TRUSS BEHAVIOR CONSIDERING GEOMETRIC NONLINEARITY AND ELASTOPLASTIC CONSTITUTIVE RELATION WITH DAMAGE.

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Abstract. *Trusses are structures widely used in civil construction mainly because it is a very light solution compared to other types of structures. On the other hand, its behavior when taken into account the geometric and / or material nonlinearities may present complex phenomena. It is in this context that this paper proposes to make a nonlinear analysis of the behavior of a Mises truss using the Finite Element Method. In the study, geometric and physical nonlinearities were considered, observing their behavior through their equilibrium trajectory. Geometric non-linearity was considered using Green-Lagrange deformations and the equilibrium in the deformed configuration, using a Total Lagrangian formulation. Physical nonlinearities were characterized by elastoplastic behavior with material damage. The method used to solve the system of non-linear equilibrium equations was the Newton-Raphson method using different types of control, such as load control, displacement control and arc length control. A parametric analysis of the structure identifies the most appropriate controls for each type of dominant nonlinearity. It was verified that the effect that manifests first along the equilibrium trajectory, the plastic hardening or the damage, will influence in the classification of the structure as to its ductile or fragile behavior, and consequently in the choice of the control that allows to reproduce the equilibrium trajectory.*

Keywords: *Finite Elements, Plasticity, Continuous Damage Mechanics, Geometric Nonlinearity.*

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INTRODUCTION

The study of trusses is an important topic in structural engineering, since this kind of structure are characterized by a strong nonlinear behavior. Turner et al (1960) and Gallagher (1962) were one of the first studies about geometric nonlinearities and physic nonlinearities, respectively, using the Finite Element Method. Leite (2000), Primeiro and Silveira (2004) and Lacerda (2014), made important studies about the nonlinear solution of 2D and 3D trusses.

If one chooses to analyze a nonlinear structural problem numerically it is necessary to use solution methods for the solution of nonlinear systems of equations. Crisfield (1991, 1997) addressed this issue in two of his books. Forde e Stiemer (1987), Shi (1994), Yang and Kuo (1994), Krenk e Hededal (1995), Oñate e Matias (1996) studied solutions for nonlinear system too. In this paper, the Newton-Raphson's Method was used to solve system of equations, with control by force, displacement and arc-length increments. A comparison of the convergence of the obtained results was made according to the type of control applied.

Two possible types of nonlinearity in truss analysis are geometric and physical nonlinearities. The geometric nonlinearity was applied by considering the Green-Lagrange strains and a total Lagrangean finite element formulation. The physicals nonlinearities were approached by considering an elastoplastic constitutive model with damage.

1 GEOMETRIC NONLINEARITY

Structures of elastic materials are said to be linear when they undergo minor deformations and engineering strain measures can be used: $\varepsilon = \frac{l-L}{L}$, where, L is the initial length and l the final length. In this case, equilibrium can be stablished considering the undeformed configuration.

However, some structures, such the Mises's Truss, Fig. 1, has by nature large displacements, which make it more appropriate to consider the deformed configuration for equilibrium and to use, for instance, Green-Lagrange strains, $\varepsilon = \frac{1}{2} \cdot \frac{l^2 - L^2}{L^2}$.

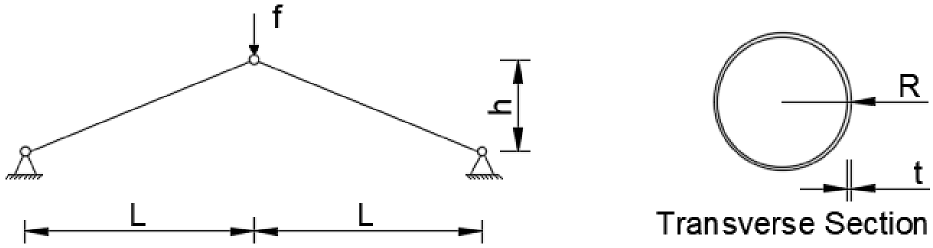

Figure 1. Mises's Truss.

Considering that the truss consists of an elastic material, with its characteristic in Table 1, the equilibrium path was determined in relation to the central node, Fig. 2, using the three increment control methods.

Table 1. Characteristics of the elastic truss.

Description	value
L	4 m
h	1 m
R	15 cm
t	8 mm
Elastic modulus	200GPa

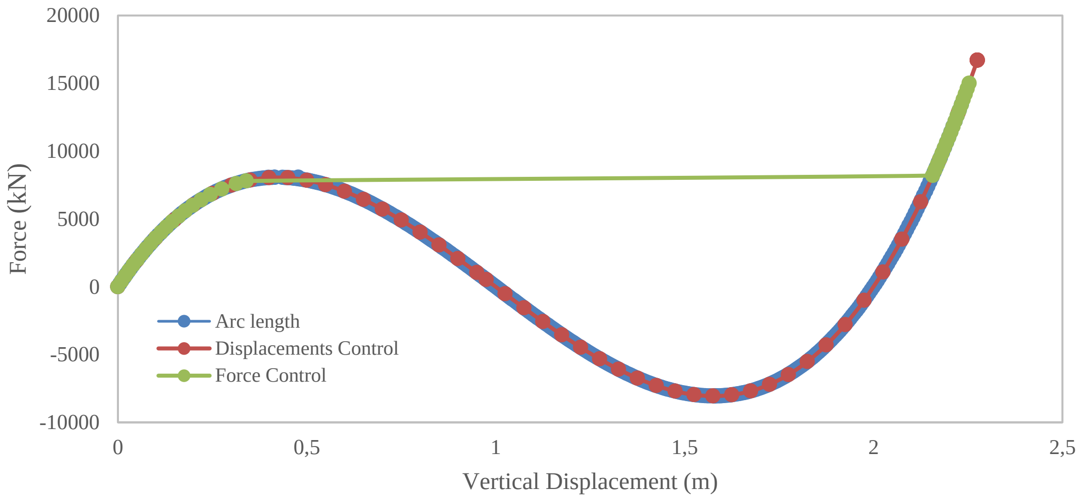

Figure 2. Equilibrium Path of an elastic Mises's Truss.

The elastic characteristics of this material are similar to a structural steel and the structure has a local buckling load of 9089 kN, so this phenomenon doesn't happen in this analysis. Observing the equilibrium path for the different controls, it is verified that the structure initially gains stiffness until it reaches a certain configuration and then it begins to lose stiffness, followed by a gain in stiffness again, this phenomenon of instability is called Snap -through. It is also observed that the load control cannot capture the trajectory of this phenomenon of instability while the others succeed, being, therefore, a deficiency in this method of control.

2 PHYSICAL NONLINEARITY

In this paper, the physical nonlinearity can be given by a constitutive relation that uses the plasticity theory, the damage mechanics theory or both simultaneously. For its computational implementation it is necessary to use integration algorithms for the plastic/damage variables.

2.1 Plasticity

Simo and Hughes (1998) addressed the plasticity theory and its computational implementation in general and consistent way. And this paper based the develop of the unidimensional plasticity nonlinearity based, mainly, on this work.

According to the theory of uniaxial plasticity, an element when reaching its elastic limit starts to have plastic strains. That is, the strain of the element becomes composed of two portions, the elastic (reversible) and the plastic (irreversible).

$$\varepsilon = \varepsilon_e + \varepsilon_p . \tag{1}$$

As plastic strains develop the elastic limit changes according to a hardening law. This work considers the isotropic hardening behavior.

In order to evaluate whether the element is in the plastic or elastic regime, it is necessary to define the stress domain, $f(\sigma)$, which depends on the yield function, $G(\alpha)$, responsible for the behavior of the material after yield, and the flow rule respectively:

$$f(\sigma) = |\sigma| - G(\alpha) \leq 0 ; \tag{2}$$

$$\dot{\varepsilon}_p = \gamma \cdot sign(\sigma). . \tag{3}$$

For implementation, it is necessary to define the elastoplastic modulus of the material. When the element starts yield $f = 0$ and $\gamma > 0$, then the variation of f in time is zero, and then:

$$\dot{f} = \frac{\partial f}{\partial \sigma}\frac{\partial \sigma}{\partial t} + \frac{\partial f}{\partial G}\frac{\partial G}{\partial \alpha}\frac{\partial \alpha}{\partial t} = 0 ; \tag{4}$$

$$\dot{f} = sign(\sigma) \cdot \dot{\sigma} - \frac{\partial G}{\partial \alpha}\dot{\alpha} = 0 . \tag{5}$$

As $\dot{\alpha} = \gamma$, and Hooke's law, $\sigma = E \cdot \varepsilon_e = E \cdot (\varepsilon - \varepsilon_p)$, derivating σ and substituting in f results:

$$\dot{f} = sign(\sigma) \cdot E \cdot (\dot{\varepsilon} - \gamma \cdot sign(\sigma)) - \frac{\partial G}{\partial \alpha}\gamma = 0 . \tag{6}$$

Isolating γ:

$$\gamma = \frac{sign(\sigma) \cdot E}{E + \frac{\partial G}{\partial \alpha}} \cdot \dot{\varepsilon} . \tag{7}$$

Applying Hooke's Law:

$$\dot{\sigma} = E \cdot (\dot{\varepsilon} - \dot{\varepsilon}_p) = E \cdot (\dot{\varepsilon} - \gamma \cdot sign(\sigma)) = E \cdot \left(1 - \frac{E}{E + \frac{\partial G}{\partial \alpha}}\right) . \tag{8}$$

Comparing with Hooke's law, the elastoplastic modulus is defined by:

$$C_{ep} = \frac{E \cdot \frac{\partial G}{\partial \alpha}}{E + \frac{\partial G}{\partial \alpha}} . \tag{9}$$

Defined the Elastoplastic Modulus, the predictor algorithm will be presented for the incremental application in Finite Elements. Point of departure of the algorithm is a known stat at a Gauss point at an incremental step n and the strain increment at step n+1.

1. From $u_{(n)}$ obtain $\varepsilon_{(n)}$.
2. Known variables: $\varepsilon_{(n)}, \varepsilon_{p(n)}, \alpha_{(n)}$.
3. Calculate trial stress using Hooke's Law and check the yield condition:

$$\sigma_{n+1}^{trial} = E \cdot \left(\varepsilon_{(n+1)} - \varepsilon_{p(n)}\right); \tag{10}$$

$$f_{n+1}^{trial} = \left|\sigma_{n+1}^{trial}\right| - G(\alpha_n). \tag{11}$$

If $f_{n+1}^{trial} \leq 0$ then, $\sigma_{n+1} = \sigma_{n+1}^{trial}$, $\varepsilon_{p(n+1)} = \varepsilon_{p(n)}$, $\alpha_{(n+1)} = \alpha_{(n)}$ e $C_{ep} = E$.

If $f_{n+1}^{trial} > 0$ then, solve $\Delta\gamma$:

$$f_{n+1}^{trial} - \Delta\gamma \cdot E - G(\alpha_{n+1}) + G(\alpha_n) = 0. \tag{12}$$

In case the previous equation is not explicit in $\Delta\gamma$, the Newton-Rhapson method is used.

Determine plastic state.

$$\sigma_{n+1} = \left[1 - \frac{\Delta\gamma}{\left|\sigma_{n+1}^{trial}\right|} \cdot E\right] \cdot \sigma_{n+1}^{trial}; \tag{13}$$

$$\varepsilon_{p(n+1)} = \varepsilon_{p(n)} + \Delta\lambda \cdot sign\left(\sigma_{n+1}^{trial}\right); \tag{14}$$

$$\alpha_{n+1} = \alpha_n + \Delta\gamma; \tag{15}$$

$$C_{ep} = \frac{E \cdot \dfrac{\partial G}{\partial \alpha}}{E + \dfrac{\partial G}{\partial \alpha}}. \tag{16}$$

End.

To exemplify the computational implementation, the same Mises's truss was analyzed considering geometric nonlinearity and physical nonlinearities with hardening and softening material response as presented in Muñoz and Roehl (2017).

Table 2. Characteristic of elastoplastic truss with hardening.

Description	Value
L	4 m
h	1 m
R	15 cm
t	8 mm
Yield stress	500 MPa
K	20Gpa
Elastic modulus	200GPa

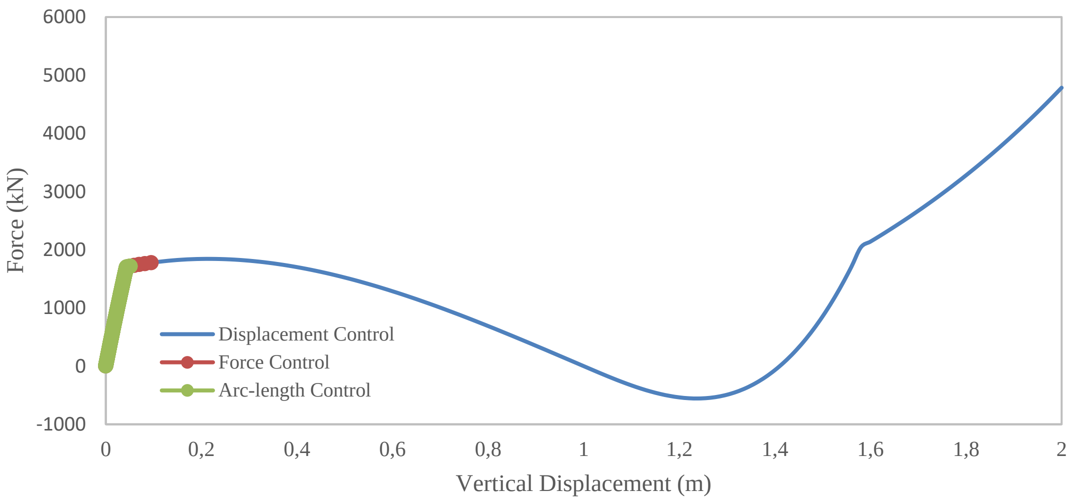

Figure 3. Equilibrium Path of an Elastoplastic Truss with Hardening.

This is a typical example of trusses composed of ductile materials. From the equilibrium path presented in Fig. 3, it is observed that only the displacement control method was able to represent the entire equilibrium path. For the given parameters the truss presents plastic strain before the limit load is reached. In the post-buckling path, the truss continues to stiffen, but at a reduced gradient compared to the elastic truss.

Next the reference truss is analysed for a softening material behavior with the material properties given in Table 3. Linear isotropic softening was considered. Figure 4 shows the equilibrium path obtained with the arc-length method and with displacement control.

Analyzing the two Fig. 3 and Fig. 4, it is possible to observe the limitation in the application of controls methods whereas only the displacements control was able to represent the whole equilibrium path.

Table 3. Characteristic of elastoplastic truss with softening.

Description	Value
L	4 m
h	1 m
R	15 cm
t	8 mm
Yield stress	500 MPa
K	-20Gpa
Elastic modulus	200GPa

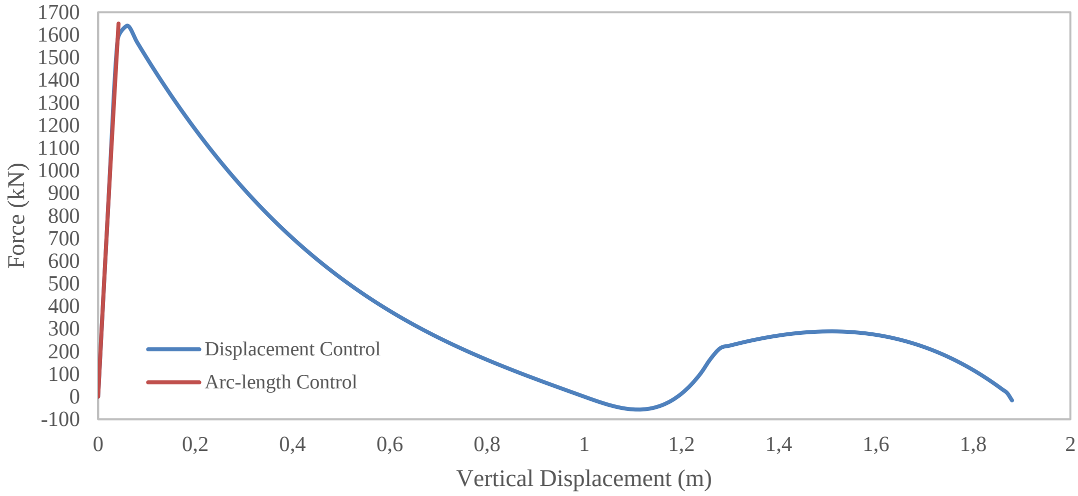

Figure 4. Equilibrium Path of an Elastoplastic Truss with Softening.

This is a typical example of fragile material behavior, which only the displacement control method provided the whole path. Regarding the peak load, even less compression load is required in the region of instability than for a ductile material, and the truss breaks even before reach the of the snap through.

2.2 Constitutive Model with Damage

Material softening can also be described, besides through plasticity, with the theory of the continuous damage, Lemaitre (1978, 1984, 1992). The main difference between these two models is that there are no residual strains for the damage theory, besides that in an unloading process, an elastoplastic material returns by a trajectory parallel to the elastic loading trajectory, and the material with damage returns by a rectilinear trajectory that connects the current point to the point of zero strains, Fig 5.

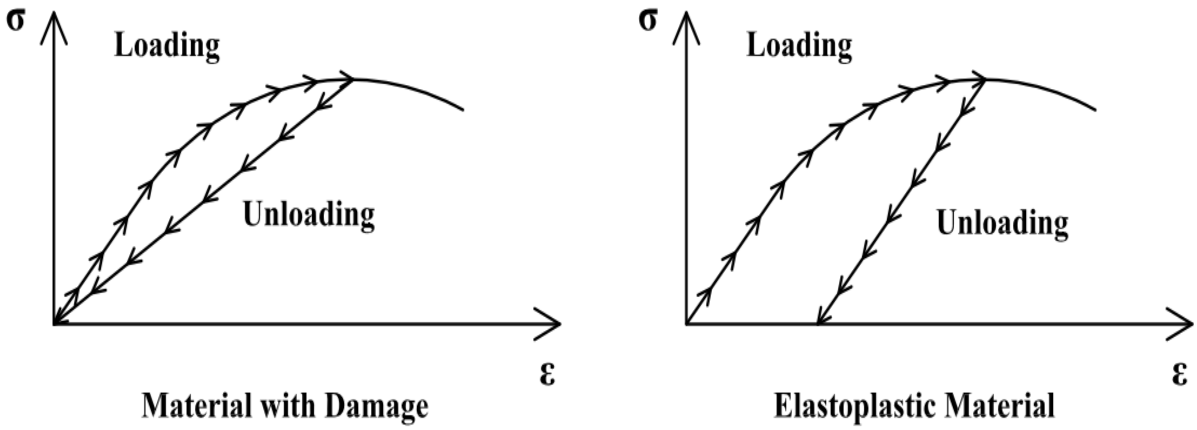

Figure 5. Differences between a stress-strain diagram of elastoplastic material and a material with damage.

To apply the Mechanics of Continuous Damage it is needed to define the damage $d = \frac{A_D}{A}$, where A_D is the area of the defects of the cross section and A is the cross-sectional area. In this way it is possible to see that *d* is variable with values between *0* and *1*, and for *d = 0*, the material has not yet been damaged and for *d = 1*, the material presents damages throughout its transversal area. It is considered that only the undamaged area $\bar{A} = A - A_D$ will be able to resist the load. The relationship between undamaged area and total area can be obtained by:

$$d = \frac{A_D}{A} = \frac{A - \bar{A}}{A} = 1 - \frac{\bar{A}}{A}\,. \tag{17}$$

It also defines the effective and apparent stresses as $\bar{\sigma} = \frac{F}{\bar{A}}$ and $\sigma = \frac{F}{A}$, where F is the applied force. The relationship between these two stresses can be expressed as a function of the damage variable.

$$\frac{\sigma}{\bar{\sigma}} = \frac{\bar{A}}{A} = 1 - d\,. \tag{18}$$

In this way, it is possible to say that the apparent stress will be equal to effective stress when $d = 0$, and the effective stress tends to infinity when $d = 1$.

The strain equivalence hypothesis proposed by Lemaitre (1984), states that the strain associated with a damaged state subjected to the apparent stress, σ, is equivalent to the strain associated with the undamaged state submitted to the effective stress, $\bar{\sigma}$, as illustrated in Fig 6. This hypothesis allows the definition of the constitutive relation in terms of the effective stress and the apparent stress:

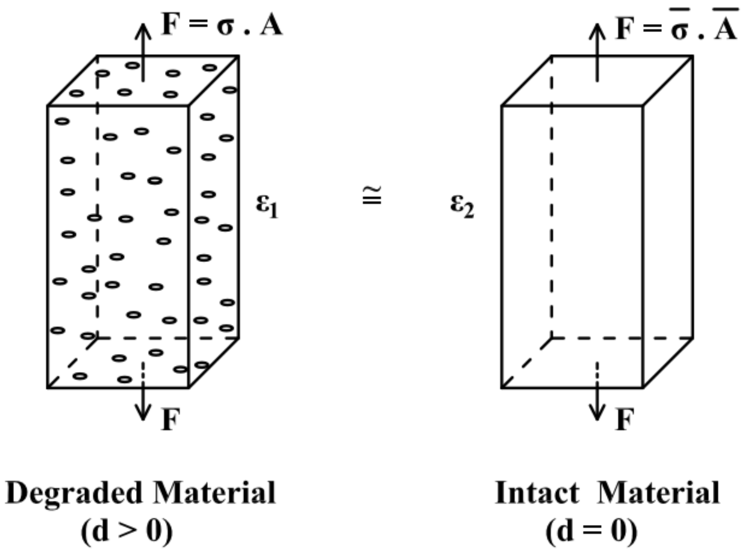

Figure 6. Equivalent strain hypothesis, Lemaitre and Chaboche (1978).

$$\varepsilon_1 = \frac{\sigma}{E_D} = \varepsilon_2 = \frac{\bar{\sigma}}{E}, \tag{19}$$

Where E_D is the degraded elastic modulus that relates to the intact elastic modulus through the expression: $E_D = (1-d)E$.

We can then write the constitutive relation between the apparent stress and the strain as:

$$\sigma = (1-d) \cdot E \cdot \varepsilon, \tag{20}$$

Similarly to plasticity, there is a damage function $F(\sigma) = \tau(\sigma) - q \leq 0$, where τ is the norm of the equivalent stress and q is the damage limit variable. The damage function can be written as a function of the effective stress, $F(\bar{\sigma}) = \tau(\bar{\sigma}) - r \leq 0$, where $r = \frac{q}{1-d}$ and $\tau(\bar{\sigma}) = \sqrt[2]{\frac{\bar{\sigma}^2}{E}}$.

The Kuhn-Tucker relations define conditions of loading and unloading

- $F(\bar{\sigma}) \leq 0;$ (21)
- $\dot{d} \geq 0;$ (22)
- $\dot{d} \cdot F(\bar{\sigma}) = 0;,$ (23)

From the Kuhn-Tucker relations follows that if $F < 0$ then $\dot{d} = 0$, that is, there is no evolution of the damage. In the same way, if $\dot{d} > 0$ then $F = 0$, that is, the element is under loading and has deterioration due to active damage. If $\dot{F} < 0$ then $\dot{d} = 0$, ie, the body is under unloading without the damage evolution. According to Rodrigues (2011), depending on the characteristic of the material, the evolution of damage can produce different behaviors after the elasticity limit. This behavior can be attributed to a variable H, called the hardening/softening modulus. According to Manzoli (1998), for the isotropic damage model the hardening/softening law can be expressed in relation to the internal variables of stress strain type:

$$\dot{q} = \frac{H}{1+H} \cdot \dot{r} \,, \tag{24}$$

where, $\dot{r} \geq 0$.

As $q = (1-d) \cdot r$, it is possible to write the the law of evolution of the damage variable as a function of the damage limit variable and the hardening / relaxation modulus:

$$\dot{d} = \left(\frac{1}{1+H} - d \right) \cdot \frac{\dot{r}}{r}, \tag{25}$$

According to the principle of thermodynamics, which governs the phenomenon of damage, the deformation process must be irreversible, so $\dot{d} \geq 0$, and H must be contained, at any point in the loading process, in the following interval: $-1 \leq H < (1-d)$, where $H(r)$ can be varied by any function in terms of the limit of damage variable Rodrigues (2011).

Solving the differential relation, the expression for the evolution of damage is obtained:

$$d(r) = \left(1 - \frac{q(r)}{r} \right), \tag{26}$$

with $q(r) = \int \frac{H(r)}{1+H(r)} dr$.

If H is constant, then:

$$d(r) = \frac{r - r_0}{r(1+H)}, \tag{27}$$

In the previous expression., r_0 is the damage thereshold that is a property of the material and can be related to the proportionality limit stress f_0 by the expression:

$$r_0 = \frac{f_0}{\sqrt{E}}. \tag{28}$$

Alternatively, an exponential law was proposed.

$$H(r) = \frac{1}{1 + A \cdot e^{A \cdot \left(1 - \frac{r}{r_0}\right)}} - 1; \tag{29}$$

$$d = 1 - \frac{r}{r_0} \cdot e^{A\left(1 - \frac{r}{r_0}\right)}. \tag{30}$$

The parameter A depends on the fracture energy of the material. For the implementation it is necessary to use a predictor algorithm similar to the one used for plasticity:

1. Initial values d_n, r_n ;
2. Calculate the norm of the equivalent stress, τ_{n+1},

If $F(\bar{\sigma}) \leq 0$:

if $\tau_{n+1} > r_n$ then $r_{n+1} = \tau_{n+1}$

if $\tau_{n+1} \leq r_n$ then $r_{n+1} = r_n$

3. Calculate d_{n+1};
4. Calculate the stress σ_{n+1}
5. Determine $C_{eq} = E_D = (1 - d_{n+1})E$

To exemplify the computational implementation, the same Mises truss was analyzed for two sets of parameters for the exponential $H(r)$ function, one in which the damage is more severe, with higher of A, and one in which it is milder, lower value of A.

Table 4. Characteristics of the truss with severe damage model.

Description	Value
L	4 m
h	1 m
R	15 cm
t	8 mm
Limit stress	500 MPa
A	0,2
Elastic modulus	200GPa

Table 5. Characteristics of the truss with mild damage model.

Description	Value
L	4 m
h	1 m
R	15 cm
t	8 mm
Limit stress	500 MPa
A	0,05
Elastic modulus	200GPa

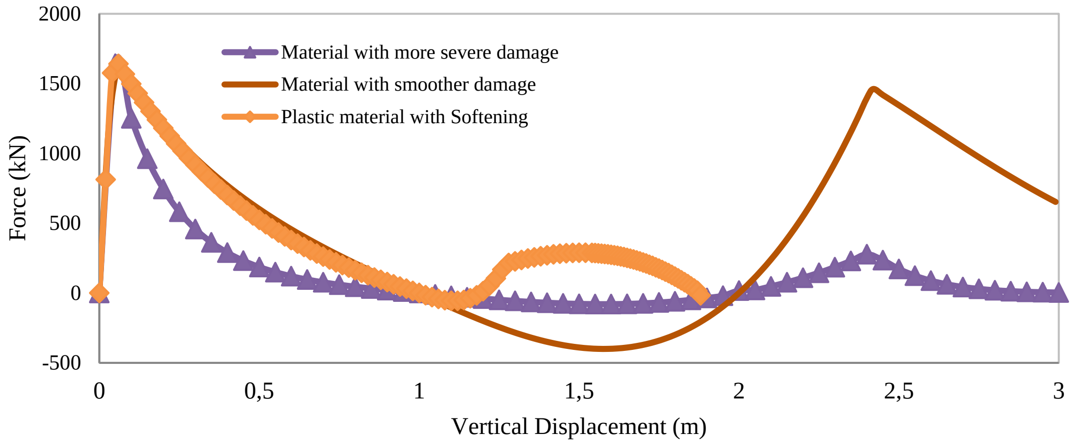

Figure 7. Comparison between plastic truss with softening and trusses with damage model.

In the analysis of a truss with a more severe damage, displacements imposed at the central node control the analysis. It is not possible to clearly perceive the snap-through, since due to the effect of the severe damage after the bars are submitted to tension they will no longer be able to withstand the stresses and with a low load stiffness is greatly reduced.

In the case of less sever damage, the snap-through ismore pronounced, resembling the response of the elastoplastic material with softening. However, when reaching the limit by tensile, the truss with the damage material model has a much sharper stiffness drop.

3 ELASTOPLASTIC ELEMENT WITH DAMAGE AND NONLINEAR GEOMETRY

Two examples have been analyzed, one in which the material has plastic hardening before damage begins, then it is expected a ductile behavior and another where damage occurs prior to yield, then it is expected a fragile beahavior. The Characteristics of the first material is0 similar to a structural steel A36 and the structure has a local buckling load of 9089 kN, so this phenomenon doesn’t happen in this analysis.

The control method used was the displacement control in order to capture the loading and unloading equilibrium path in the next two exemples.

Table 6. Characteristic of elastoplastic truss with ductile damage model

Description	Value
L	4 m
h	1 m
R	15 cm
t	8 mm
Yield stress	250 MPa
k	10GPa
Limit Stress	500MPa
A	0,5
Elastic modulus	200GPa

Table 7. Characteristic of elastoplastic truss with fragile damage model

Description	Value
L	4 m
h	1 m
R	15 cm
t	8 mm
Yield stress	500MPa
k	10GPa
Limit Stress	250 MPa
A	0,5
Elastic modulus	200GPa

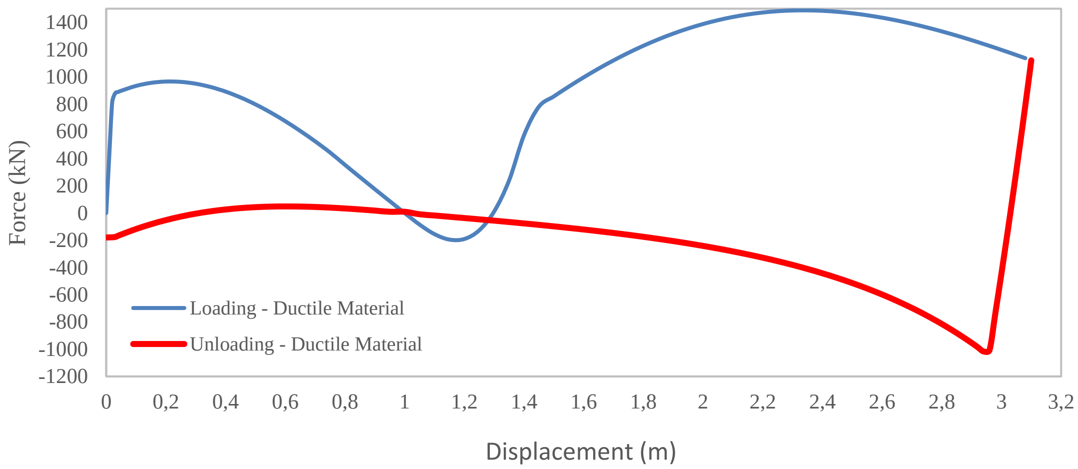

Figure 8. Loading and unloading of ductile elastoplastic truss with damage model.

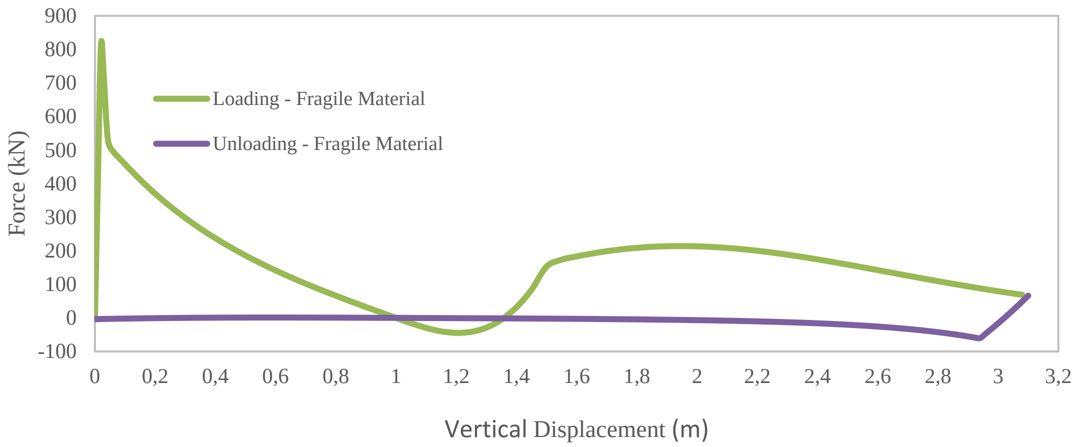

Figure 9. Loading and unloading of fragile elastoplastic truss with damage model.

The first exemple presented characteristics of a ductile material while the second of a fragile material with loss of stiffness shortly after peak load. It is clear that the characteristic that manifests first will be dominant in the behavior of the structure.

The unloading process was also different between the adopted models, the ductile truss to return to the initial position, without deformations it is necessary that a vertical force was applied upwards tensiling the structure, effect caused by the plastic strain better noticeable in this model. It is also noticeable that the unloading trajectory bears a great resemblance to the one of loading, but displaced in the horizontal and vertical axis, being possible to make an analogy with the test of traction for cyclical loads. For a fragile material, the truss returns to the undisturbed position with the discharge path quite different from that of the load and without needing any residual force, this occurs because when a material is damaged, the cracks close, despite the material remains degraded, and there are no residual strains.

4 CONCLUSION

From the analyses of the trusses with physics nonlinearities, it is emphasized that nonlinear structural problems require adequate controls of the incremental solution of the nonlinear equilibrium equations, as can be seen in Fig. 3 and Fig 4. It is emphasized, too, that for problems that present some type of unloading the control method applied was the displacement control in order to easily choose the point in the equilibrium path to start the unloading.

In addition, with respect to the physical behavior of the materials, the difference between the use of ductile and fragile materials in which the second type has a rather abrupt loss of stiffness without the presence of residual stresses or strains in the unloading process became clear, as can be seen comparing the Fig. 8 and Fig. 9.

For future studies it is suggested to advance in the analysis of more complex trusses and observe the effect of the nonlinearities in their behavior, as well as the robustness of the solution schemes of the nonlinear systems.

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