$$S^3 = 2\kappa_1 S^2 - \kappa_2 S + \kappa_3 I, \tag{17}$$

where the variables κ_1, κ_2 and κ_3 can be obtained from matrix S (11):

$$\kappa_1 = \frac{1}{2} tr\,(S) = \epsilon, \qquad \kappa_2 = tr\,[adj\,(S)], \qquad \kappa_3 = det\,(S).$$

From the Eq. (17), a *Meromorphic Function* of S can be expressed by:

$$\left[\left(\lambda + \kappa_1\right) I - S\right]^{-1} = \zeta^{-1}\left(\kappa_4 I + \kappa_5 S + S^2\right). \tag{18}$$

where,

$$\kappa_4 = \lambda_{max}^2 - \kappa_1^2 + \kappa_2, \qquad \kappa_5 = \lambda_{max} - \kappa_1, \qquad \zeta = \left(\lambda_{max} + \kappa_1\right)\kappa_4 - \kappa_3.$$

The Eq. (18) applied to Eq. (16) results in a more convenient expression for the characteristic equation of the eigenvalues of the matrix K. The polynomial of λ with coefficients formed by S and Z is:

$$P(\lambda) = \lambda^4 - \left(\omega_1 + \omega_2\right)\lambda^2 - \omega_3\lambda + \left(\omega_1\omega_2 + \omega_3\kappa_1 - \omega_4\right), \tag{19}$$

where,

$$\omega_1 = \kappa_1^2 - \kappa_2, \qquad \omega_2 = \kappa_1^2 + Z^{\mathrm{T}} Z, \qquad \omega_3 = \kappa_3 + Z^{\mathrm{T}} S Z, \qquad \omega_4 = Z^{\mathrm{T}} S^2 Z.$$

It is known that the maximum eigenvalue of Eq. (19) is close to unity. Thus, its calculation using the *Newton-Raphson Method* for an initial unit value requires few iterations to acquire a high accuracy of the maximum eigenvalue of the K matrix. In this work, it was considered that only two directional sensors are available. In this case, λ_{max} can be written as:

$$\lambda_{max} = \sqrt{a_1^2 + 2a_1 a_2 cos\left(\theta_b - \theta_{b^*}\right) + a_2^2},$$

where,

$$cos\left(\theta_b - \theta_{b^*}\right) = \left(b_1 \bullet b_2\right)\left(b_1^* \bullet b_2^*\right) + \|b_1 \times b_2\|\|b_1^* \times b_2^*\|.$$

It important to note that the QUEST algorithm is less robust than the TRIAD algorithm (Black, 1964), and its other forms (Shuster, 2007) and (Guangjun, 2012). However, the QUEST algorithm remains an important method, due its lower computational cost compared with q-method, mainly in the context when more then two sensors are available.

The QUEST algorithm, with only two sensors, can be summarized by the flowchart:

Start

↓

$$\left(\sigma_{total}^2\right)^{-1} = \sum_{i=1}^{2} \left(\sigma_i^2\right)^{-1},$$
$$a_1 = \frac{\sigma_{total}^2}{\sigma_1^2} \quad e \quad a_2 = \frac{\sigma_{total}^2}{\sigma_2^2}$$

Computation of the weighting vector

↓

$$B = \sum_{i=1}^{2} a_i b_i^* b_i^{\mathrm{T}}$$

Computation of the attitude profile matrix

↓

$$\epsilon = tr\left(B\right) = \frac{1}{2} tr\left(S\right)$$
$$S = B + B^{\mathrm{T}}$$
$$Z = \begin{bmatrix} B_{23} - B_{32} \\ B_{31} - B_{13} \\ B_{12} - B_{21} \end{bmatrix}$$

Computation of the ϵ and matrices S and Z

↓

$$\kappa_1 = tr\left(B\right) = \frac{1}{2} tr\left(S\right)$$
$$\kappa_2 = tr\left[adj\left(S\right)\right]$$
$$\kappa_3 = det\left(S\right)$$

Computation of values κ_1, κ_2 e κ_3

↓

$$cos\left(\theta_b - \theta_{b^*}\right) = \left(b_1 \bullet b_2\right)\left(b_1^* \bullet b_2^*\right) + \|b_1 \times b_2\| \|b_1^* \times b_2^*\|$$
$$\lambda_{max} = \sqrt{a_1^2 + 2 a_1 a_2 cos\left(\theta_b - \theta_{b^*}\right) + a_2^2}$$

Computation of maximum eigenvalue

↓

$$\kappa_4 = \lambda_{max}^2 - \kappa_1^2 + \kappa_2$$
$$\kappa_5 = \lambda_{max} - \kappa_1$$
$$\zeta = \left(\lambda_{max} + \kappa_1\right) \kappa_4 - \kappa_3$$

Computation of values κ_4, κ_5 e ζ

↓

$$\vec{\mathbf{X}} = \left(\kappa_4 I + \kappa_5 S + S^2\right) Z$$

Computation of the *Gibbs's vector*

↓

$$q_{opt} = \frac{1}{\sqrt{\zeta^2 + \|\vec{\mathbf{X}}\|^2}} \begin{bmatrix} \vec{\mathbf{X}} \\ \zeta \end{bmatrix}$$

Computation of the optimal quaternion

↓

End

4 METHODS AND COMPUTATIONAL SIMULATIONS

It is possible that two different quaternions represent the same attitude of the reference system $\{B\}$ relative to the system $\{A\}$. The relationship between these quaternions is:

$$ {}^{A}_{B}q = -{}^{A}_{B}q. \tag{20} $$

In a geometric point of view, the Eq. (20) can represents an attitude with parameters of axis $\vec{k}$ and angle θ, and the same attitude can be represented by the opposite axis and angle $-\theta$. This duality was detect during computational simulation of the QUEST algorithm, and must be considered when using the output QUEST attitude in control systems loops. In fact, in the context of control, the reference signal must be represented by smooth signals, which means signals without abrupt variations. Due the duality property of QUEST algorithm, an attitude estimation could result in points with inversion of angle/axis.

In this work it was used three accelerometers and threes magnetometers orthogonally placed in a tridimensional frame. Figure 1 shows these signals collected during 60 seconds.

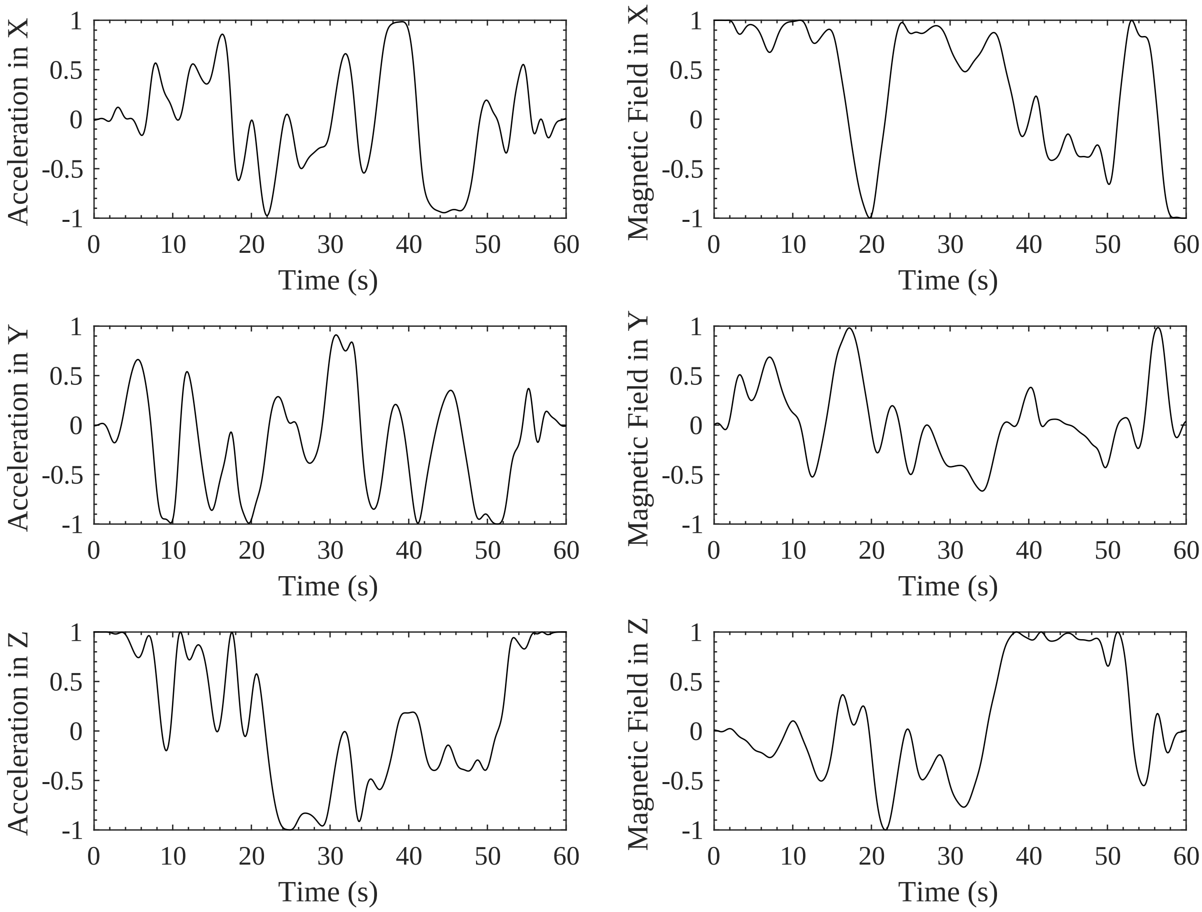

Figure 1: Sensory data of the accelerometer and magnetometer

To verify the duality problem and the variations of quaternions components, the data shown in Fig. 1 were applied to QUEST algorithm. The computational results confirm that the real component ζ of the resulting quaternion is always positive. It means that rotation will always be in anti-clockwise direction along the base axis. The components that describes the attitude is showed in Fig. 2. Notice the occurrence of discontinuity points in signals of components q_1, q_2 e q_3. This occurs due the duality problem in results of QUEST algorithm, as related before.

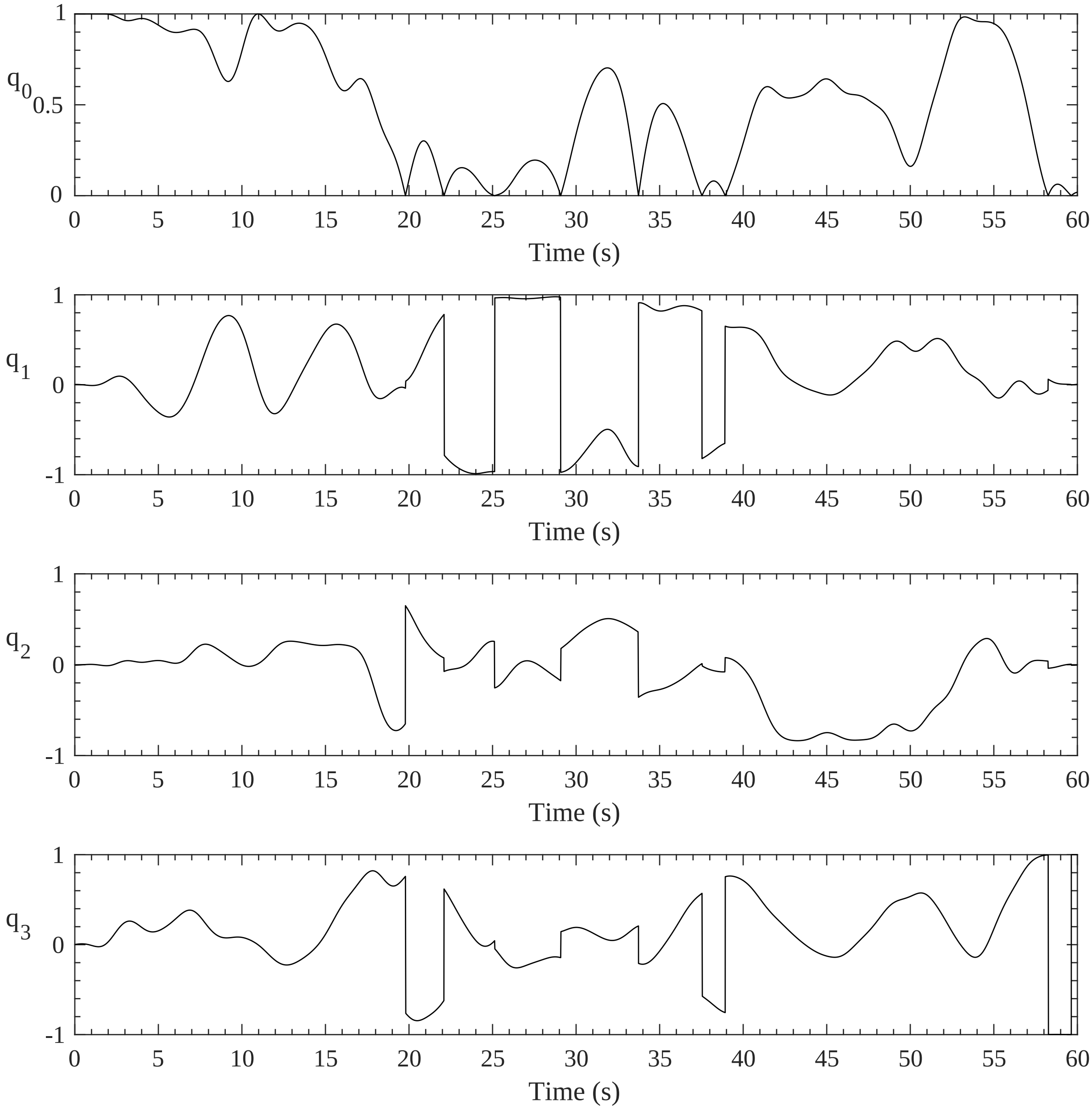

Figure 2: Quaternions components calculated without correction

To avoid discontinuities as illustrated in Fig. 2, firstly was implemented a method to monitor signal of imaginary components of quaternion, aiming detect undesirable signal changing, when real component is zero, $q_0 = 0$. However, this method fails in same cases, specially when real quaternion component is nearly zero, but not zero. Then, signal changes in imaginary components, but this instant of the discontinuity is not detected.

Thus, a second method was implemented, based on the second derivative of q_0 as a function of time, and after that, a filtering process is used. In fact, with this method was possible identify abrupt variation in q_0 and, more important, in which time instants these variations occurs.

The first part of the method aim to find the time instants in which there is discontinuity in the signal of the quaternion components. Thus, it is necessary to define a limit value Δ to compare with the second derivative of q_0. If the absolute value of the second derivative of q_0 is higher than Δ, then the signal of the quaternion components is inverted,

$$\frac{d^2 q_0}{dt^2} > \Delta \tag{21}$$

The time instants in which the Eq. (21) is true represents the instants of the critical points of the signal of the quaternion component q_0.

After the first step, residues were observed on the signals of the imaginary components of q. In the instants of discontinuity and in the following the signal presents characteristics as shown in Fig. 3. The rising edge represents the instant that signal discontinuity occurs and the falling edge the next instant. The points marked with ▼ denotes the necessity of signal inversion.

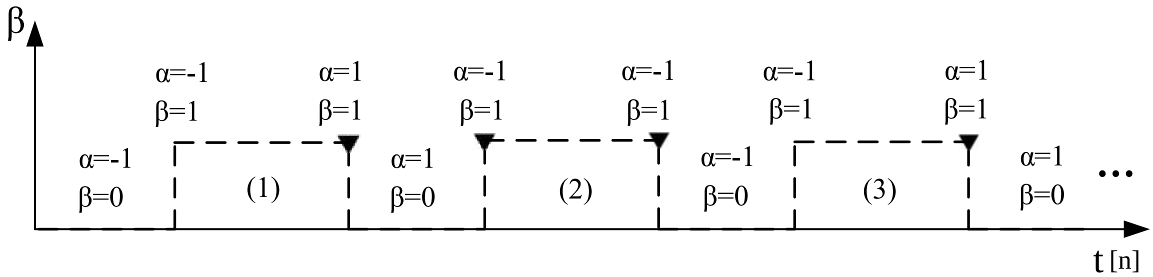

Figure 3: Logic used in the point filter

From the logic observed in Fig. 3, a point filter was created to correct the observed residues. Whenever there is a discontinuity of q in time $t[n]$, verified by Eq. (21), the α signal is inverted from $t[n+1]$ until the next discontinuity, for odd discontinuities. For even discontinuities the value of α is inverted from $t[n]$ until the next discontinuity. Initially α takes -1. A value of 1 for β is given at instants $t[n]$ and $t[n+1]$, from the moment of discontinuity, and 0 at the other instants of time.

The logic for signal inversion, executed by the point filter, can be given by:

$$sign = \left| \frac{\{\alpha\,(t[n]) + \alpha\,(t[n+1])\} \cdot \beta\,(t[n])}{2} \right|. \tag{22}$$

From Eq. (22), it is possibly to write:

$$\Im m\,(q) = \begin{cases} -\Im m\,(q)\,, if \;\; sign = 1, \\ \;\;\Im m\,(q)\,, if \;\; sign = 0. \end{cases}$$

The effectiveness of method proposed is dependent of the comparative limit Δ. In fact, as the real component of quaternion is used to define the rotation angle of the vehicle in space, it is important arbitrate a value higher than maximum angular acceleration estimated to the spacecraft. Notice that, this acceleration must be given as a quaternion function.

5 NUMERICAL RESULTS

The method proposed to duality correction was applied in output data obtained from QUEST algorithm, showed in Fig. 2. The results are illustrated in Fig. 4.

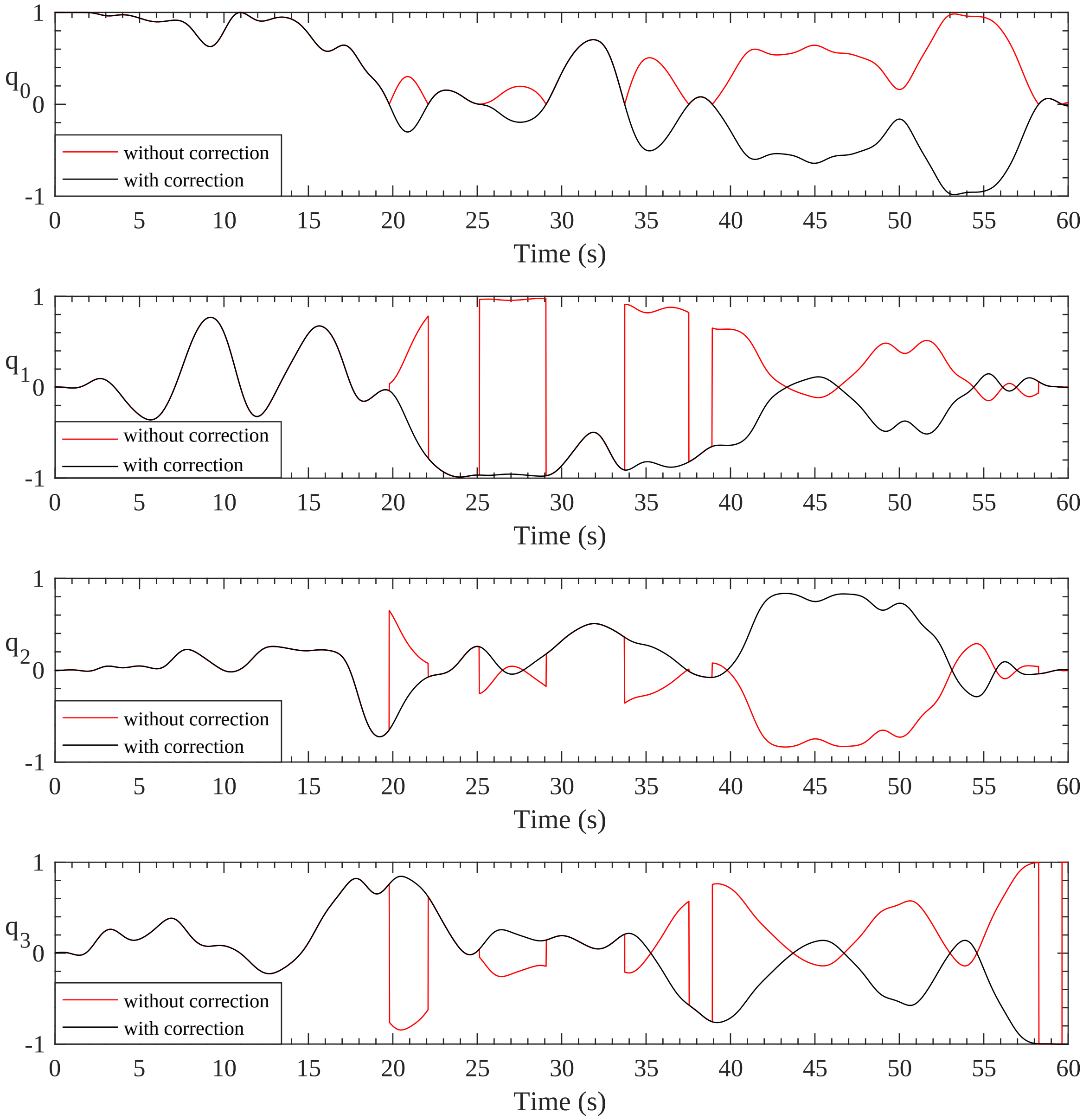

Figure 4: Quaternions components computed before and after the duality correction routine

As can be seen, the duality problem was not detected after correction. The variation of the rotation base axis $\vec{k}$, at time instant $t[n]$, occurs consistently with variation of quaternion components q_1, q_2 e q_3. Which means that there is no change to the octant of the tridimensional frame, used to represents the attitude axis of the vehicle in space, shown in the Fig. 5. In the same way, there is no abrupt variation in the θ angle that rotates the base axis $\vec{k}$.

The method proposed duplicates the domain of the space used to represent the attitude. It can be show by the analysis of quaternion components during the numerical simulation time.

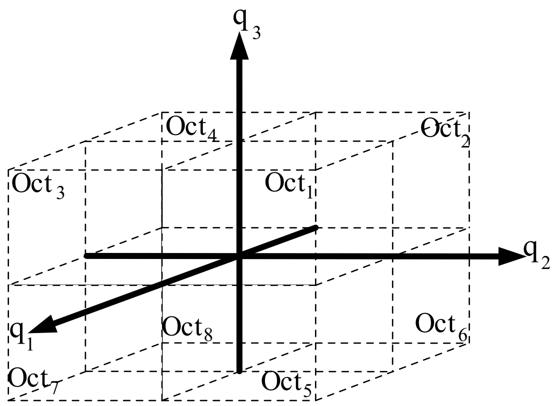

Figure 5: Octants formed by the components q_1, q_2 and q_3 in tridimensional frame

From Eq. (2), the angle of θ is given by the arccosine of the real component of q. Table 1 illustrate the signal of each quaternion component in the octants of Fig. 5. It is important to note that rotation occurs in the counter-clockwise, when using only the QUEST algorithm.

Table 1: Signals values of the quaternions components in each octant of Fig. 5

	Oct_1	Oct_2	Oct_3	Oct_4	Oct_5	Oct_6	Oct_7	Oct_8
q_1	$+$	$-$	$+$	$-$	$+$	$-$	$+$	$-$
q_2	$+$	$+$	$-$	$-$	$+$	$+$	$-$	$-$
q_3	$+$	$+$	$+$	$+$	$-$	$-$	$-$	$-$

From Eq. (2), when the q_0 component is negative the angle θ will also be negative, and the rotation on the base axis $\vec{k}$ occurs in the clockwise direction. Table 2 shows the signal of q_0 and the rotation direction of θ in the octants of Fig. 5 with and without duality correction.

Table 2: Signal values of q_0 and the direction of rotation in each octant of Fig. 5

	Oct_1	Oct_2	Oct_3	Oct_4	Oct_5	Oct_6	Oct_7	Oct_8
q_0 (without correction)	$\underset{+}{\curvearrowleft}$	$\underset{+}{\curvearrowleft}$	$\underset{+}{\curvearrowleft}$	$\underset{+}{\curvearrowleft}$	$\underset{+}{\curvearrowleft}$	$\underset{+}{\curvearrowleft}$	$\underset{+}{\curvearrowleft}$	$\underset{+}{\curvearrowleft}$
q_0 (with correction)	$\underset{+}{\curvearrowleft}/\underset{-}{\curvearrowright}$	$\underset{+}{\curvearrowleft}/\underset{-}{\curvearrowright}$	$\underset{+}{\curvearrowleft}/\underset{-}{\curvearrowright}$	$\underset{+}{\curvearrowleft}/\underset{-}{\curvearrowright}$	$\underset{+}{\curvearrowleft}/\underset{-}{\curvearrowright}$	$\underset{+}{\curvearrowleft}/\underset{-}{\curvearrowright}$	$\underset{+}{\curvearrowleft}/\underset{-}{\curvearrowright}$	$\underset{+}{\curvearrowleft}/\underset{-}{\curvearrowright}$

A Gaussian white noise, with different (*SNR*), was added to the trajectory data and then submitted to QUEST algorithm with the correction method proposed. Fig. 6a shows the quaternion components estimated from data with $SNR = 80\text{dB}$. As can be seen, the estimation presents high imprecision. Only for a $SNR = 160\text{dB}$ (Fig. 6b) is it possible a accurate attitude estimation. Thus, this analysis can indicate high sensitivity of QUEST algorithm, with the duality correction, to data corrupted with noise.

In order to visualize imprecision in Fig. 6a, the Fig. 7 show only the q_0 component from time $16s$ to $16.04s$.

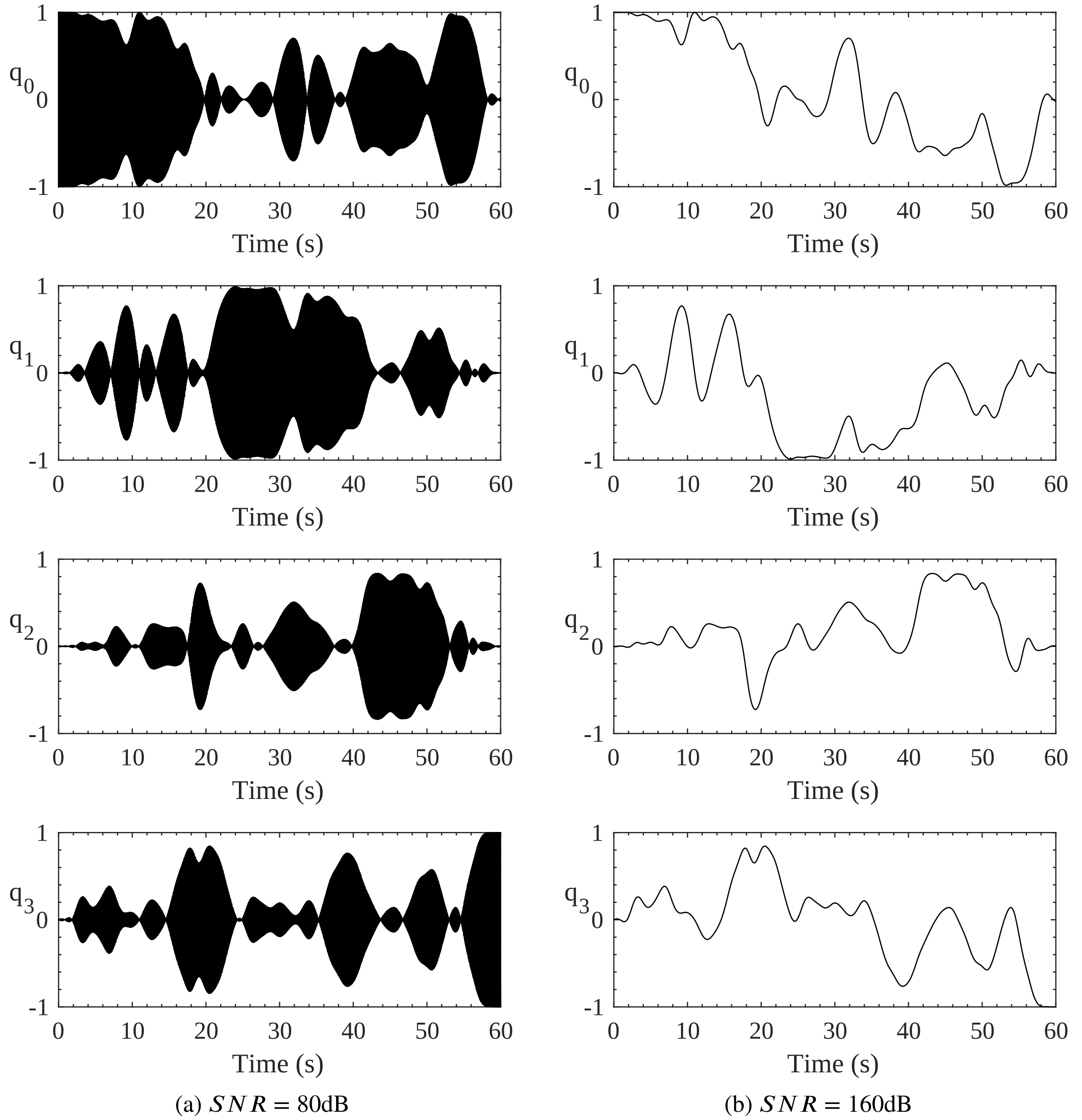

Figure 6: Quaternions components computed with the routine of duality correction from sensory signals contaminated by Gaussian white noise

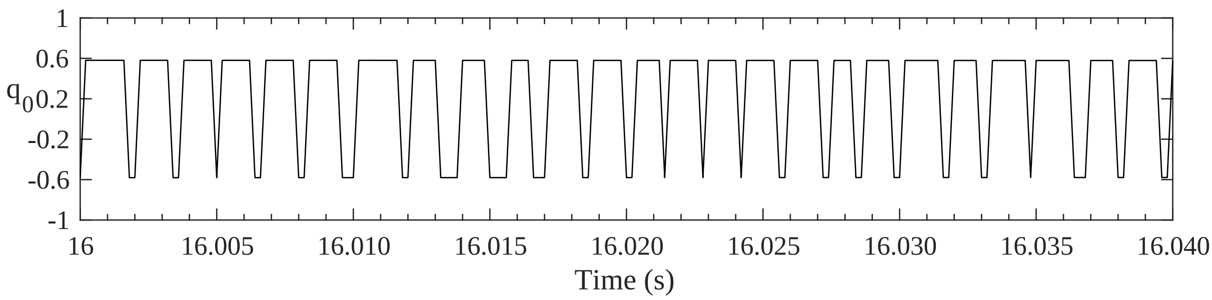

Figure 7: Zoom $1500\times$ **on the representation of quaternion component** q_0 **with signal-to-noise ratio of 80dB**

6 CONCLUSION

In this work was proposed a method to solve the duality problem in the results of QUEST algorithm. This method was applied to estimate the attitude of an vehicle along an trajectory in the space. The analysis of the quaternion components indicates the efficiency and low complexity of the method in solving the duality.

In simplest way, the duality can be monitored by the signal of the q_0 component. Although, numerical problems can introduce an important limitation, for example, the sampling time in controllers/computers, which could lead to errors in the attitude estimation.

The problem of noise in sensory data was also analyzed. The results indicates the limitation of proposed method with the QUEST algorithm to deal with noise contamination.

Finally, the estimation of the optimal quaternion using the QUEST algorithm and the proposed method present a solution of continuous curves of the quaternion components. However, this solution is valid only for deterministic systems.

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