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IMPEDANCE COMPUTATION USING THE SPECTRAL ELEMENT METHOD AND THE RICCATI EQUATION

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Abstract. *The reduction of structural vibrations has always been a difficult task. One approach to achieve this goal using a wave approach consists in reducing wave reflections at the boundaries. Without reflections, standing waves cannot build up and, therefore, vibration is attenuated. Reflection can be avoided by using anechoic devices at the structure extremities. The Acoustic Black Hole (ABH) theory was established based on the property that longitudinal and flexural waves do not reflect from boundaries with thickness decaying with a power law. The dynamics of tapered structures can be formulated using a state space approach. The elastodynamic equations in the frequency domain (spectral formulation) are written as a first order state space differential equation system. For a tapered structure, the space state formulation leads to a Riccati equation that can be numerically solved. Thus, any geometrical law for the structure thickness can be treated. The intent of this paper is to investigate the numerical solution of the Riccati equation. Furthermore, the ABH effect using graded material properties instead of tapered geometries is proposed. Elementary rod model are treated. Results using the proposed formulation are validated by comparison with semi analytical results obtained using the Spectral Element Method (SEM) for constant and linearly varying properties. SEM solutions for rods with linearly varying coefficients formulated using Bessel functions are recalled. Numerical issues when using the Ricatti equation are addressed.*

Keywords: *Impedance, Rod, Spectral Element Method, Riccati Equation.*

1 INTRODUCTION

Linear elastodynamic problems are usually formulated as boundary value problems,which may be solved analytically or numerically. Analytical solutions are often written in the frequency domain, and may be called spectral relations. For relavively simple structures it is straightforward to write the solution in terms of finite elements but keeping the exact solutions in the frequency domain. This approach is mainly known as dynamic stiffness method or spectral element method. Given a spectral element a built up structure global dynamic stiffness matrix ca be assembled using the direct stiffness method, also used in finite element analysis.

To derive a spectral element for a straight homogeneous rod with constant material and geometrical properties along the rod is straightforward. However, for rods with varying properties along their length this can be quite cumbersome. Analytical solutions exist for a few cases, such as linearly varying (tapered) and exponentially varying rods.

In the acoustics of waveguides (such as gases in pipes), some authors have developed a clever method of solution consisting of formulating the dynamic problem in the space state (using pressure and velocity along the tube). This transforms the Helmholtz spectral equation - a boundary value problem - into an initial value problem. The issue is that one does not have the full initial value,*i.e.*, the state at one end of the waveguide, to use this equation directly. Therefore, the problem is transformed and written in terms of the ratio between effort and velocity, i.e., the impedance. This yields a Riccati matrix equation that can be solved analytically for simple problems or numerically for more complex problems.

The matrix Riccati differential equation plays an important role in many engineering science applications [Lancaster and Rodman, 1995]. Among those applications, the linear quadratic optimal control problem [Kučera, 1973], which is an outcome of the calculus of variations, can be solved using a Riccati differential equation (RDE). This type of equation [Levin, 1959, Reid, 1946], which is quadratic and thus nonlinear, can be reduced to a linear system of twice the size that can be efficiently solved using numerical algorithms [Davison and Maki, 1973, Kenney and Lepnik, 1985].

Recently, some authors [Georgiev et al., 2011] used this impedance method to solve the problem of a beam with cross section varying in a power law with the Riccati equation. The shape of this special tapering can be tailored to minimize wave reflection caused by impedance change, thus creating a very absorbing termination, which they called an Acoutic Black Hole (ABH).

This paper reviews the technique consisting of writing the elastodynamic equations as a state space equation in the frequency domain, *i.e.*, as a function of the spatial variable only. Thus, reformulating it in terms of impedance, yields a Riccati equation that can be numerically solved. The implemented method is verified using a linearly tapered rod, for which an analytical spectral element exists. The impedance is compared with excellent agreement.

Some issues concerning the solution of the Riccati equation are addressed. The proposed method may be complemented by a back propagation from the force and displacement (state) at one end of the rod to compute the full state at the other end. From the relations between these states one can compute the dynamic stiffness or the transfer matrix. With these matrices the forced responses may be computed.

Given the possibilities allowed by additive manufacturing technologies, the shape and material properties of structures can be easily varied along their length. The proposed technique should be useful to optimize the waveguide shape aiming at reduced or enhanced vibration energy propagation.

2 IMPEDANCE THROUGH THE SPECTRAL ELEMENT METHOD

This section describes how the impedance of a tapered rod can be computed (Ahmida et al. [2003], Banerjee and Williams [1985]) using the spectral element method (SEM).

Figure 1 shows a schematic representation of a tapered rod with a cross section area $A(x)$, linearly varying with the displacement x, given by

$$A(x) = \epsilon(x+\xi) \qquad \text{with} \qquad \xi = \frac{A_0 l}{A_l - A_0} \qquad \text{and} \qquad \epsilon = \frac{A_0}{\xi} \tag{1}$$

where l is the element length, A_0 and A_l are the cross section areas in the positions $x = 0$ and $x = l$, respectively.

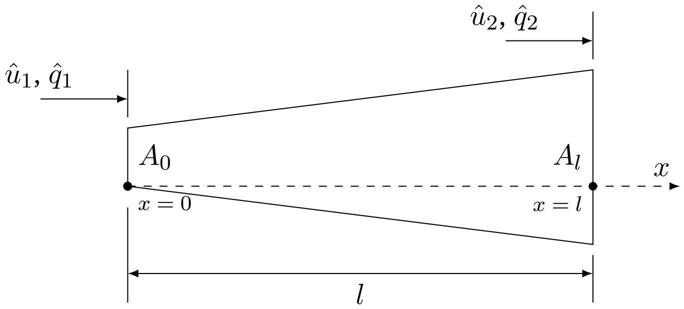


Figure 1: Two-nodes tapered rod spectral element with linearly varying cross section area.

The elementary rod theory considers the model of a long and slender structure that supports only axial stress, neglecting the lateral contraction (Poisson's effect). As shown in Doyle [1989], the continuous-time homogeneous differential equation of motion for a tapered rod is given by

$$\frac{\partial}{\partial x}\left[EA(x)\frac{\partial u(x,t)}{\partial x}\right] = \rho A(x)\frac{\partial^2 u(x,t)}{\partial t^2}, \tag{2}$$

where $u(x) := u(x,t)$ is the axial displacement, ρ is the mass density, $A(x)$ is the cross section area, E is the complex (or dynamic) Young's modulus, to account for energy dissipation, which is given by $E = \bar{E}(1 + j\eta)$, with $\bar{E}$ the Young's modulus, η the loss factor and $j = \sqrt{-1}$.

Substituting the expression for $A(x)$ from (1) in (2), the rod homogeneous differential equation of motion becomes

$$\epsilon(x+\xi)\frac{\partial^2 u(x,t)}{\partial x^2} + \epsilon\frac{\partial u(x,t)}{\partial x} - \epsilon\frac{\rho}{E}(x+\xi)\frac{\partial^2 u(x,t)}{\partial t^2} = 0. \tag{3}$$

For this equation, the solution $u(x,t)$ has the following form

$$u(x,t) = \hat{u}(x)\cos(\omega t) \tag{4}$$

where $\hat{u}(x) := \hat{u}(x,\omega)$ is the displacement in the frequency domain, *i. e.*, the Fourier transform of $u(x,t)$. Substituting this solution into the above equation of motion, one obtains, after some manipulation, the following equation:

$$\frac{d^2\hat{u}(x,\omega)}{dx^2} + \frac{1}{(x+\xi)}\frac{d\hat{u}(x,\omega)}{dx} + k^2\hat{u}(x,\omega) = 0, \tag{5}$$

where $k = \omega\sqrt{\rho/E}$ is the wavenumber. As shown in McLachlan [1955], a Bessel type solution for (5) is given by

$$\hat{u}(x) = \alpha_1 J_0(\gamma) + \alpha_2 Y_0(\gamma), \qquad \gamma = k(x+\xi) \tag{6}$$

where γ, α_1 and α_2 are constants determined from the boundary conditions, J_0 and Y_0 are Bessel function of first and second kind of order zero, respectively. The displacement boundary conditions for the two-nodes element shown in Figure 1, are

$$\begin{aligned} \hat{u}_1 &= \hat{u}(0) = \alpha_1 J_0(k\xi) + \alpha_2 Y_0(k\xi) \\ \hat{u}_2 &= \hat{u}(l) = \alpha_1 J_0(kl+k\xi) + \alpha_2 Y_0(kl+k\xi) \end{aligned} \tag{7}$$

where, $\hat{u}_1$ and $\hat{u}_2$ are the nodal displacements. The equation above can be rewritten in the following matrix form

$$\hat{u}_i = \Phi\alpha, \qquad \text{with} \quad \hat{u}_i = \begin{bmatrix} \hat{u}_1 \\ \hat{u}_2 \end{bmatrix}, \quad \Phi = \begin{bmatrix} J_0(k\xi) & Y_0(k\xi) \\ J_0(kl+k\xi) & Y_0(kl+k\xi) \end{bmatrix}, \quad \alpha = \begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix} \tag{8}$$

Noting that the rod axial force is given by

$$\hat{q}(x) = EA(x)\frac{\partial\hat{u}(x)}{\partial x} \tag{9}$$

and calculating the forces at the element boundaries, ie, $\hat{q}_1 = \hat{q}(x)$ for $x = 0$ and $\hat{q}_2 = \hat{q}(x)$ for $x = l$, the following equation is obtained:

$$\begin{aligned} \hat{q}_1 = \hat{q}(0) = EA_0\frac{\partial\hat{u}(x)}{\partial x}\bigg|_{x=0} &= -kEA_0\left[\alpha_1 J_1(k\xi) + \alpha_2 Y_1(k\xi)\right], \\ \hat{q}_2 = \hat{q}(l) = EA_l\frac{\partial\hat{u}(x)}{\partial x}\bigg|_{x=l} &= -kEA_l\left[\alpha_1 J_1(kl+k\xi) + \alpha_2 Y_1(kl+\xi)\right]. \end{aligned} \tag{10}$$

where J_1 and Y_1 are Bessel functions of first and second kind of first order. Rewriting this equation in the matrix form gives:

$$\hat{q}_i = \Psi\alpha, \qquad \text{with} \quad \hat{q}_i = \begin{bmatrix} \hat{q}_1 \\ \hat{q}_2 \end{bmatrix}, \quad \Psi = -kE\begin{bmatrix} A_0 J_1(k\xi) & A_0 Y_1(k\xi) \\ A_l J_1(kl+k\xi) & A_l Y_1(kl+k\xi) \end{bmatrix}, \quad \alpha = \begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix} \tag{11}$$

Now, by substituting α from (8) in (11), one obtains:

$$\hat{q}_i = K\hat{u}_i, \qquad K(\omega) = \Psi\Phi^{-1} \tag{12}$$

where K is a 2×2 dynamic stiffness matrix of a two-noded element with cross section area varying linearly.

The shape functions are derived in the same manner, solving the coefficients α_1 and α_2 in terms of the nodal displacements, the internal longitudinal displacement along the finite rod is given by:

$$\hat{u}(x) = g_1(x)\hat{u}_1 + g_2(x)\hat{u}_2 \tag{13}$$

where the element shape functions are

$$\begin{aligned} g_1(x) &= \left[J_0(kl + k\xi)Y_0(kx + k\xi) + Y_0(kl + k\xi)J_0(kx + k\xi)\right]/\Delta \\ g_2(x) &= \left[J_0(k\xi)Y_0(kx + k\xi) - Y_0(k\xi)J_0(kx + k\xi)\right]/\Delta \end{aligned} \tag{14}$$

with $\Delta = Y_0(kl + k\xi)J_0((k\xi)) - J_0(kl + k\xi)Y_0(k\xi)$.

By using these same shape functions and the nodal values of displacement, the internal forces are obtained easily as:

$$\hat{q}(x) = EA(x)\left[\frac{\partial g_1(x)}{\partial x}\hat{u}_0 + \frac{\partial g_2(x)}{\partial x}\hat{u}_l\right] \tag{15}$$

The response of a structure to a external force applied at any node, $\hat{q}_1$ or $\hat{q}_2$, can be expressed in terms of the mechanical impedance $\hat{z}$

$$\hat{q}(x) = \hat{z}(x)\hat{v}(x) = j\omega\hat{z}(x)\hat{u}(x) \tag{16}$$

since the internal velocity $\hat{v}(x)$ is given by $\hat{v}(x) = j\omega\hat{u}(x)$, the impedance $\hat{z}(x)$ at any internal point between the nodes can be evaluated.

3 IMPEDANCE THROUGH RICCATI'S EQUATION

This section shows how to compute the mechanical impedance matrix $\hat{z}(x)$ using a Riccati type equation. By applying the Fourier transform to the homogeneous differential equation of motion (2) and the traction force (9), one obtains in the frequency domain [Arruda et al., 2007] the following set of equations:

$$\frac{\partial \hat{q}(x)}{\partial x} = -\rho A(x)\omega^2\hat{u}(x) \qquad \text{and} \qquad \frac{\partial \hat{u}(x)}{\partial x} = \frac{\hat{q}(x)}{EA(x)} \tag{17}$$

which can be written in the state-space form

$$\frac{\partial \hat{p}}{\partial x} = H\hat{p}, \qquad \hat{p}(0) = \hat{p}_0 \tag{18}$$

with $\hat{p}(x) := \hat{p}(x, \omega)$ and $H(x, \omega)$ having the following partitions:

$$\hat{p}(x) = \begin{bmatrix} \hat{u}(x) \\ \hat{q}(x) \end{bmatrix} \qquad \text{and} \qquad H(x, \omega) = \begin{bmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{bmatrix} = \begin{bmatrix} 0 & \dfrac{1}{EA(x)} \\ -\rho A(x)\omega^2 & 0 \end{bmatrix} \tag{19}$$

Note that this is a linear time variant system, whose exact (symbolic) solution usually does not exists. One practical approach to numerically solve this linear system of equation is to rewrite it as an equivalent Riccati type equation [Georgiev et al., 2011].

Substituting the relation between force and displacement at a point x along the rod given by (16) into (18), one obtains the following set of equations:

$$\frac{\partial \hat{u}(x)}{\partial x} = H_{11}\hat{u}(x) + H_{12}\hat{q}(x) \qquad \text{and} \qquad j\omega \frac{\partial \hat{z}(x)\hat{u}(x)}{\partial x} = H_{21}\hat{u}(x) + H_{22}\hat{q}(x) \tag{20}$$

which lead to:

$$j\omega \left(\frac{\partial \hat{z}(x)}{\partial x}\hat{u}(x) + \hat{z}(x)H_{11}\hat{u}(x) + \hat{z}(x)H_{12}j\omega\hat{z}\hat{u}(x) \right) = H_{21}\hat{u}(x) + H_{22}j\omega\hat{z}(x)\hat{u}(x) \tag{21}$$

Since $\hat{u}(x)$ cannot be identically zero, one finally obtains the following Riccati type equation

$$\frac{\partial \hat{z}(x)}{\partial x} + \hat{z}(x)H_{11} - H_{22}\hat{z}(x) + j\omega\hat{z}(x)H_{12}\hat{z}(x) = (j\omega)^{-1}H_{21} \tag{22}$$

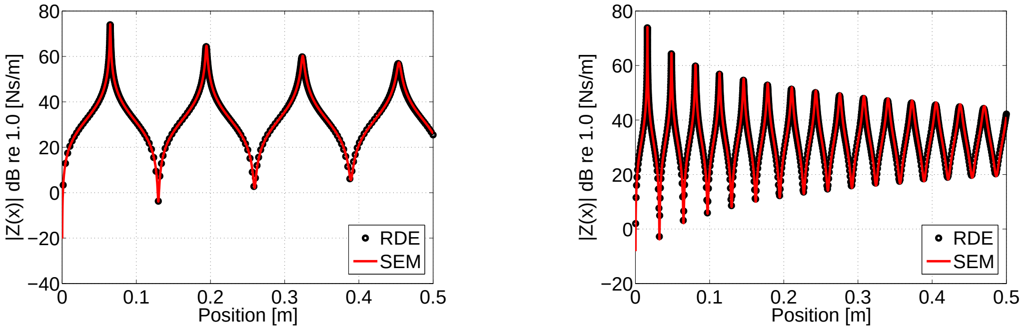
whose initial condition $\hat{p}(0) = \hat{p}_0$ must satisfy $\hat{q}_0 = j\omega\hat{z}_0\hat{u}_0$.

4 NUMERICAL RESULTS

In this section the two methods for computing the impedance for a simple rod element with constant and linearly varying cross section area are simulated. One of them is analytical (SEM) and the second one is numerical (RDE).

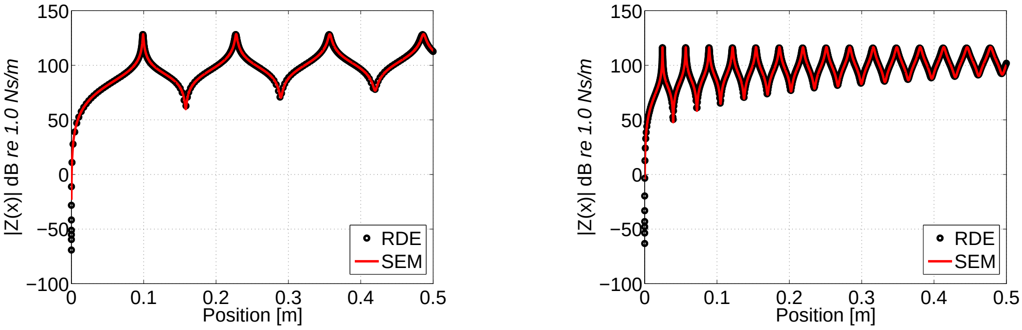
The rod element with constant cross section area is simulated using the following geometry and material properties: length $l = 0.5$ [m], constant cross section area $A_0 = 10^{-7}$ [m^2], Young's modulus $\bar{E} = 210 \times 10^9$ [Pa], mass density $\rho = 7800$ [kg/m^3], and loss factor $\eta = 0.01$. For the rod element with a linearly varying cross section area it is assumed an initial cross section area $A_0 = 10^{-7}$ [m^2] and final area $A_l = 4 \times 10^{-4}$ [m^2]. The response for both cases were analyzed for $\omega = 2\pi f$ with $f = 20$ [kHz] and $f = 80$ [kHz].

For both cases, a free-free boundary condition is assumed. To compute the internal forces by SEM, an external axial force was applied at the second node, *i.e.*, $\hat{q}_1 = 0$ and $\hat{q}_2 = 1$ [N]. After solving the system given by (12) and obtaining the nodal displacements $\hat{u}_1$ and $\hat{u}_2$, the shape functions are used to compute the internal displacement ($\hat{u}(x)$) and force ($\hat{q}(x)$), allowing to evaluated the velocity $\hat{v}(x)$ and the impedance $\hat{z}(x)$. To compute the numerical solution of (22) with RDE, it is necessary to use an algorithm such as the trapezoidal rule integration, which determines the impedance solution. An initial condition is required, $\hat{z}_0 = \hat{z}(x = 0) = 0$ in our case, which describes a free edge at $x = 0$.



(a) 20 kHz (b) 80 kHz

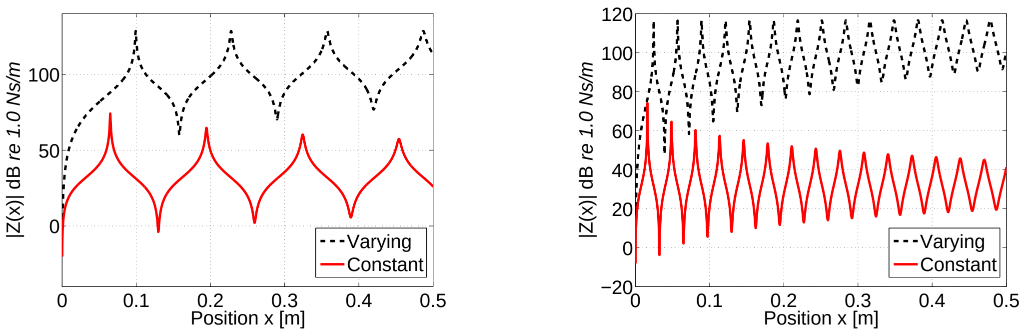
Figure 2: Impedance for constant cross section rod using RDE and SEM.



(a) 20 kHz (b) 80 kHz

Figure 3: Impedance for varying cross section rod using RDE and SEM.

Figure 2 and Fig.3 show the impedance computed using SEM and RDE for a rod with constant and linearly varying cross section area, respectively. The initial difference is due the initial condition established for RDE, but as it can be seen, the results present a very good agreement forboth frequencies evaluated.



(a) 20 kHz (b) 80 kHz

Figure 4: Impedance for constant and varying cross section rod using RDE and SEM.

Figure 4 shows the impedance computed using SEM for a rod with constant cross section area and linearly varying area. Since both methods are in agreement, only one is shown when comparing both types of elements.

5 CONCLUSION

Recent work has shown that the elastodynamic equilibrium equations expressed in the frequency domain can be formulated as a state space equation in terms of the spatial variable. However, the initial condition for a given value of the spatial variable cannot be guessed. This problem can be reformulated in terms of the mechanical impedance, for which the initial value can be obtained directly from the boundary condition. Therefore, the boundary value problem is transformed into an initial value problem. This method, recently proposed in the literature for elastic structures, allows computing the impedance of a structural waveguide with varying material or geometrical properties in a straightforward way. The impedance of a waveguide (in our case a rod) with a free termination (zero impedance) is computed as a function if the length x.

In the literature, this technique has been used to find spatial varying laws yielding an impedance variation similar to a non-reflecting boundary (called an acoustic black hole). In this paper this technique has been reviewed and validated using analytical solutions for a tapered rod.

This technique may be used to compute the dynamic response of one dimensional structures such as rods and beams with arbitrarily varying cross section properties (geometrical and material). This is a valuable tool for the design of non reflecting and periodic structures that can be used to attenuate vibrations.

Future work will develop on the use of this technique to compute the forced response of periodic structures.

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