CILAMCE 2017
Florianópolis - Santa Catarina - Brazil

ASSESSMENT OF FINITE ELEMENTS FOR THE ANALYSIS OF MULTILAYERED BEAMS WITH PARTIAL INTERACTION

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Abstract. *Multilayered partially connected beams are present in many structural applications, such as the classic composite steel-concrete beam (2 layers), the bi-steel panels (3 layers), as well as timber or composite beams composed by several layers, glued or connected mechanically. The main characteristic of these elements is the allowance for relative displacements between the layers, which is usually termed as partial interaction, incomplete interaction, partial shear connection or interlayer slip. Finite element solutions for these kinds of problems have been developed for some time and the classic Euler-Bernoulli as well as Timoshenko beam models are the most employed. It is well known that these kinds of beams are prone to slip locking for some interpolation schemes. The purpose of this work is to present the formulation for the numerical linear analysis of* n*-layered composite beams and to perform a comparative study of the quality of the response of some finite elements specifically designed to solve this problem.*

Keywords: *multilayered beam, partial interaction, interlayer slip (up to 5 keywords)*

1 INTRODUCTION

It is common practice in civil engineering to employ members which consist of an assemblage of two or more different elements connected by a deformable interface. Examples are composite steel-concrete beams, where the connection is normally done by shear connectors welded to the steel beam and embedded in the concrete, bi-steel steel-concrete three-member panels, composite timber beams, normally glued or connected by nails.

This interesting problem has been addressed analytically as well as numerically by many researchers. The differential equations for the case of linear elastic composite beams with linear connection stiffness were developed already in the early 50's (Newmark et al., 1951). Commonly named as the Newmark model for composite beams, this basic formulation has been improved and extended by a large number of research works. Analytical improvements include extensions to Timoshenko beam theory (Xu and Wu, 2007; Schnabl et al., 2007; Xu and Wang, 2012), non-linear geometric and buckling effects (Girhammar and Pan, 2007), consideration of multiple layers (Sousa Jr. and da Silva, 2010), occurrence of uplift (Kroflič et al., 2010), among others.

Simultaneously, numerical methods have been employed to solve more general problems of composite beams, mostly by the finite element method. Pure displacement-based (Dall'Asta and Zona, 2002; Gara et al., 2006), force-based and mixed (Dall'Asta and Zona, 2004; Schnabl et al., 2007a) finite elements have been largely used in problems with linear and nonlinear material behaviour.

The main issue is that under some circumstances the displacement-based elements exhibit curvature (or slip) locking (Dall'Asta and Zona, 2004b; Erkmen and Bradford, 2010, 2011; Erkmen and Attard, 2011), under Euler as well as Timoshenko beam theories, and a lot of work has been carried out to overcome these drawbacks, in order to recover the ideal rates of convergence of the FE method. The locking problems to which the conventional FE formulations of composite beams are susceptible stem from the multifield nature of the problem, where axial, transverse displacements and interlayer slip and derivatives contribute to the strain energy (within Timoshenko kinematics, also the independent section rotations). After the field interpolations, non-matching terms on the polynomial expansions give rise to inconsistent expansions. In the context of Euler-beam theory the quadratic axial displacement, combined with cubic transverse displacement, has been shown to produce a locking-free element, at the expense of employing a hierarchical mid-element degree of freedom. Nonetheless, these elements frequently require static condensation of internal nodes, which may be cumbersome especially when extended to nonlinear analysis. However, the linear axial displacement gives rise to curvature locking, which may be alleviated by different strategies such as assumed strain, discrete strain gap or the kinematic interpolatory technique (Erkmen and Bradford, 2011).

Within Timoshenko beam theory the locking appears in the form of shear combined with slip locking. The low-order beam elements which are known to lock when the elements become slender also give rise to spurious distributions of slip.

This paper provides detailed FE formulations for the implementation of multilayered partially connected beam elemens, for the Euler-Bernoulli as well as Timoshenko beam theories. These elements are employed to solve a typical problem and the quality of the elements is assessed by comparison with analytical results.

2 MULTILAYERED BEAM FORMULATION

The development of the governing equations for multilayered beams may be carried out directly by the establishment of the equations of equilibrium and compatibility (i.e. the strain-displacement relations) along with the hypotheses for material behaviour. On the other hand, one may depart from variational formulations such as Principle of Virtual Work or Minimum Potential Energy (as long as elastic behaviour is concerned, which is the case here). Figure 1 shows the notation and the kinematics of a multilayered composite beam with interlayer slips.

Henceforth the assumptions are: (a) plane sections remain plane, with small displacements and rotations; (b) linear elastic material properties, with $\sigma_i = E_i \varepsilon_i$ and $\tau_i = G_i \gamma_i$ for material of layer i; (c) the transverse displacement v and rotation θ are shared by every layer, see discussion and references below; (d) each axis passes through the centroid of the respective component, i.e. the first moment of area is zero eliminating axial and flexure generalized constitutive coupling; (e) the interface force per unit length is proportional to the interlayer slip: $\frac{\mathrm{d}S_i}{\mathrm{d}x} = K_i s_i$.

These assumptions have been widely employed in analytical and numerical formulations for two-layered beams. In the case of shear-flexible beams, some authors have considered unequal rotations at the cross sections whilst considering the same transverse displacement. This would lead to different curvatures in each layer, resulting in a more complex formulation. Martinelli and coworkers (2012) investigated three possibilities of kinematic representation of two-layer composite beams and identified the main parameters that are relevant to each of them.

2.1 Timoshenko beam formulation

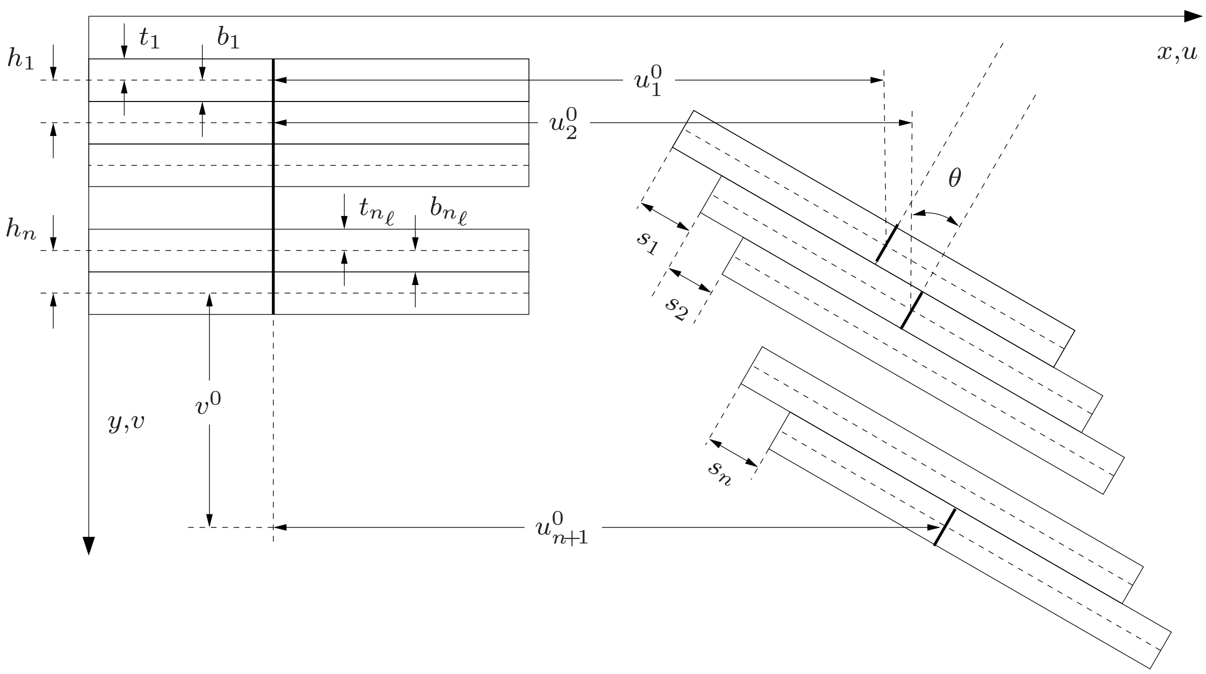

Figure 1: kinematics of multilayered beam with interlayer slip

According to the assumptions the expression for the displacement field is (Figure 1):

$$u_i(x, y) = u_i^0(x) - (y - y_i)\,\theta(x) \tag{1}$$

$$v_i(x, y) = v^0(x) \tag{2}$$

where $i = 1..n{+}1$ represents each beam element, and y_i is the reference axis for each constituent element, usually taken at the centroid. In this text, superscript 0 indicates quantities dependent on x only, measured at $y = 0$. These kinematic assumptions have been employed to derive mostly two-layered composite beams with partial interaction.

The general, two-dimensional displacements are then related to the one-dimensional (dependent only on x) displacements as

$$\begin{Bmatrix} u_1(x,y) \\ u_2(x,y) \\ \vdots \\ u_n(x,y) \\ v(x,y) \end{Bmatrix} = \begin{bmatrix} 1 & & & & -(y-y_1) \\ & 1 & & & -(y-y_2) \\ & & \ddots & & \vdots \\ & & & 1 & -(y-y_n) \\ & & & & 1 \quad 0 \end{bmatrix} \begin{Bmatrix} u_1^0(x) \\ u_2^0(x) \\ \vdots \\ u_n^0(x) \\ v^0(x) \\ \theta(x) \end{Bmatrix} \tag{3}$$

or in matrix form

$$\boldsymbol{u}(x, y) = \boldsymbol{L}(y)\ \boldsymbol{u}^0(x) \tag{4}$$

The slip between layers i and $i{+}1$ is given by

$$s_i(x) = u_{i+1}^0(x) - u_i^0(x) + h_i\theta(x) \tag{5}$$

where h_i is the distance between the reference axes of consecutive layers.

The 2D strains are obtained from the displacements as usual. The normal strains for each layer is given by

$$\varepsilon_{xi} = \frac{\partial u_i}{\partial x} = \frac{\mathrm{d}u_i^0}{\mathrm{d}x} - (y - y_i)\ \frac{\mathrm{d}\theta}{\mathrm{d}x} = \varepsilon_{0i}(x) - (y - y_i)\ \kappa(x) \tag{6}$$

where $\kappa(x) = \frac{\mathrm{d}\theta}{\mathrm{d}x}$ is the curvature. The shear strain is the same for every layer and is given by

$$\gamma_{xy} = \gamma_{xy_i} = \frac{\partial v}{\partial x} + \frac{\partial u_i}{\partial y} = -\theta(x) + \frac{\mathrm{d}v^0(x)}{\mathrm{d}x} \tag{7}$$

At this point the generalized strains, which depend only on x and allow the FE formulation to be one-dimensional, may be introduced. These strains are the normal strains at each reference axis, the curvature, the shear strain and the slip distributions, and they relate to the 2D strains $\boldsymbol{\varepsilon}(x, y)$ by

$$\begin{Bmatrix} \varepsilon_{x1}(x,y) \\ \varepsilon_{x2}(x,y) \\ \vdots \\ \varepsilon_{xn}(x,y) \\ \gamma_{xy}(x,y) \end{Bmatrix} = \begin{bmatrix} 1 & & & & -(y-y_1) & \\ & 1 & & & -(y-y_2) & \\ & & \ddots & & \vdots & \\ & & & 1 & -(y-y_n) & \\ & & & & & 1 \end{bmatrix} \begin{Bmatrix} \varepsilon_{0,1}(x) \\ \varepsilon_{0,2}(x) \\ \vdots \\ \varepsilon_{0,n}(x) \\ \kappa(x) \\ \gamma(x) \end{Bmatrix} \tag{8}$$

or in matrix terms

$$\boldsymbol{\varepsilon}(x,y) = \boldsymbol{M}(y)\ \boldsymbol{\varepsilon}^0(x) \tag{9}$$

The generalized strain-displacement relation, to be further inserted into the virtual work expression, is

$$\left\{\begin{array}{c} \varepsilon_{0,1} \\ \varepsilon_{0,2} \\ \vdots \\ \varepsilon_{0,n} \\ \kappa \\ \gamma \\ s_1 \\ s_2 \\ \vdots \\ s_{n-1} \end{array}\right\} = \begin{bmatrix} \frac{\mathrm{d}}{\mathrm{d}x} & & & & & \\ & \frac{\mathrm{d}}{\mathrm{d}x} & & & & \\ & & \ddots & & & \\ & & & \frac{\mathrm{d}}{\mathrm{d}x} & & \\ & & & & & \frac{\mathrm{d}}{\mathrm{d}x} \\ & & & & \frac{\mathrm{d}}{\mathrm{d}x} & -1 \\ -1 & 1 & & & & h_1 \\ & -1 & 1 & & & h_2 \\ & & \ddots & & & \vdots \\ & & -1 & 1 & & h_{n-1} \end{bmatrix} \left\{\begin{array}{c} u_1^0 \\ u_2^0 \\ \vdots \\ u_n^0 \\ v^0 \\ \theta \end{array}\right\} \tag{10}$$

or in matrix notation

$$\boldsymbol{\varepsilon}^0(x) = \boldsymbol{\partial}\ \boldsymbol{u}^0(x) \tag{11}$$

The internal work includes the work of the axial stress components (for each layer), the shear stress and the slip forces. The work of the normal and shear stresses must be integrated along the volume of the beam while the slip force contribution may be evaluated by integration along each of the interlayer connections.

$$\delta W_i = \int_V (\sum_i \delta\varepsilon_{xi}\ \sigma_{xi} + \delta\gamma_{xy}\ \tau_{xy})\ \mathrm{d}V + \int_L (\sum_i \delta s_i F_i(s_i))\ \mathrm{d}x \tag{12}$$

The typical term which involves the normal stress is expanded as follows for each layer

$$\int_V \delta\varepsilon_{xi}\ \sigma_{xi}\mathrm{d}V = \int_L \int_A (\delta\varepsilon_{0,i} - (y - y_i)\delta\kappa)\ \sigma_i\ \mathrm{d}A\mathrm{d}x \tag{13}$$

Defining the normal force in a layer and the bending moment as

$$N_i(x) = \int_A \sigma_{xi}\ \mathrm{d}A \tag{14}$$

$$M(x) = \sum M_i(x) = \sum_{i=1}^{n} \left[\int_A \sigma_{xi}(y - y_i)\ \mathrm{d}A\right] \tag{15}$$

the virtual work term on Eq. (13) reads

$$\int_V \delta\varepsilon_{xi}\,\sigma_{xi}\,\mathrm{d}V = \int_0^L \left[\delta\varepsilon_{0i}\,N_i(x) + \delta\kappa\,M(x)\right]\,\mathrm{d}x \tag{16}$$

Integration over the cross section allows the internal work to be expressed in terms of generalized strains and corresponding forces

$$\delta W_i = \int_0^L \left[\sum_i \delta\varepsilon_{0i}\,N_i(x) + \chi\delta\gamma\,Q(x) + \delta\kappa\,M(x) + \sum_i \delta s_i\,F_i(s_i)\right]\,\mathrm{d}x \tag{17}$$

or in compact matrix notation as

$$\delta W_i = \int_0^L \delta\boldsymbol{\varepsilon}^{0T}\boldsymbol{\sigma}^0\,\mathrm{d}x \tag{18}$$

where the shear correction factor χ was applied as is usual on shear flexible beam analysis. This factor aims to correct the wrong consideration of constant shear strain along the section thickness. The generalized stresses are collected into the vector $\boldsymbol{\sigma^0}$

$$\boldsymbol{\sigma}^0 = \left\{\begin{matrix} N_1(x) & \dots & N_n(x) & Q(x) & M(x) & F_1(x) & \dots & F_{n-1}(x)\end{matrix}\right\}^T \tag{19}$$

The external work is given by the contributions of the transverse forces $q(x)$, and the axial forces $n_i(x)$ along the element as well as end forces contributions which will contribute to the boundary terms.

Assuming linear elastic material properties the following relations are obtained

$$N_i(x) = \int_x\int_A E_i\left[\varepsilon_{0i} + (y-y_i)\kappa\right]\,\mathrm{d}A = E_iA_i\,\varepsilon_{0i} \tag{20}$$

$$M_i(x) = \int_A \left[\varepsilon_{0i} + (y-y_i)\kappa\right]_i (y-y_i)\,\mathrm{d}A = E_iI_i\,\kappa \tag{21}$$

and the constitutive relation between generalized stresses and strains is obtained:

$$\boldsymbol{\sigma}^0 = \boldsymbol{C}\boldsymbol{\varepsilon}^0 \tag{22}$$

where C is a diagonal matrix

$$\boldsymbol{C} = \mathrm{diag}(E_1A_1,\dots,E_nA_n,\sum E_iI_i,\sum G_iA_i,F_1,\dots,F_{n-1}) \tag{23}$$

FE interpolation schemes

In order to develop a FE formulation, the crucial step is the definition of a interpolation scheme which comprises the selection of the degrees of freedom and the related polynomial functions. Timoshenko beams require only C^0 continuity between elements, which allows one to employ simple and independent lagrangian interpolations for the primary fields u^0, v^0 and θ.

Let n_u, n_v and n_θ be the number of parameters in each of the primary independent fields, then the FE displacement interpolation is given by

$$u_k^0(x) = \sum_{i=1}^{n_u} N_{u,i}(x) u_{k,i}^0 \tag{24}$$

$$v^0(x) = \sum_{1}^{n_v} N_{v,i}(x) v_i^0 \tag{25}$$

$$\theta(x) = \sum_{1}^{n_\theta} N_{\theta,i}(x) \theta_i \tag{26}$$

$$\boldsymbol{u}^0(x) = \boldsymbol{N}(x)\, \boldsymbol{d} \tag{27}$$

$$\boldsymbol{N}(x) = \mathrm{blockdiag}(\boldsymbol{N}_u, ..., \boldsymbol{N}_u, \boldsymbol{N}_v, \boldsymbol{N}_\theta) \tag{28}$$

It should be noted that the present implementation permits any combination of parameters n_u, n_v and n_θ, i.e. the order of polynomials is independent and is a choice of the analyst in order to obtain the best element numerical behaviour. Furthermore, integration using Gauss quadrature is performed separately for each strain term, allowing the use of full, reduced or selective integration. The aspect of the finite element is depicted in Figure 2.

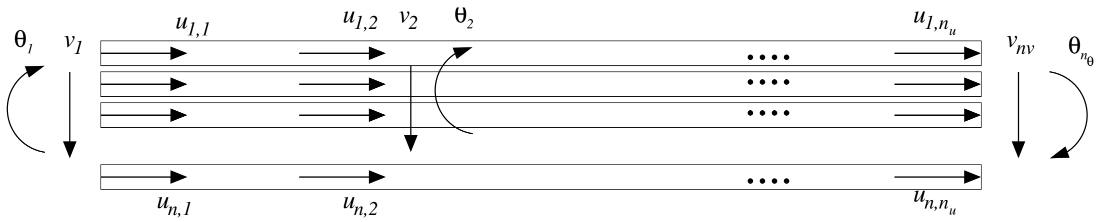

Figure 2: degrees of freedom for interpolation

Due to the possibility of different number of interpolation points, the concept of node is associated with the element endpoints where there will always be all the possible degrees of freedom. The element dof's may be collected in the following vector:

$$\boldsymbol{d}^T = \left\{ \begin{array}{cccccccccccc} u_{1,1}^0 & \dots & u_{1,n_u}^0 & \dots & u_{n,1}^0 & \dots & u_{n,n_u}^0 & v_1 & \dots & v_{n_v} & \theta_1 & \dots & \theta_{n\theta} \end{array} \right\} \tag{29}$$

Taking into account the previous expressions it is possible to define the strain-displacement matrix for the finite element as

$$\boldsymbol{\varepsilon}^0(x) = \boldsymbol{\partial}\, \boldsymbol{u}^0(x) = \boldsymbol{\partial}\, \boldsymbol{N}(x)\, \boldsymbol{d} = \boldsymbol{B}(x)\, \boldsymbol{d} \tag{30}$$

which upon substitution in the virtual work expression leads to

$$\delta W_i = \int_0^L \delta \boldsymbol{\varepsilon}^{0T} \boldsymbol{\sigma}^0 \mathrm{d}x = \boldsymbol{d}^T \int_0^L \boldsymbol{B}^T(x)\, \boldsymbol{\sigma}^0(x)\, \mathrm{d}x = \boldsymbol{d}^T\, \boldsymbol{g} \tag{31}$$

from which one ay identify the internal force

$$\boldsymbol{g} = \int_0^L \boldsymbol{B}^T(x)\, \boldsymbol{\sigma}^0(x)\, \mathrm{d}x \tag{32}$$

and due to the linearity of the problem, the stiffness matrix $\boldsymbol{k}$:

$$\boldsymbol{g} = \int_0^L \boldsymbol{B}^T \boldsymbol{C} \boldsymbol{B} \boldsymbol{d}\, \mathrm{d}x = \left[\int_0^L \boldsymbol{B}^T \boldsymbol{C} \boldsymbol{B}\, \mathrm{d}x \right] \boldsymbol{d} = \boldsymbol{k}\, \boldsymbol{d} \tag{33}$$

Numerical aspects

It was established already in the early days of the FE Method that Shear-flexible (Timoshenko) beams are prone to shear locking when used to model slender beams, and this behaviour was readily identified as recurrent on Reissner-Mindlin Plates and Shells. Spurious energy terms arise in the formulation resulting in stiff solutions which has to be dealt with. The two-layered Newmark model or the multilayered model, translated to the Timoshenko beam theory, are not exception to this drawback, and it remains to see if a locking behaviour of other nature, characteristic of partially connected beams, also shows up.

The strain-displacement matrix may be decomposed into its various strain related components (transposition made to save space in the text)

$$\boldsymbol{B}^T = \left[\begin{array}{cccc} \boldsymbol{B}_\varepsilon^T & \boldsymbol{B}_\kappa^T & \boldsymbol{B}_\gamma^T & \boldsymbol{B}_s^T \end{array} \right] \tag{34}$$

where the matrices $\boldsymbol{B}_\varepsilon$, $\boldsymbol{B}_\kappa$, $\boldsymbol{B}_\gamma$ and $\boldsymbol{B}_s$ are respectively the membrane, bending, shear and slip strain-displacement relations. The stiffness matrix may be decomposed so that

$$\boldsymbol{k} = \int_0^L \left[\boldsymbol{B}_\varepsilon^T \boldsymbol{C}_\varepsilon \boldsymbol{B}_\varepsilon^T \,\mathrm{d}x + \boldsymbol{B}_\gamma^T \boldsymbol{C}_\gamma \boldsymbol{B}_\gamma^T \,\mathrm{d}x + \boldsymbol{B}_\kappa^T \boldsymbol{C}_\kappa \boldsymbol{B}_\kappa^T \,\mathrm{d}x + \boldsymbol{B}_s^T \boldsymbol{C}_s \boldsymbol{B}_s^T\right] \,\mathrm{d}x \tag{35}$$

This is the basis for a selective integration scheme, which means employing different number of integration points for the strain-related matrices. Usually, to avoid shear locking the shear part of the stiffness is underintegrated while the other terms use the full integration scheme.

As the FE formulation developed here is a multifield one, different requirements for the numerical integration may be needed. The simplest formulation uses the linear interpolation for every field, and requires two gauss points in the full integration context. As this element is particularly prone to shear locking, a selective integration with only one point for the shear part may improve the results, maybe at the expense of possible mechanism formations.

Let a row vector of a generic interpolation function with n_a interpolation points be given as

$$\boldsymbol{N}_a = \left[\begin{array}{ccc} N_{a,1} & \dots & N_{a,n_a} \end{array} \right] \tag{36}$$

and its derivative expressed as

$$\boldsymbol{N}_{a,x} = \left[\begin{array}{ccc} N_{a,1_{,x}} & \dots & N_{a,n_a,x} \end{array} \right] \tag{37}$$

The strain-displacement matrix for the membrane strain contribution is given by the following expression:

$$\boldsymbol{B}_\varepsilon = \left[\begin{array}{cccc} \boldsymbol{N}_{u,x} & & & \boldsymbol{0}_{1\times(n_v+n_\theta)} \\ & \ddots & & \boldsymbol{0}_{1\times(n_v+n_\theta)} \\ & & \boldsymbol{N}_{u,x} & \boldsymbol{0}_{1\times(n_v+n_\theta)} \end{array} \right] \tag{38}$$

while the flexure-related strain-displacement matrix is

$$\boldsymbol{B}_\kappa = \left[\begin{array}{cccc} \boldsymbol{0}_{1\times n.n_u+n_v} & N_{\theta,1_{,x}} & \dots & N_{\theta,n_\theta,x} \end{array} \right] \tag{39}$$

and the transverse shear counterpart reads

$$\boldsymbol{B}_\gamma = \begin{bmatrix} \mathbf{0}_{1\times n.n_u} & N_{v,1,x} & \dots & N_{v,n_v,x} & N_{\theta,1} & \dots & N_{\theta,n_\theta} \end{bmatrix} \tag{40}$$

and, finally, the slip-related strain displacement operator is given by

$$\boldsymbol{B}_s = \begin{bmatrix} -\boldsymbol{N}_{u,x} & \boldsymbol{N}_{u,x} & & & & \dots & h_1\boldsymbol{N}_\theta \\ & -\boldsymbol{N}_{u,x} & \boldsymbol{N}_{u,x} & & & \dots & h_2\boldsymbol{N}_\theta \\ & & \ddots & \ddots & & & \vdots \\ & & & -\boldsymbol{N}_{u,x} & \boldsymbol{N}_{u,x} & \dots & h_n\boldsymbol{N}_\theta \end{bmatrix} \tag{41}$$

With respect to the numerical integration, in order to reproduce the analyical results (full numerical integration), the minimum number of gauss integration points is a function of the degree of the correspondent interpolation order and the order of derivation in each interpolation function.

2.2 Euler-Bernoulli element formulation

Departing form the shear-flexible theory, the Euler-Bernoulli or classical beam theory introduces a restriction on the rotation field θ, namely, after deformation the cross section remains normal to the deformed axis (the curve $v^0(x)$). Along with the small strain-small displacement assumption, the result is that $\theta(x) = \mathrm{d}v^0(x)/\mathrm{d}x$ and the strain field is simpler, with $\gamma_{xy} = 0$, i.e. no shear strains or stresses develop.

$$\varepsilon_{xi} = \frac{\partial u_i}{\partial x} = \frac{\mathrm{d}u_i^0}{\mathrm{d}x} - (y - y_i)\,\frac{\mathrm{d}^2 v^0}{\mathrm{d}x^2} = \varepsilon_{0i}(x) - (y - y_i)\,\kappa(x) \tag{42}$$

where a different expression for κ appears.

The analytical formulation proceeds in a similar way to the Timoshenko elements of the previous sections.

The generalized strain-displacement relation is now

$$\begin{Bmatrix} \varepsilon_{0,1} \\ \varepsilon_{0,2} \\ \vdots \\ \varepsilon_{0,n} \\ \kappa \\ s_1 \\ s_2 \\ \vdots \\ s_{n-1} \end{Bmatrix} = \begin{bmatrix} \frac{\mathrm{d}}{\mathrm{d}x} & & & & \\ & \frac{\mathrm{d}}{\mathrm{d}x} & & & \\ & & \ddots & & \\ & & & \frac{\mathrm{d}}{\mathrm{d}x} & \\ & & & & \frac{\mathrm{d}^2}{\mathrm{d}x^2} \\ -1 & 1 & & & h_1\frac{\mathrm{d}}{\mathrm{d}x} \\ & -1 & 1 & & h_2\frac{\mathrm{d}}{\mathrm{d}x} \\ & & \ddots & & \vdots \\ & & -1 & 1 & h_{n-1}\frac{\mathrm{d}}{\mathrm{d}x} \end{bmatrix} \begin{Bmatrix} u_1^0 \\ u_2^0 \\ \vdots \\ u_n^0 \\ v^0 \end{Bmatrix} \tag{43}$$

or, with the same notation as for the Timoshenko formulation

$$\boldsymbol{\varepsilon}^0(x) = \boldsymbol{\partial}\,\boldsymbol{u}^0(x) \tag{44}$$

The EB beam element requires that the displacement and its first derivative are interelement continuous, therefore for the transverse displacement interpolation the hermitian cubic interpolation is the simplest option.

$$v^0(x) = H_1(x)\, v_1^0 + H_2(x)\, \theta_1 + H_3(x)\, v_2^0 + H_4(x)\, \theta_2 \tag{45}$$

The virtual work expression from Equation (13) without the shear strain contribution is employed in the development of the analytical model, resulting in similar expressions in terms of generalized forces, but without the shear force contribution $Q(x)$.

Interpolation schemes

For the Euler-Bernoulli composite beams, it has already been established for the conventional two-layer scheme that the displacement-based formulations are able to achieve fast convergence to the exact results if the interpolation for the axial degree of freedom is at least quadratic.

These elements obviously do not present the shear locking problem but may be prone to the slip locking issue, which shows up for high levels of connection stiffness causing a spurious distribution of slip.

In view of these observations in this work the implementation of the Euler-Bernoulli elements will employ cubic interpolation for the transverse displacement $v^0(x)$ and variable possibilities of interpolation for the axial displacements $u_i^0(x)$.

3 APPLICATIONS

The assessment of the numerical results for the displacement-based element formulations may be made by comparison with exact analytical results. These results may be obtained using the results of FE analyses using special finite elements derived from analytical solutions (Sousa Jr., 2013).

3.1 Three-layer partial composite beam

A two-span continuous three-layer composite beam was analyzed by Keo et al. (2016) to assess the results of an exact formulation for multilayered shear-flexible composite beams. The same example will be employed here to compare the results of the displacement-based formulations and provide an evaluation of the response of the finite elements. The geometry of the beam is shown in Fig. 9 where H is the total height of the section. The beam section is composed of two equal layers with 10 cm wide and 20 cm thick attached to the core of the same width and a depth of 40cm. The beam has two equal spans L and is subjected to a uniformly distributed load of 10 kN/m. The same elastic modulus 8000MPa, Poisson coefficient 0.3 and shear correction factor 5/6. is employed for each layer. The shear connection stiffness(K) for each slipping plane is considered to have the same value, and the numerical analysis is carried out for different values of shear connection..

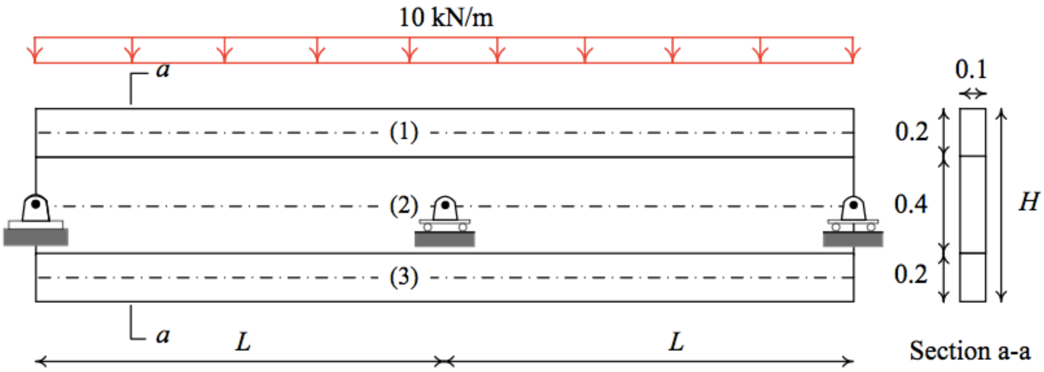

Figure 3: Three-layered beam

The beam is analyzed by the aforementioned finite elements under different interpolation schemes, span/depth ratios, and beam theories. The exact solution for the comparisons is obtained by employing the finite element exact solution through formulations based on the analytical expressions (Sousa Jr., 2013). Full integration of the FE matrices was employed in the numerical solution.

The graphs show the displacement at midspan (symmetric) normalized by the exact results. Figures 4 and 5 show the results for linear and quadratic Timoshenko beam elements respectively. These elements are clearly prone to shear locking.

Figure 6 displays the results for the Euler beam element with quadratic interpolation for the axial displacements and cubic for the transverse displacements. It is noticeable the very good quality of the solution for the Euler elements already with coarse meshes.

In terms of convergence of the displacement no clear tendency is observed with respect to the variation of the connection stiffness. It is known, however, that high levels of connection stiffness may lead to erroneous slip distributions (slip locking), but that does not seem to affect the displacement solution too much.

4 CONCLUSIONS

Multilayered beams have important practical applications in structural engineering and their numerical analysis deserves attention by researchers and engineers. This paper showed the implementation and test of a finite element model for the analysis of multilayered composite beams under shear-flexible (Timoshenko) beam assumptions. The Timoshenko beam elements clearly present locking for coarse meshes, and the Euler beam elements provide very good results already with poor discretizations.

ACKNOWLEDGEMENTS

The author wishes to acknowledge the financial support from Brazilian Agency CNPq (under grants 461810/2014-0 and 307643/2016-6).

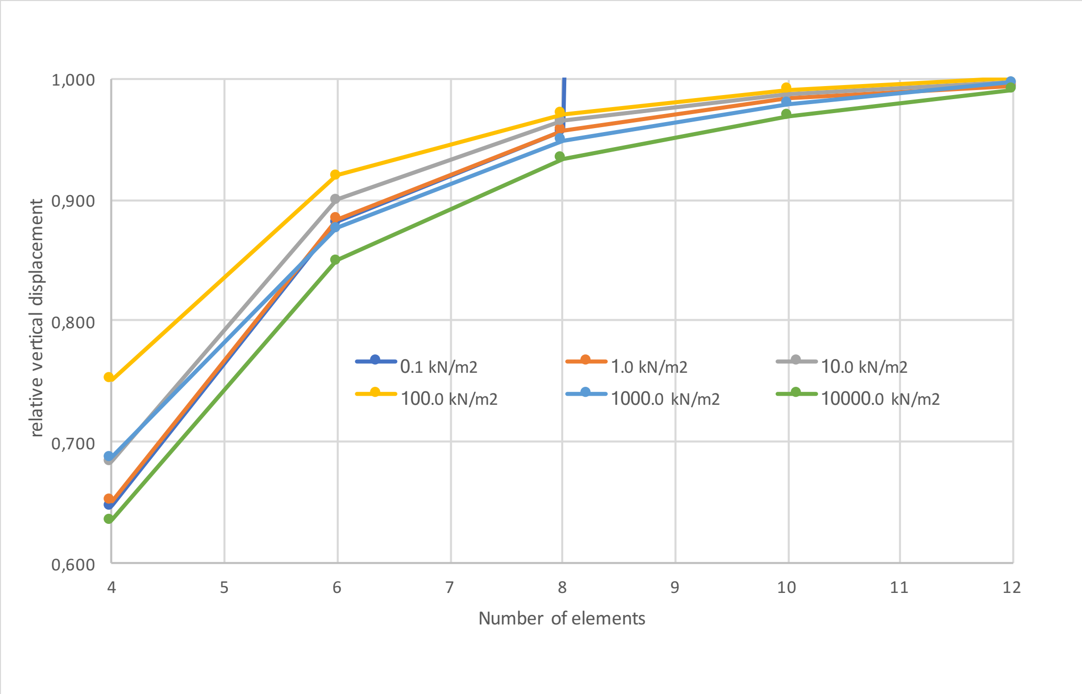

Figure 4: results for Timoshenko linear element for L/H=5

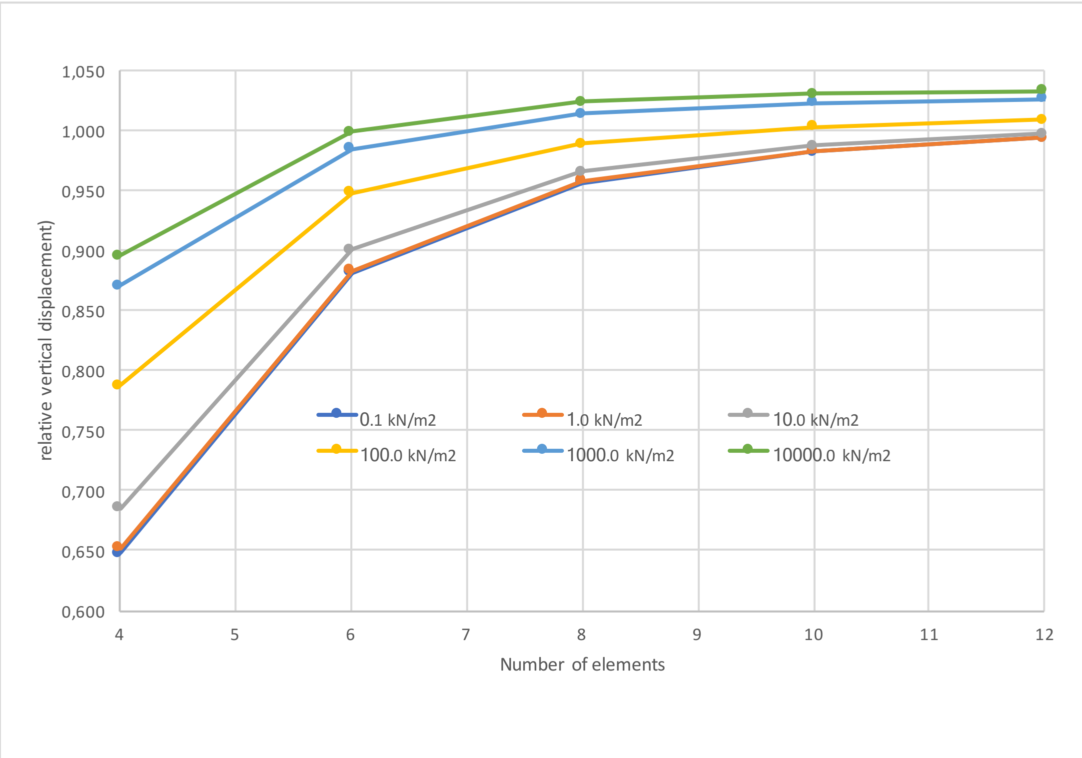

Figure 5: results for Timoshenko quadratic element for L/H=5

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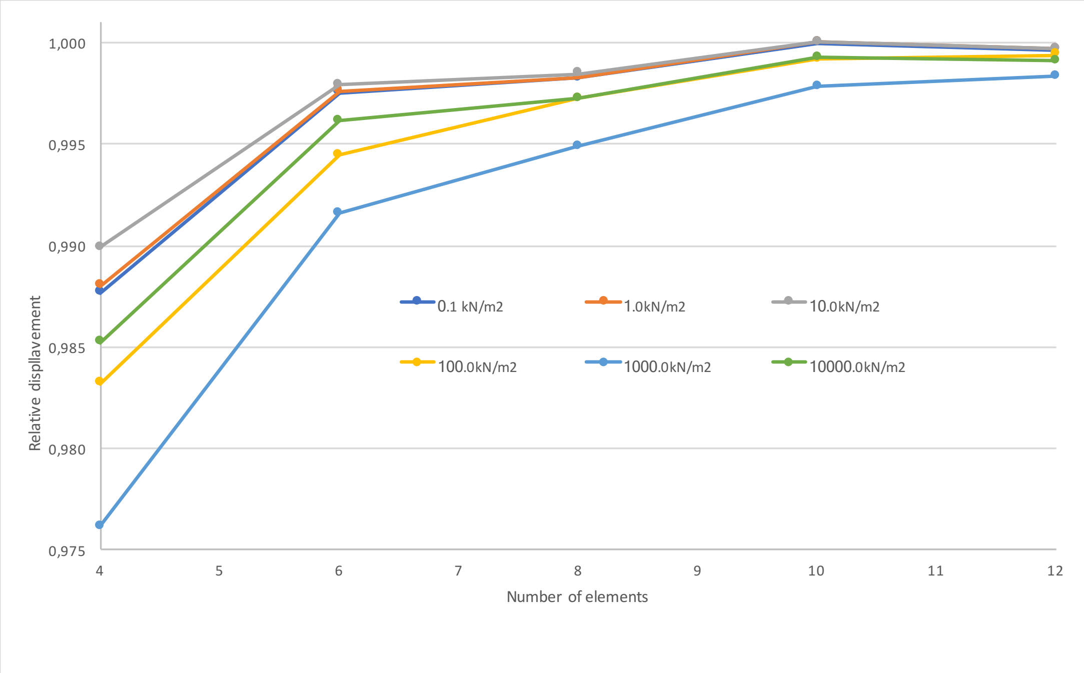

Figure 6: results for Euler quadratic/cubic element for L/H=5

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