ANALYTICAL AND NUMERICAL COMPARATIVE STUDY OF PROBLEM IN 2D ELASTICITY OF A HOLLOW TUBE SUBJECT TO NORMAL AND SHEAR STRESSES IN THE INTERNAL CONTOUR.

Thiago Arnaud Abreu de Oliveira

eng.thiagoarnaud@gmail.com

Rodolfo de Azevedo Palhares

rodolfo.palhares@hotmail.com

Danilo Carvalho de Moura

danilo.cmoura@hotmail.com

Luís Vinícius Pereira Silva

luis.vinicius.ucb@gmail.com

Luciano Lins Vieira

lucianovieira.lv@outlook.com

Gilberto Gomes

ggomes2007@gmail.com

Universidade de Brasília – UNB

Programa de Pós-graduação em Estruturas e Construção Civil, Prédio SG–12, Campus Darcy Ribeiro, Universidade de Brasília, CEP 70910-900, Brasília - DF, Brasil.

Abstract. *Structures with hollow circular cross-sections are often used in engineering. The necessity to study pipes is considerably important in the nuclear, aerospace and petrochemical industries. Since the problem of hollow pipes as structural components of great importance to explain the behavior of several practical situations of engineering, this study allows a better understanding of the structural behavior of this element. This paper presents a problem of Elasticity Theory that is not commonly found in the literature and it aims not only the solution, but also to evidence the theoretical concepts involved. The problem consists of a hollow tube subjected to internal pressure as well as internal shear, with boundary conditions of free radial displacement and tangential displacement restrained. The methodology of this paper consists of a linear static analysis through analytical formulations and numerical simulations using Finite Element Method (FEM). For the analysis of the results, an analytical formulation of the problem was developed based on the elasticity theory in polar coordinates, and its results were* *compared with those obtained numerically, using the software ANSYS. The results analyzed were longitudinal and radial displacement and the acting tensions. After the analysis, the results were very approximate, and this approach occurred due the level of refinement of the finite element mesh. For greater precision of the numerical results it would be required an increase in the number of elements of the mesh, although it would increase the processing time. However, for engineering purposes and the purpose of this paper, the numerical results obtained were valid.*

Keywords: *Hollow Tube, Elasticity Theory, Tension Field, Displacement Field, Finite Element Method.*

1 INTRODUCTION

When applying load on a body, it tends to undergo displacements and/or deformations. However, the tendency of a body is to promote balance, through the internal efforts that oppose the deformations and that create a state of internal tensions.

According to Timoshenko and Goodier (1980), most materials used in engineering show elastic behavior up to a certain level of external loading. This means that after ceased the external actions, it returns to its natural state, that is, not deformed. This phenomenon is studied by the so-called elasticity theory, which there are many applicabilities in general engineering problems.

Among the simplifications of elasticity, there are other usual considerations such as the homogeneity of the material (same physical properties at any point in the body) and isotropy (equal elastic properties in all directions).

It is understood that matter is formed by molecules, but in more classic problems, approached by the theory of elasticity, within the mechanics of the continuum, it is not considered its atomic structure.

According to Alves (2007), considering the continuous material is consistent with experimental observations. In addition, it is possible to use an infinitesimal volume of matter to characterize a particle.

Unlike the problems of the classical theory of elasticity, in which the behavior of the material is elastic, following Hooke's Law. This paper will be different from the classical approaches due to the introduction of tangential stresses in the peripheries of the element, being indispensable application of polar coordinates. Thus, the paper will deal with a study of the tension and displacement fields in a tube in the field 2D, and present all mathematical construction to acquire the analytical solutions of the fields under study.

Using the contour conditions, it will be possible to determine the fields of tension and displacement, however, in order to be able to define the necessary fields, they must satisfy the mathematical formulations, so that, when reaching the periphery of the tube, such conditions must be in equilibrium with the external conditions, making the external forces as a continuation of the internal distribution of the tensions.

2 LITERATURE REVIEW

2.1 Plane State of Tensions

In many elasticity problems, the geometric characteristics turn the use of cylindrical coordinates r, θ and z rather than cartesian x, y, z, more especially when dealing with cylindrical shaped bodies such as tubes, bars and rings. In plane problems in which works with plane state of stress or strain, the shear stress in the plane orthogonal to the z axis is zero, so no variable will depend on the z coordinate. Hence, the cylindrical coordinates can be reduced to polar coordinates r, and θ. Figure 1 illustrates the polar coordinate system and its relation to the cartesian system.

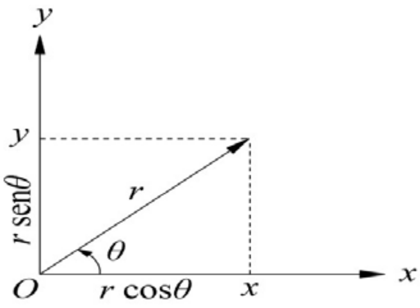

Figure 1: Polar coordinates

The relationships between polar and rectangular coordinates are shown below:

$$x = r \cdot \cos\theta \qquad r = \sqrt{x^2 + y^2} \tag{1}$$

$$y = r \cdot \cos\theta \qquad \tan\theta = \frac{y}{x}$$

2.1.1 Hooke's Law

Most of the engineering materials present a linear relationship between tension and deformation in the region of elasticity. According to Hibbeler (2004), an increase in tension causes a proportional increase in deformation. A fact known as Hooke's Law, where up to a certain limit of proportionality, the deformation of the element is given by:

$$\varepsilon_x = \frac{\sigma_x}{E} \tag{2}$$

This deformation of the element in the x direction is followed by a lateral deformation component:

$$\varepsilon_y = -v\frac{\sigma_x}{E} \tag{3}$$

In this expression, E represents the proportionality constant, it is called the modulus of elasticity or Young's modulus, and ν is the Poisson's coefficient, where both have the same value either in traction or compression.

By overlapping the transversal deformation components obtained by the tensions at x and y, the following equations are obtained:

Proceedings of the XXXVIII Iberian Latin-American Congress on Computational Methods in Engineering
P.O. Faria, R.H. Lopez, L.F.F. Miguel, W.J.S. Gomes, M. Noronha (Editores), ABMEC, Florianópolis, SC, Brazil, November 5-8, 2017.

$$\varepsilon_x = \frac{1}{E}(\sigma_x - v\sigma_y) \tag{4}$$

$$\varepsilon_y = \frac{1}{E}(\sigma_y - v\sigma_x) \tag{5}$$

The shear strain depends only on a single component of shear stress, being proportional to it, as can be seen in the equation:

$$\gamma_{xy} = \frac{\tau_{xy}}{G} = \frac{2(1+v)}{E}\tau_{xy} \tag{6}$$

Considering the orthogonality of the polar coordinate system and using the isotropy of the material, the equations for deformation in polar coordinates can be obtained by simply replacing x by r and y by θ in Eqs. (4), (5) e (6).

$$\varepsilon_r = \frac{1}{E}(\sigma_r - v\sigma_\theta) \tag{7}$$

$$\varepsilon_\theta = \frac{1}{E}(\sigma_\theta - v\sigma_r) \tag{8}$$

$$\gamma_{r\theta} = \frac{2(1+v)}{E}\tau_{r\theta} \tag{9}$$

2.1.2 Relation Displacement – Deformation

As result of loading, solids with elastic behavior change their shape. These deformations can be quantified by displacements at points in the material. Thus, to consider the displacement in polar coordinates, there are the components of the displacement in the radial and tangential directions, being represented by u_r and u_θ respectively. The relationship between deformation in the radial direction and the displacement field is

$$\varepsilon_r = \frac{\partial u_r}{\partial r} \tag{10}$$

In the tangential direction, the deformation depends on both displacement components, u_r and u_θ:

$$\varepsilon_\theta = \frac{u_r}{r} + \frac{1}{r}\frac{\partial u_\theta}{\partial \theta} \tag{11}$$

The total variation in the angle, representing the angular deformation, is:

$$\gamma_{r\theta} = \frac{1}{r}\frac{\partial u_r}{\partial \theta} + \frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r} \tag{12}$$

2.1.3 Equilibrium Equations

The equations of equilibrium in polar coordinates can be obtained based on an infinitesimal element in equilibrium as can be seen in the Figure 2:

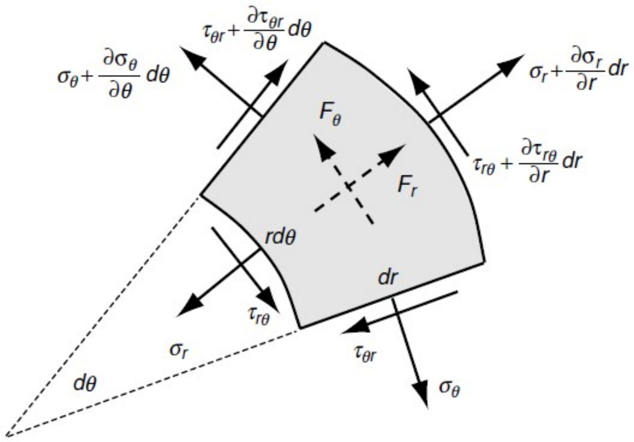

Figure 2: Infinitesimal element in polar coordinates

In which the mass forces in the radial and tangential directions are represented by F_r and F_θ respectively.

If the element is in equilibrium, then the sum of the forces in the radial direction must be zero. Making this sum, assuming that

$$\operatorname{sen}\left(\frac{d\theta}{2}\right)\approx\frac{d\theta}{2},\qquad \cos\left(\frac{d\theta}{2}\right)\approx 1$$

and neglecting the infinitesimal second order, we have the equation of equilibrium in the radial direction

$$\frac{\partial\sigma_r}{\partial r}+\frac{1}{r}\frac{\partial\tau_{r\theta}}{\partial\theta}+\frac{1}{r}(\sigma_r-\sigma_\theta)+F_R=0 \qquad (13)$$

At the same way for the sum of forces in the tangential direction, we obtain the other equation of equilibrium in polar coordinates

$$\frac{1}{r}\frac{\partial\sigma_\theta}{\partial\theta}+\frac{\partial\tau_{r\theta}}{\partial r}+\frac{2}{r}\tau_{r\theta}+F_\theta=0 \qquad (14)$$

When solving two-dimensional problems by means of polar coordinates with no mass forces, these equations are satisfied by

$$\sigma_r=\frac{1}{r}\frac{\partial\phi}{\partial r}+\frac{1}{r^2}\frac{\partial^2\phi}{\partial\theta^2} \qquad (15)$$

$$\sigma_\theta=\frac{\partial^2\phi}{\partial r^2} \qquad (16)$$

$$\tau_{r\theta}=-\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial\phi}{\partial\theta}\right) \qquad (17)$$

Where Ø is the Airy Tension Function, expressed in terms of r and θ.

2.1.4 Formulation in tensions

In the case of constant mass forces, the service by the stress function Ø(x,y) to the biharmonic equation

$$\nabla^2 \nabla^2 \emptyset = 0 \text{ ou } \nabla^4 = 0 \tag{18}$$

guarantees, within the hypotheses of plane state, the service to the balance and the compatibility of deformations, being $\emptyset$, in the case of zero mass forces, related to the tensions by

$$\sigma_x = \frac{\partial^2 \emptyset}{\partial y^2}; \qquad \sigma_y = \frac{\partial^2 \emptyset}{\partial x^2}; \qquad \tau_{xy} = -\frac{\partial^2 \emptyset}{\partial x \partial y} \tag{19}$$

with the use of polar coordinates, $\emptyset = \emptyset$ (r, θ); the Laplace operator turns,

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial x^2} = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} \tag{20}$$

therefore, the biharmonic equation Eq. (18) takes on the aspect:

$$\nabla^4 \phi = \nabla^2 \nabla^2 \phi = \left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}\right)\left(\frac{\partial^2 \phi}{\partial r^2} + \frac{1}{r}\frac{\partial \phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \phi}{\partial \theta^2}\right) = 0 \tag{21}$$

2.2 Airy Tension Function

Solving a problem of elasticity requires the integration of the differential equations of equilibrium, along with the compatibility equation and the boundary conditions. In polar coordinates, the resolution of two-dimensional problems occurs through the partial differential equation Eq. (18), which in the form of a series given by Michell (1899), is expressed as

$$\emptyset = \sum_{n=2}^{\infty} (a_{n1} r^n + a_{n1} r^{-n} + a_{n1} r^{2-n}) \cos(n\theta) + \sum_{n=2}^{\infty} (a_{n1} r^n + a_{n1} r^{-n} + a_{n1} r^{2-n}) sen(n\theta) \tag{22}$$

The boundary conditions in terms of stresses are not always sufficient for the determination of all the coefficients of this expression. Further investigations concerning displacements may be required.

3 METHODOLOGY

The present paper presents a problem of Elasticity Theory that is not commonly approached in the literature. It consists of a hollow tube subjected to internal pressure as well as internal shear, with boundary conditions of free radial displacement and impedance tangential displacement. The input data of the problem are shown in Figure 3.

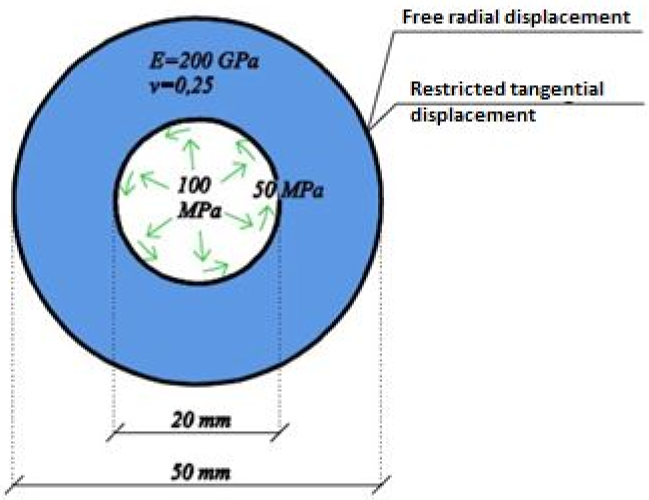

Figure 3: Input data of the problem

The methodology of this paper is composed of a linear static analysis through numerical simulations, through the Finite Element Method (MEF) and analytical formulations for such problem.

3.1 Formulation of the Analytical Solution

The analytical solution was performed by overlapping effects, separating it into two problems. The first consists of the individual analysis only of internal and external pressures, and the second only pure shear. At the end, both solutions are overlapped, being possible because it is an analysis in which the material is under a linear elastic regime. As shown in Figure 4. The origin of the polar coordinate system lies in the center of the cast disk.

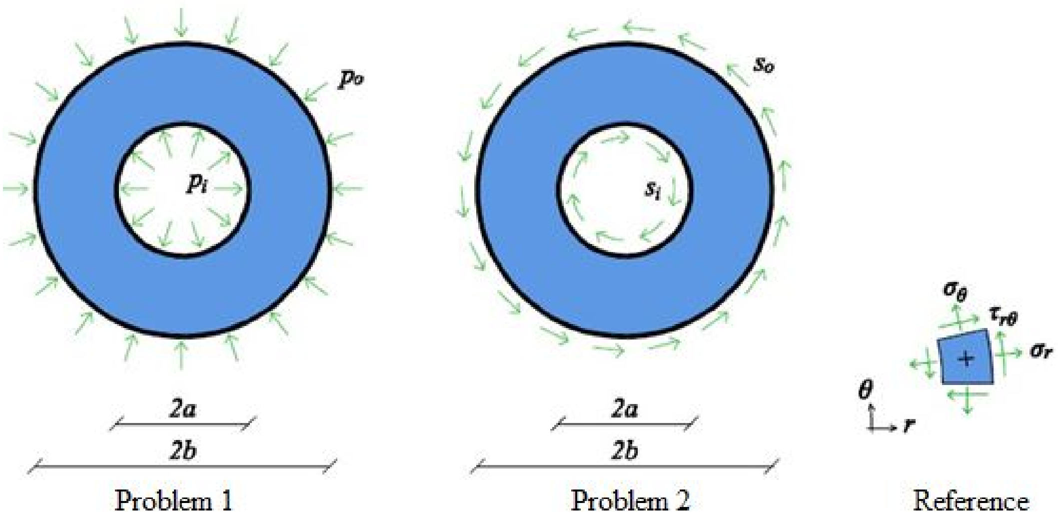

Figure 4: Separation of the problem into two parts

For problems with symmetry with respect to an axis, the stresses and, therefore, the Airy function depends only on the radius r. Represented by the following equation:

Proceedings of the XXXVIII Iberian Latin-American Congress on Computational Methods in Engineering
P.O. Faria, R.H. Lopez, L.F.F. Miguel, W.J.S. Gomes, M. Noronha (Editores), ABMEC, Florianópolis, SC, Brazil, November 5-8, 2017.

$$\left(\frac{d^2}{dr}+\frac{1}{r}\frac{d}{dr}\right)\left(\frac{d^2\phi}{dr^2}+\frac{1}{r}\frac{d\phi}{dr}\right)=\frac{d^4\phi}{dr^4}+\frac{2}{r}\frac{d^3\phi}{dr^3}-\frac{1}{r^2}\frac{d^2\phi}{dr^2}+\frac{1}{r^3}\frac{d\phi}{dr}=0 \qquad (23)$$

3.1.1 Problem 1 resolution

The solution for the tensions only in the radial direction departs with the following terms from the Airy Tension Function:

$$\phi = A_1\log r + A_2 r^2\log r + A_3 r^2 + A_4 \qquad (24)$$

Where A1, A2, A3 and A4 are integration constants, obtained with the boundary conditions. The resulting stresses are determined by substituting Eq. (14) and Eq. (23) in Eqs. (15), (16) and (17):

$$\sigma_r = \frac{A_1}{r^2} + A_2(1+2\log r) + 2A_3 \qquad (25)$$

$$\sigma_\theta = \frac{A_1}{r^2} + A_2(3+2\log r) + 2A_3 \qquad (26)$$

$$\tau_{r\theta} = 0 \qquad (27)$$

As the problem does not involve shear stresses and assuming that there are no volume forces, the radial equilibrium equation, Eq. (13), results in:

$$\frac{d\sigma_r}{dr} + \frac{\sigma_r-\sigma_\theta}{r} = 0 \qquad (28)$$

The equilibrium equation Eq. (14) is automatically satisfied because no tension changes in θ. Applying the strain-displacement relations Eqs. (10) and (11) to the problem, results:

$$\frac{d}{dr}\left[\frac{E}{1-v^2}\left(\frac{du}{dr}+v\frac{u}{r}\right)\right]+\frac{1}{r}\left[\frac{E}{1-v^2}\left(\frac{du}{dr}+v\frac{u}{r}-\frac{u}{r}-v\frac{du}{dr}\right)\right]=0$$

$$\frac{d^2u}{dr^2}+\frac{1}{r}\frac{du}{dr}-\frac{u}{r^2}=0 \qquad (29)$$

The solution for this ODE is:

$$u = C_1 r + \frac{C_2}{r} \qquad (30)$$

From this solution, the tensions result in:

$$\sigma_r = \frac{E}{1-v^2}\left[C_1(1+v) - C_2\left(\frac{1-v}{r^2}\right)\right] \qquad (31)$$

$$\sigma_\theta = \frac{E}{1-v^2}\left[C_1(1+v) + C_2\left(\frac{1-v}{r^2}\right)\right] \qquad (32)$$

To find constants C1 and C2, since there is no prescribed radial displacement, it is necessary to impose the known boundary conditions in terms of tension. If the internal radius from the tube is “a” and the external “b”, then

$$\sigma_r(a) = -p_i \quad e \quad \sigma_r(b) = -p_e$$

Where and p_e are the internal and external pressures.

$$\sigma_r = \frac{a^2p_i-b^2p_e}{b^2-a^2} - \frac{p_i-p_e}{r^2}\frac{a^2b^2}{b^2-a^2} \qquad (33)$$

$$\sigma_\theta = \frac{a^2p_i-b^2p_e}{b^2-a^2} - \frac{p_i-p_e}{r^2}\frac{a^2b^2}{b^2-a^2} \qquad (34)$$

With these tensile solutions, the resulting radial displacement is:

$$u = \frac{(1-v)}{E}\left(\frac{a^2p_i - b^2p_e}{b^2-a^2}\right)r + \frac{1+v}{E}\frac{(p_i-p_e)a^2b^2}{b^2-a^2}\frac{1}{r} \tag{35}$$

3.1.2 Problem 2 resolution

Differently from Problem 1, in Problem 2 there are components of shear and thus the term Z_θ must be inserted in the function of Airy so that the function $\tau_{r\theta}$ is not null.

$$\phi = A\log r + Br^2\log r + Cr^2 + D + Z\theta \tag{36}$$

$$\sigma_r = \frac{A}{r^2} + B(1 + 2\log r) + 2C \tag{37}$$

$$\sigma_\theta = -\frac{A}{r^2} + B(3 + 2\log r) + 2C \tag{38}$$

$$\tau_{r\theta} = \frac{Z}{r^2} \tag{39}$$

To determine the constant Z, the boundary condition $\tau_{r\theta} = s_i$ is applied. Therefore, $Z = s_i a^2$. Equating Eqs.(7), (8) and (10), (11), obtain the following first order ordinary equations:

$$u_r = \frac{1}{E}\left[\frac{-A(1+v)}{r} - Br(1+v) + 2Br\log r\,(1-v)2Cr(1-v)\right] + f(\theta) \tag{40}$$

$$u_\theta = \frac{4Br\theta}{E} - \int f(\theta)d\theta + f_1(r) \tag{41}$$

Where (θ) and $f_1(r)$ are determined by equating Eq. (9) to (12)

$$\frac{1}{r}\frac{d[f(\theta)]}{d\theta} + \frac{1}{r}\int f(\theta)\,d\theta + \frac{d[f_1(r)]}{dr} - \frac{f_1(r)}{r} = \frac{2(1+v)s_i a^2}{E}\frac{1}{r^2} \tag{42}$$

3.1.3 Effects Overlay

Thus, with the results obtained and from the overlap of effects, obtains the following field of tensions:

$$u_r = \frac{1}{E}\left[-\frac{a^2b^2(p_0-p_i)(1+v)}{(b^2-a^2)}\frac{1}{r} + \frac{(a^2p_i - b^2p_0)(1-v)}{(b^2-a^2)}r\right] \tag{43}$$

$$u_\theta = \frac{-(1+v)s_i a^2}{E}\frac{1}{r} + \left[\frac{u_{\theta,r=\rho}}{\rho} + \frac{(1+v)s_i a^2}{E}\frac{1}{\rho^2}\right]r \tag{44}$$

3.2 Numerical Modeling (FEM)

The modeling using numerical simulations using the Finite Element Method (MEF) was developed through ANSYS Workbench, version 18.0 computational tool.

4 RESULTS AND DISCUSSION

4.1 Analytical Results

In this section, the analytical results obtained through the effect superposition formulation are presented. For all θ, the tensions vary with r as shown in Figure 5.

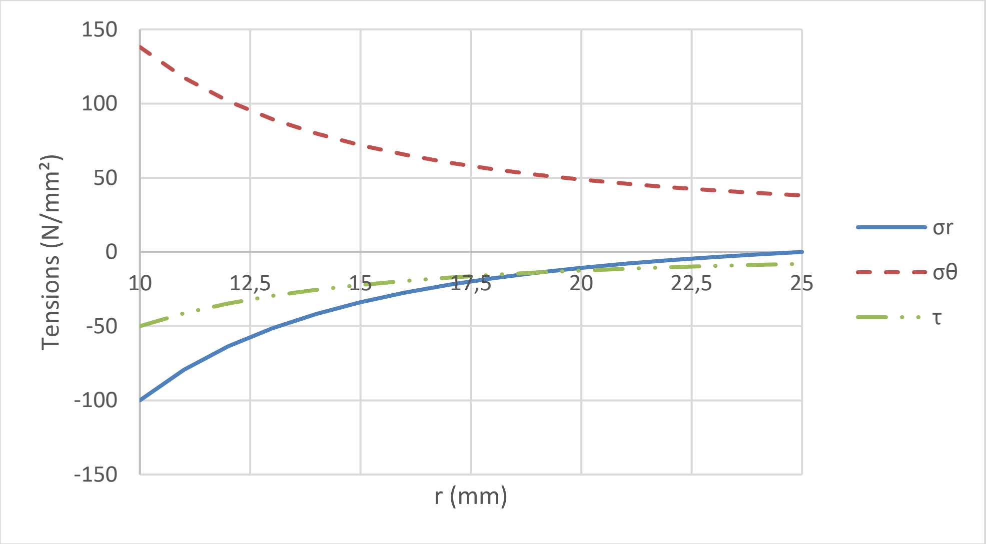

Figure 5: Tension field

Analogously, for all θ, the displacements vary with r as shown in Figure 6.

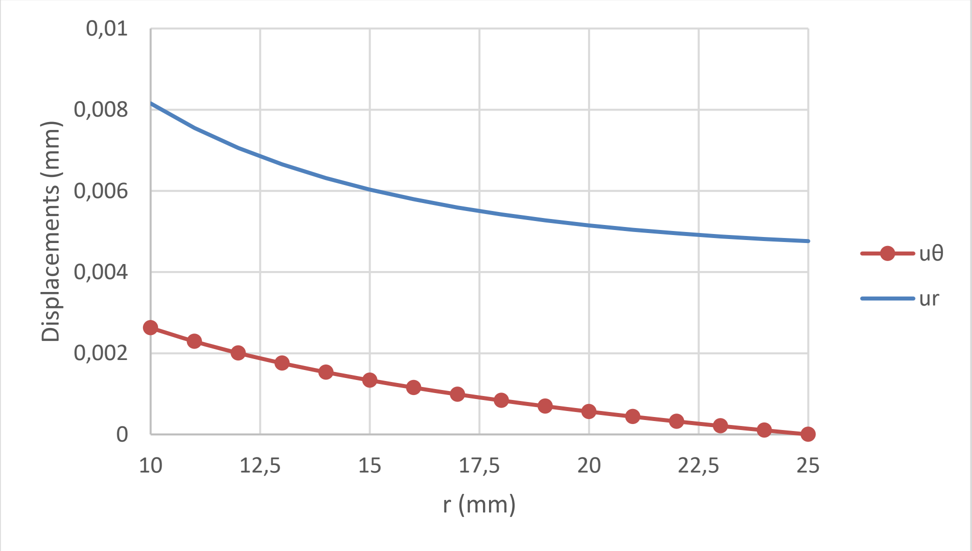

Figure 6: Displacement field

4.2 Numerical Results (FEM)

In this section it is presented, visually, the numerical results obtained by the FEM. The analyzes presented here were obtained for a mesh that presented a statistic of 28963 nodes, and 4039 elements. The processing time was 1 second for each exposed analysis. For radial normal stress σ_r, a maximum of approximately 0.007 MPa was obtained at the outer edge, and a minimum of 100.14 MPa negative at the inner edge, varying between the ends as shown in Figure 7.

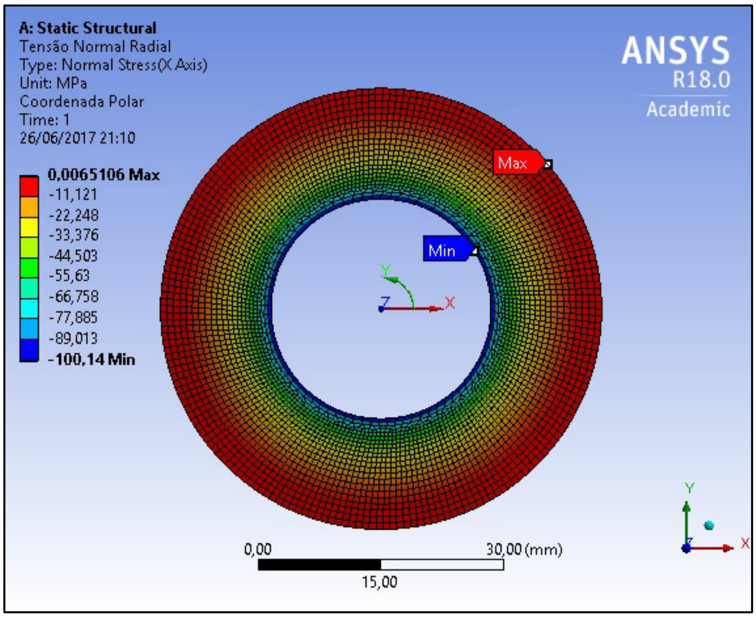

Figure 7: Radial normal stress

For the tangential normal stress σ_{θ}, a maximum of 138.22 MPa was obtained at the inner edge and a minimum of 38.979 MPa at the outer edge, varying between the ends as shown in Figure 8.

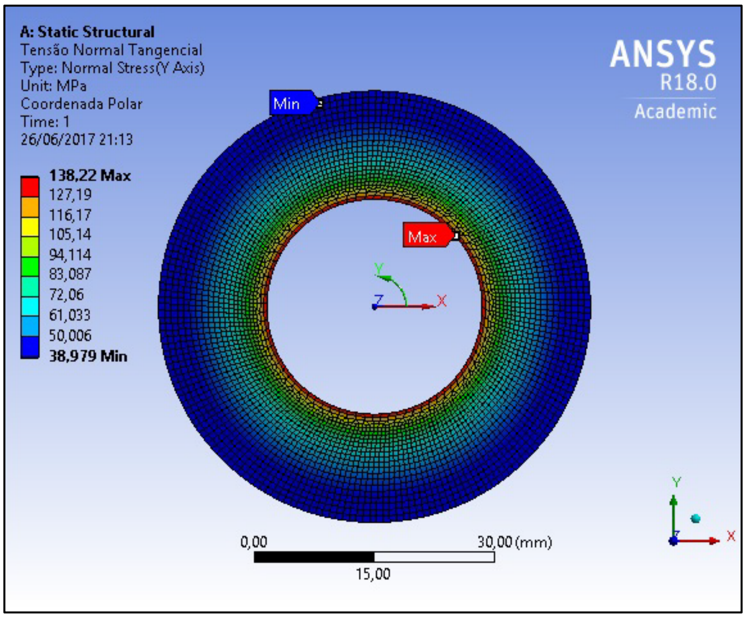

Figure 8: Tangential normal stress

For the shear stress $\sigma_{r\theta}$, a maximum of approximately 8 MPa negative, at the outer edge and a minimum of 50,149 MPa negative at the inner edge, ranging between the ends as shown in Figure 9.

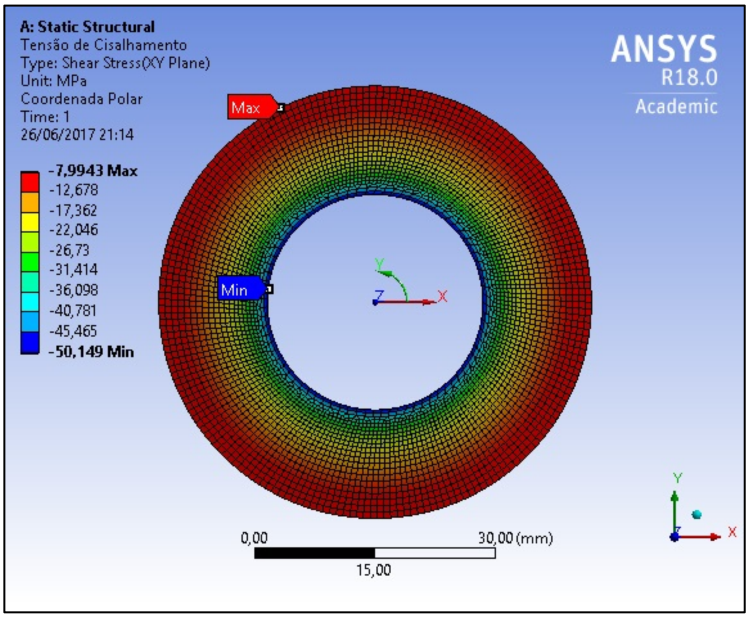

Figure 9: Shear stress

Regarding the displacement in the radial direction, the maximum value obtained was approximately 0.0085 mm at the inner edge. The variations of displacements along the section are shown in Figure 10.

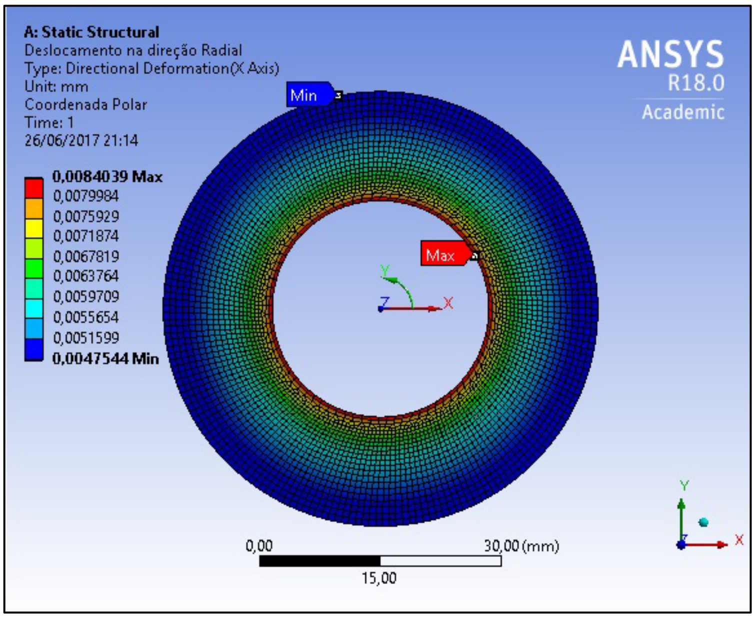

Figure 10: Radial displacement

Regarding the displacement in the tangential direction, the maximum displacement obtained was approximately 0.003 mm at the inner edge. The variations of displacements along the section are shown in Figure 11.

Proceedings of the XXXVIII Iberian Latin-American Congress on Computational Methods in Engineering
P.O. Faria, R.H. Lopez, L.F.F. Miguel, W.J.S. Gomes, M. Noronha (Editores), ABMEC, Florianópolis, SC, Brazil, November 5-8, 2017.

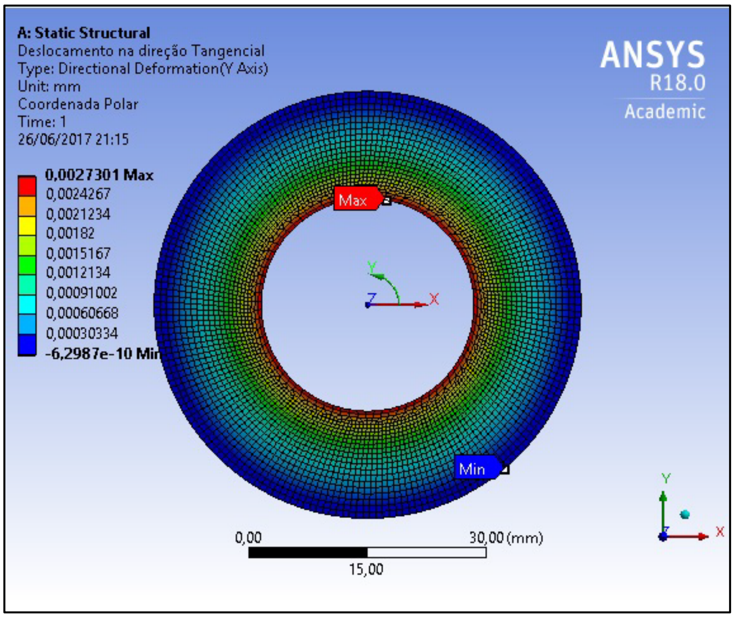

Figure 11: Tangential displacement

4.3 Analytical and Numerical Comparison

In order to compare the results, four points of the coordinate problem A (10,0), B (15,0), C (20,0) and D (25,0) were defined to compare the results obtained in each analysis.

These results are presented in Tables 1 and 2.

Table 1: Results of the tension field at the points A, B, C e D.

Points e coordinates r (mm), θ (rad)		σ_r (MPa)		σ_θ (MPa)		τ (MPa)	
		Analytical	FEM	Analytical	FEM	Analytical	FEM
A	r = 10 θ = 0	-100,00	-100,14	138,09	138,22	-50,00	-50,15
B	r = 15 θ = 0	-33,85	-33,78	71,95	71,86	-22,21	-22,22
C	r = 20 θ = 0	-10,70	-10,73	48,80	48,66	-12,49	-12,49
D	r = 25 θ = 0	0,00	0,007	38,09	38,98	-7,99	-7,99

Proceedings of the XXXVIII Iberian Latin-American Congress on Computational Methods in Engineering
P.O. Faria, R.H. Lopez, L.F.F. Miguel, W.J.S. Gomes, M. Noronha (Editores), ABMEC, Florianópolis, SC, Brazil, November 5-8, 2017.

Table 2: Results of the displacement field at points A, B, C and D

Points e coordinates r (mm), θ (rad)		u_r (10^{-3} mm)		u_θ (10^{-3} mm)	
		Analytical	FEM	Analytical	FEM
A	r = 10 θ = 0	8,15	8,40	2,63	2,73
B	r = 15 θ = 0	6,03	6,08	1,33	1,40
C	r = 20 θ = 0	5,15	5,16	0,56	0,57
D	r = 25 θ = 0	4,76	4,75	0,00	0,00

After analysis, these were well approximated. For FEM this approach was due to the level of refinement of the mesh, where for greater precision numerical results would require an increase in the number of elements of the mesh, which would increase the processing time.

Thus, for engineering purposes and the purpose of this work, the numerical results obtained are considered valid.

5 FINAL CONSIDERATIONS

By using a reference system based on polar coordinates followed by a set of symmetrical aspects of the tube, it is possible to observe a facility in the analytical formulation used in this problem of elasticity, through the superposition of effects. One in which only the internal pressure acts and another in which the shear performance is preponderant.

It was also verified that the bi-harmonic equation is fundamental in solving the problem of elasticity.

In the sequence, it is apparent that the analytical solution confronted with the numerical solution in finite elements, are quite approximate. This approximation was due to the use of the element mesh used by ANSYS. The differences between the analytical and numerical values are insignificant, which makes the result valid for practical purposes.

Therefore, it is possible to state that the analytical solution is harder comparing to the numerical solution. However, the latter requires the user's sensitivity, critical sense and experience to evaluate the results obtained by these powerful tools.

6 ACKNOWLEDGMENT

The authors thank CAPES and CNPq for the resources received and the University of Brasília.

7 REFERENCES

Alves, L.M. *Introdução a Mecânica do Contínuo:* uma abordagem moderna. 2001. 329 f. Notas de Aula (Introdução a Mecânica do Contínuo) – Departamento de Engenharia Civil, Universidade do Paraná, Curitiba, 2001.

Michell, J. H. *Proc. London MAth. Soc*. v. 31. p. 100. Londres, Inglaterra, 1899.

Hibbeler, R.C. Resistência dos Materiais, Quinta Edição, Livros Técnicos e Científicos Editora, Rio de Janeiro, 2004.

Timoshenko, S. P.; Goodier, S.P. *Teoria da Elasticidade*. ed. 3. Traduzida. Guanabara Dois. Rio de Janeiro, Brasil, 1980.