VIBRATION OF BUCKLED COMPOSITE THIN-WALLED BEAMS

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Abstract. *In this work, a generalization of the Timoshenko-Vlasov theory for composite beam of arbitrary cross-sectional shapes and taking into account shear, warping and geometrically non-linear effects is developed. This model is employed to analyze the vibrations of small amplitude of buckled composite thin-walled beams.*

The corresponding equations of motion are first used to describe the non-linear static behavior. Therefore, the dynamic equations with respect to the buckled configuration are obtained. These last ones can be solved for determining the natural frequencies and modal shapes. Numerical examples are given to show the potential of the present model.

Keywords: *Vibration, Buckling, Post-buckling, Initial Stresses.*

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INTRODUCTION

Structural members made of composites are increasingly used in aeronautical, mechanical and civil engineering applications where high strength and stiffness and low weight are of primary importance. Other advantages that motivate some applications are corrosion resistance, enhanced fatigue life, low thermal expansion, etc. (Barbero, 1999). Many structural members made of composites have the form of thin-walled beams. Accordingly, a significant amount of research has been conducted in recent years toward the development of theoretical and computational methods for analyzing the structural behavior of such members.

The structural analysis of isotropic thin-walled open beams is appropriately performed by means of Vlasov's theory. This theory considers the warping effect that is of great importance in this type of structures (Vlasov, 1961).

Vlasov's theory was extended to composites by several authors (Bauld and Tzeng, 1984; Ghorbanpoor and Omidvar, 1996; Estivalezes and Barrau, 1998). However, the above-mentioned works do not take into account the influence of the shear flexibility on the dynamics of the member.

For the case of composite thin-walled beams, the shear effect may be very important owing to the high value of the ratio between the longitudinal elasticity modulus and the transverse elasticity modulus (Sherbourne and Kabir, 1995; Godoy et al., 1995).

Cortínez and Piovan (2002) developed a theoretical model for the dynamic analysis of composite (symmetric balanced laminates and especially orthotropic laminates) thin-walled beams with initial stresses. This model takes into account, in a full form, the shear flexibility (bending and warping shear). This model is valid for open or one-cell closed cross section.

Recently, the authors (Cortínez et al., 2017, Dominguez et al., 2017) have developed an extension of this model in order to take into account completely arbitrary cross-sectional shapes, open, multi-cell, combined closed and open, thin or thick-walled sections. Here, this theory is mentioned as Timoshenko-Vlasov model.

Slender thin-walled structures employed in several technological situations present non-linear responses in operation conditions. Accordingly, the understanding of different aspects of the non-linear dynamics, such as large amplitudes, natural frequencies dependent of amplitudes or dynamic stability, is of great importance.

In particular, the dynamic behavior of buckled beams has attracted the attention of several researchers. In this connection, there exist interesting applications of post-buckled beams in vibration control devices.

In other cases, buckling is an undesired state related with anomalous operation conditions. In order to identify post-buckled states for beam located, in non-accessible places, measurements of natural frequencies may be useful because these values may be correlated with the amplitude of post-buckling displacements.

In this work, the vibrations of small amplitude of buckled composite thin-walled beams are analyzed. In order to do this, an extension of the Timoshenko-Vlasov theory for considering geometrically non-linear effects is developed.

The present theory can be considered as a generalization of that approach developed by Mohri et al. (Mohri, 2004) for analyzing vibration of buckled isotropic thin-walled beams.

The corresponding equations of motion are first used to analyze the non-linear static behavior. Therefore, the dynamic equations with respect to the buckled configuration are obtained. These last ones can be solved for determining the natural frequencies and modal shapes.

Some numerical examples are given in order to show the possibilities of the proposed model.

1 STRUCTURAL MODEL

The structural system shown in Fig. 1 is considered. It is a non-homogeneous beam of arbitrary cross section (open, closed, open-closed, multi-cell, thin or thick-walled). They can present thin walls made of specially orthotropic or symmetric balanced composite materials.

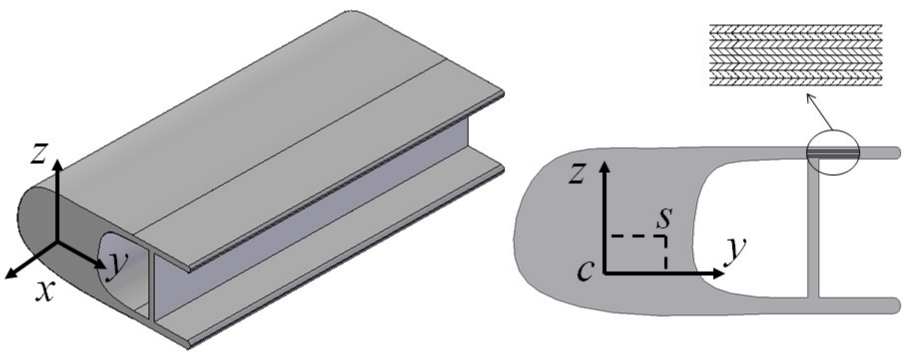

Figure 1. Structural model

1.1 Variational formulation

The dynamics of the analyzed structure may be formulated by means of the principle of Virtual Works as follows:

$$\int_V \left(\sigma_x \delta\varepsilon_x + 2\tau_{xy}\delta\varepsilon_{xy} + 2\tau_{xz}\delta\varepsilon_{xz} + f_x\delta u + f_y\delta v + f_z\delta w\right)dV + \int_S \left(T_x\delta u + T_y\delta v + T_z\delta w\right)dS = 0 \quad (1)$$

where $\sigma_x, \tau_{xy}, \tau_{xz}$ are the non-null components of the stress tensor, ε_x, $\varepsilon_{xy}, \varepsilon_{xz}$ are the corresponding components of the Green-Lagrange strain tensor, *u*, *v* and *w* are the displacements in *x*, y, *z* directions, respectively, f_x, f_y, f_z are components of volumetric forces

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(including inertial forces), and T_x, T_y, T_z are components of the stress vector applied at the surfaces.

The Green-Lagrange tensor components are defined in terms of displacements, as follows:

$$\varepsilon_x = \frac{\partial u}{\partial x} + \frac{1}{2}\left(\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2 + \left(\frac{\partial w}{\partial x}\right)^2\right)$$

$$\varepsilon_{xy} = \frac{1}{2}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} + \frac{\partial u}{\partial x}\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\frac{\partial v}{\partial y} + \frac{\partial w}{\partial x}\frac{\partial w}{\partial y}\right) \tag{2}$$

$$\varepsilon_{xz} = \frac{1}{2}\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} + \frac{\partial u}{\partial x}\frac{\partial u}{\partial z} + \frac{\partial v}{\partial x}\frac{\partial v}{\partial z} + \frac{\partial w}{\partial x}\frac{\partial w}{\partial z}\right)$$

The displacement field is assumed to be given by:

$$u \cong u_0 - \theta_z y - \theta_y z + \theta\omega + \phi\theta_z \tilde{z} + \phi\theta_y \tilde{y} - \omega\frac{1}{2}\left(\theta_z\theta_y^{'} - \theta_y\theta_z^{'}\right)$$

$$v \cong v_s - \phi\tilde{z} + \frac{1}{2}\left(-\phi^2\tilde{y} - \theta_z^2 y - \theta_z\theta_y z\right)$$

$$w \cong w_s + \phi\tilde{y} - \frac{1}{2}\left(-\phi^2\tilde{z} - \theta_y^2 z - \theta_z\theta_y y\right) \tag{3}$$

$$\begin{cases} \tilde{y} = y - y_s \\ \tilde{z} = z - z_s \end{cases}$$

They are based on the concept of semi-tangential rotation defined by Argyris (1982). In these expressions u_0, v_s, w_s, θ_y, θ_z, ϕ and θ may be understood as the infinitesimal generalized displacements. That is to say, they represent the axial displacement in *x*-longitudinal direction, transverse displacements, in *y*- and *z*-directions, bending rotations with respect to y- and z- axes, twisting rotation and a measure of the intensity of warping along the beam, respectively, when non-linear terms are neglecting. On the other hand, ω is the Saint-Venant torsional warping function, and (y_s, z_s) correspond to the shear center (Cortínez y Dominguez, 2017).

Substituting expressions (3) into (2) one obtains:

$$\varepsilon_x = u_0' - \left(\theta_z' - \theta_y'\phi\right)y - \left(\theta_y' + \theta_z'\phi\right)z + \left(\theta' - \frac{1}{2}\left(\theta_z\theta_y'' - \theta_y\theta_z''\right)\right)\omega + \frac{1}{2}\left(v_s'^2 + w_s'^2 + \left(\tilde{y}^2 + \tilde{z}^2\right)\phi'^2\right) + y_s\theta_y'\phi - z_s\theta_z'\phi$$

$$\varepsilon_{xy} = \frac{1}{2}\left(\left(v_s' - \theta_z + z_s\frac{1}{2}\left(\theta_z\theta_y' - \theta_y\theta_z'\right)\right) + \left(\phi' - \frac{1}{2}\left(\theta_z\theta_y' - \theta_y\theta_z'\right)\right)\left(\frac{\partial\omega}{\partial y} - \tilde{z}\right) - \frac{\partial\omega}{\partial y}(\phi' - \theta)\right) \tag{4}$$

$$\varepsilon_{xz} = \frac{1}{2}\left(\left(w_s' - \theta_y - y_s\frac{1}{2}\left(\theta_z\theta_y' - \theta_y\theta_z'\right)\right) + \left(\phi' - \frac{1}{2}\left(\theta_z\theta_y' - \theta_y\theta_z'\right)\right)\left(\frac{\partial\omega}{\partial z} + \tilde{y}\right) - \frac{\partial\omega}{\partial z}(\phi' - \theta)\right)$$

Substituting now the above expressions into the Principle of Virtual Work (Eq. 1) and integrating over the cross section one obtains the following one dimensional expression:

$$\begin{aligned}
&\int_0^L \Bigg(N\delta u_0' + \left(-M_z + \left(M_y - Nz_s\right)\phi + \frac{1}{2}\left(Q_z y_s - Q_y z_s + T_{sv}\right)\theta_y \right)\delta\theta_z' + \delta v_s'\left(Q_y + v_s' N\right) + \\
&\delta w_s'\left(Q_z + w_s' N\right) + \left(-M_y - \left(M_z - Ny_s\right)\phi + \frac{1}{2}\left(-Q_z y_s + Q_y z_s - T_{sv}\right)\theta_z \right)\delta\theta_y' - \delta\theta_y'\frac{1}{2}B\theta_z + \\
&\delta\theta_z\left(-Q_y + \frac{1}{2}\left(-Q_z y_s + Q_y z_s - T_{sv}\right)\theta_y' - \frac{1}{2}B\theta_y'' \right) + \delta\theta_y\left(-Q_z + \frac{1}{2}\left(Q_z y_s - Q_y z_s + T_{sv}\right)\theta_z' + \frac{1}{2}B\theta_z'' \right) + \\
&\delta\theta_z'\frac{1}{2}B\theta_y + \delta\phi\left(\left(M_y - Nz_s\right)\theta_z' - \left(M_z - Ny_s\right)\theta_y'\right) + \delta\phi'\left(T_{sv} + T_w + B_1\phi'\right) + \delta\theta' B - \delta\theta T_w \Bigg) dx = \\
&= -\int_0^L \left(\ddot{u}_0\overline{\rho A}\delta u_0 + \ddot{\theta}_z\overline{\rho I_z}\delta\theta_z + \ddot{\theta}_y\overline{\rho I_y}\delta\theta_y + \ddot{\theta}\overline{\rho C_W}\delta\theta + \ddot{v}_s\overline{\rho A}\delta v_s + \ddot{w}_s\overline{\rho A}\delta w_s + \ddot{\phi}\overline{\rho I_s}\delta\phi \right) dx + \\
&+\int_0^L \left(q_x\delta u_0 - m_z\delta\theta_z - m_y\delta\theta_y + b\delta\theta + q_y\delta v_s + q_z\delta w_s + m_x\delta\phi \right) dx + \\
&\Bigg[\left(-\bar{N}\delta u_0 - \left(-\bar{M}_z + \left(\bar{M}_y - \bar{N}z_s\right)\phi \right)\delta\theta_z - \delta v_s\bar{Q}_y - \delta w_s\bar{Q}_z - \left(-\bar{M}_y - \left(\bar{M}_z - \bar{N}y_s\right)\phi \right)\delta\theta_y + \right. \\
&\left. - \; \delta\phi\left(\bar{M}_x + \left(\bar{M}_y - \bar{N}z_s\right)\theta_z - \left(\bar{M}_z - \bar{N}y_s\right)\theta_y \right) \right) \Bigg]_0^L
\end{aligned} \tag{5}$$

where the following definitions have been used:

$$\begin{aligned}
&\overline{\rho A} = \int_A \rho dA, \quad \overline{\rho C_w} = \int_A \rho\omega^2 dA, \quad \overline{\rho I_s} = \int_A \rho\left(\tilde{z}^2 + \tilde{y}^2\right)dA, \\
&\overline{\rho I_z} = \int_A \rho y^2 dA, \; \overline{\rho I_y} = \int_A \rho z^2 dA
\end{aligned} \tag{6}$$

$$N=\int_A \sigma_x dA,\ \ M_y=\int_A \sigma_x z dA,\ \ M_z=\int_A \sigma_x y dA,\ B_1=\int_A \sigma_x\left(\tilde{y}^2+\tilde{z}^2\right)dA,$$
$$B=\int_A \sigma_x \omega dA,\quad Q_y=\int_A \tau_{xy} dA,\quad Q_z=\int_A \tau_{xz} dA, \tag{7}$$
$$T_{sv}=\int_A\left(\tau_{xy}\left(\frac{\partial\omega}{\partial y}-\tilde{z}\right)+\tau_{xz}\left(\frac{\partial\omega}{\partial z}+\tilde{y}\right)\right)dA,\quad T_w=-\int_A\left(\tau_{xy}\frac{\partial\omega}{\partial y}+\tau_{xz}\frac{\partial\omega}{\partial z}\right)dA$$

$$q_x=\int_A f_x dA,\ \ q_y=\int_A f_y dA,\ \ q_z=\int_A f_z dA,\ b=\int_A f_x\omega dA$$
$$m_y=\int_A f_x z dA,\ m_z=\int_A f_z y dA,\ \ m_x=\int_A\left(-f_y\tilde{z}+f_z\tilde{y}\right)dA \tag{8}$$

Here, $q_x(x,t)$, $q_y(x,t)$ and $q_z(x,t)$ are distributed forces per unit length, $m_x(x,t)$, $m_y(x,t)$, $m_z(x,t)$ are external moments per unit length and b is the external bimoment per unit length, $\bar{N},\bar{M}_y,\bar{M}_z,\bar{Q}_y,\bar{Q}_z$ are external sectional stress resultants applied at the ends of the beam, N, M_y, M_z, B, Q_y, Q_z, T_{sv} and T_w are the sectional forces and $\overline{\rho A}$, $\overline{\rho I_z}$, $\overline{\rho I_y}$, $\overline{\rho C_w}$, $\overline{\rho I_s}$ are inertia sectional coefficients.

1.2 Constitutive equations for the sectional stress resultants

Substituting the strain components given by Eq. (4) into the Hooke's Law, $\sigma_x=E\varepsilon_x$, $\tau_{xy}=2G\varepsilon_{xy}$, $\tau_{xz}=2G\varepsilon_{xz}$, and the results into the definitions of the stress resultants (Eq. 7) one obtains the following constitutive equations:

$$N=\overline{EA}\left(u_0'+\frac{1}{2}\left(v_s'^2+w_s'^2\right)+\phi\left(y_s\theta_y'-z_s\theta_z'\right)\right)+\overline{EI_0}\frac{\phi'^2}{2}$$
$$M_y=-\overline{EI_y}\left(\theta_y'+\theta_z'\phi\right)$$
$$M_z=-\overline{EI_z}\left(\theta_z'-\theta_y'\phi\right)$$
$$B=\overline{EC_w}\left(\theta'-\frac{1}{2}\left(\theta_z\theta_y''-\theta_y\theta_z''\right)\right) \tag{9}$$
$$Q_y=\overline{GA_y}\left(v_s'-\theta_z+z_s\frac{1}{2}\left(\theta_z\theta_y'-\theta_y\theta_z'\right)\right)+\overline{GA_{yz}}\left(w_s'-\theta_y-y_s\frac{1}{2}\left(\theta_z\theta_y'-\theta_y\theta_z'\right)\right)$$
$$Q_z=\overline{GA_{yz}}\left(w_s'-\theta_y-y_s\frac{1}{2}\left(\theta_z\theta_y'-\theta_y\theta_z'\right)\right)+\overline{GA_z}\left(v_s'-\theta_z+z_s\frac{1}{2}\left(\theta_z\theta_y'-\theta_y\theta_z'\right)\right)$$

$$T_{sv} = \overline{GJ}\left(\phi' - \frac{1}{2}\left(\theta_z\theta_y' - \theta_y\theta_z'\right)\right)$$

$$T_w = \overline{GD_\omega}\left(\phi' - \theta\right) \tag{9}$$

$$B_1 = \overline{EI_0}\left(u_0' + \frac{1}{2}\left(v_s'^2 + w_s'^2\right) + \phi\left(y_s\theta_y' - z_s\theta_z'\right)\right) + \overline{EI_R}\frac{\phi'^2}{2}$$

where the sectional stiffness coefficients $\overline{EA}, \overline{EI_y}, \overline{EI_z}, \overline{EC_w}, \overline{GA_y}, \overline{GA_{yz}}, \overline{GA_z}, \overline{GJ}$ y $\overline{GD_\omega}$ are defined as:

$$\overline{EA} = \int_A E dA,\ \overline{EI_0} = \int_A E\left(\tilde{y}^2 + \tilde{z}^2\right)dA,\ \overline{EI_y} = \int_A Ez^2 dA,\ \overline{EI_z} = \int_A Ey^2 dA,$$

$$\overline{EC_w} = \int_A E\omega^2 dA,\ \overline{GA_y} = \overline{GA_z} = \int_A G dA,\ \overline{GA_{yz}} = 0,\ \overline{EI_R} = \int_A E\left(\tilde{y}^2 + \tilde{z}^2\right)^2 dA, \tag{10}$$

$$\overline{GJ} = \int_A G\left(\tilde{y}^2 + \tilde{z}^2 - \tilde{z}\frac{\partial\omega}{\partial y} + \tilde{y}\frac{\partial\omega}{\partial z}\right)dA,\ \overline{GD_\omega} = \int_A G\left(\left(\frac{\partial\omega}{\partial y}\right)^2 + \left(\frac{\partial\omega}{\partial z}\right)^2\right)dA$$

In Cortínez et al. (2016) the authors have developed improved expressions for $\overline{GA_y}$, $\overline{GA_z}$, $\overline{GA_{yz}}$, $\overline{GD_\omega}$ and $\overline{GJ}$ by formulating the theory from the Hellinger-Reissner Principle. This approach leads to a generalization of the concept of shear factors used in the Timoshenko bending theory. To obtain the present numerical results, these improved expressions have been used.

It has be noted that the above equations have been derived assuming an isotropic material. However, the authors have shown in references (Cortínez and Dominguez, 2017; Dominguez and Cortínez, 2017) that the mathematical expressions remain valid even for specially orthotropic or symmetric balanced composite materials if the following equivalent elasticity moduli is used.

$$E^* = \overline{A}_{11}/t,\quad G^* = \overline{A}_{66}/t \tag{11}$$

where $\overline{A}_{11}$ and $\overline{A}_{66}$ are extensional stiffnesses of the laminates (Barbero, 1998) and t is the wall thickness. Moreover, the approach to recover the stress components in the composite materials, if desired, is explained in Cortínez and Dominguez (2017). A particular case of the present approach for open or single-cell closed sections is the model developed by Machado et al. (2007).

It is important to mention that the formulation requires that cross-section remains unchanged after buckling, thus not addressing local buckling. This last aspect may be considered in an approximate form as indicated in reference Cortínez and Dominguez (2017).

1.3 Differential equations of motion in the case of a beam subjected to an axial force

The general equations in variational form are given by Eq. (5) and Eq. (9). Applying variational calculus, it is possible to obtain the differential form of the governing equations. As an example, this form is written below for the special case of a beam subjected to an axial force.

$$
\begin{aligned}
& -\frac{\partial N}{\partial x}+\overline{\rho A}\ddot{u}_0 = q_x \\
& \frac{\partial M_z}{\partial x}-Q_y+Nz_s\frac{\partial \phi}{\partial x}+\overline{\rho I_z}\ddot{\theta}_z = -m_z \\
& \frac{\partial M_y}{\partial x}-Q_z-Ny_s\frac{\partial \phi}{\partial x}+\overline{\rho I_y}\ddot{\theta}_y = -m_y \\
& -\frac{\partial Q_y}{\partial x}-\frac{\partial}{\partial x}\left(N\frac{\partial v}{\partial x}\right)+\overline{\rho A}\left(\ddot{v}_s+z_s\ddot{\phi}\right)=q_y \\
& -\frac{\partial Q_z}{\partial x}-\frac{\partial}{\partial x}\left(N\frac{\partial w}{\partial x}\right)+\overline{\rho A}\left(\ddot{w}_s-y_s\ddot{\phi}\right)=q_z \\
& -\frac{\partial B}{\partial x}-T_w+\overline{\rho C_w}\ddot{\theta}=b \\
& -\frac{\partial (T_{sv}+T_w)}{\partial x}+N\left(-z_s\frac{\partial \theta_z}{\partial x}+y_s\frac{\partial \theta_y}{\partial x}+\frac{\overline{EI_0}}{\overline{\overline{EA}}}\frac{\partial^2\phi}{\partial x^2}\right)+\overline{\rho A}\left(-y_s\ddot{w}_s+z_s\ddot{v}_s+\frac{\overline{\rho I_s}}{\overline{\overline{\rho A}}}\ddot{\phi}\right)=m_x
\end{aligned}
\tag{12}
$$

subject to:

$$
\begin{aligned}
& N-\bar{N}=0 && or && \delta u_0 = 0 \\
& -M_z = 0 && or && \delta\theta_z = 0 \\
& -M_y = 0 && or && \delta\theta_y = 0 \\
& Q_z+N\frac{\partial w}{\partial x}=0 && or && \delta w = 0 \\
& Q_y+N\frac{\partial v}{\partial x}=0 && or && \delta v = 0 \\
& B=0 && or && \delta\theta = 0 \\
& T_w+T_{sv}-\bar{N}\left(z_s\theta_z-y_s\theta_y-\frac{\overline{EI_0}}{\overline{\overline{EA}}}\phi'\right)=0 && or && \delta\phi = 0
\end{aligned}
\tag{13}
$$

2 SMALL VIBRATION AROUND A NON-LINEAR STATIC EQUILIBRIUM CONFIGURATION

In order to describe a small vibration around a non-linear static equilibrium configuration, the displacement vector $\mathbf{U}=\{u_0, v_s, w_s, \phi, \theta_z, \theta_y, \theta\}^T$ may be expressed in the following form:

$$\mathbf{U}=\mathbf{U}_e(x)+\mathbf{U}_d(x,t) \tag{14}$$

where $\mathbf{U}_e$ and $\mathbf{U}_d$ are static and dynamic components of displacements, respectively. The non-linear static equilibrium equations are given by Eq. (5) and Eq. (9) in a variational form (or equivalently they can be expressed in a differential form as in Eq. 12). These governing equations may be symbolically written as:

$$\mathbf{E}\left[\mathbf{U}_e\right]=0 \tag{15}$$

Once determined $\mathbf{U}_e$, the equations describing small vibrations around the non-linear equilibrium configuration may be obtained by substituting Eq. (14) into Eq. (5) and Eq. (9) and neglecting non-linear terms in $\mathbf{U}_d$. That is to say, $\mathbf{U}_d$ is assumed to be infinitesimal. The corresponding linear equations may be symbolically written as:

$$\tilde{\mathbf{E}}\left[\mathbf{U}_d, \mathbf{U}_e\right]=0 \tag{16}$$

3 VIBRATION OF POST-BUCKLED I-BEAMS

As an example of this theory, in the present section the particular case of an I-beam subjected to an axial displacement at the centroid is considered.

If the I-beam is fixed at the left end, and an axial displacement u_L is imposed at the right end, an axial force will be generated. In this way, the governing equations will given by Eq. (12) with y_s=z_s=0. It should to be noted that the beam suffers just axial displacements while $|N|<|N_{cr}|$. After reached the critical value, a transverse displacement will appear in the direction of minimum stiffness (for example in y-direction). In order to describe this situation for the static case the first of Eq. (12) is solved arriving at:

$$N=\frac{\overline{EA}}{L}\left(u_L+\frac{1}{2}\int_0^L v_{se}'^{\,2}dx\right) \tag{17}$$

Transverse motion in y-direction is governed by the second and fourth of Eq. (12). That is to say:

$$\begin{cases} \overline{GA_y}\left(v_s'' - \theta_z'\right) + \left(N v_s'\right)' = \overline{\rho A}\ddot{v}_s \\ \overline{EI_z}\theta_z'' - \overline{GA_y}\left(v_s' - \theta_z\right) = \overline{\rho I_z}\ddot{\theta}_z \end{cases} \tag{18}$$

First, the static transverse displacements are considered by neglecting the acceleration terms in the above equation:

$$\begin{cases} \overline{GA_y}\left(\bar{v}_{se}'' - \theta_z'\right) + \left(N_{cr}\bar{v}_{se}'\right)' = 0 \\ \overline{EI_z}\bar{\theta}_z'' - \overline{GA_y}\left(\bar{v}_{se}' - \bar{\theta}_z\right) = 0 \end{cases} \tag{19}$$

These equations along with the corresponding homogeneous boundary conditions describe the buckling problem. Their solution provides the critical load N_{cr} and the buckling shape. Then, the critical axial displacement at the right end is given by:

$$u_{cr} = \frac{N_{cr} L}{\overline{EA}} \tag{20}$$

It has to be noted that Eq. (19) is valid in the pre and post-buckling state. In this way, the post-buckling transverse displacement can be expressed as:

$$v_{se} = C\bar{v}_{se} \tag{21}$$

After substituting this expression into Eq. (17) for the critical load, one may obtain the amplitude of the post-buckled displacement as:

$$C = \sqrt{\frac{2\left(\frac{N_{cr} L}{\overline{EA}} - u_L\right)}{\int_0^L \bar{v}_{se}'^{\,2} dx}} \tag{22}$$

For the motion in the *x-y* plane, the small vibrations should be separated into the pre or post-buckling state. In the former the natural frequencies and modal shapes can be determined from the solution of Eq. (18) assuming harmonic displacements for v_s and θ_z, and considering the pre-buckling axial load given by Eq. (17) (with v_s=0). In the post-buckling regime N=N_{cr} and the transverse displacements are given by the sum of the static and dynamic parts. Substituting this form of displacements into Eq. (18) and considering the dynamic

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displacements to be infinitesimal, one obtains after neglecting higher order non-linear terms, the following equations:

$$\begin{cases} \overline{GA_y}\left(v_{sd}^{''}-\theta_{zd}^{'}\right)+\left(N_{cr}v_{sd}^{'}\right)^{'}+\overline{\rho A}\omega^2 v_{sd}=\Gamma v_{se}^{''} \\ \overline{EI_z}\theta_{zd}^{''}-\overline{GA_y}\left(v_{sd}^{'}-\theta_{zd}\right)+\overline{\rho I_z}\omega^2\theta_{zd}=0 \end{cases} \tag{23}$$

where:

$$\Gamma=\frac{\overline{EA}}{L}\int_0^L v_{se}^{'}v_{sd}^{'}dx \tag{24}$$

One can observe that expressions (23-24) constitute integral-differential equations.

The corresponding bending motion in the *x-z* plane and the torsional motion are governed by the third, fifth, sixth and seventh of the Eq. (12), without external loadings and with $N=N_{cr}$. It should to be noted that this critical axial load corresponds to the buckling in the *x-y* plane. That is to say, *x-z* and torsional motions are in a pre-buckling state. According to the above explanation the corresponding natural frequencies and modal shapes may be determined by solving the following equations:

$$\begin{cases} \overline{GA_z}\left(w_{sd}^{''}-\theta_{yd}^{'}\right)+\left(N_{cr}w_{sd}^{'}\right)^{'}+\overline{\rho A}\omega^2 w_{sd}=0 \\ \overline{EI_y}\theta_{yd}^{''}-\overline{GA_z}\left(w_{sd}^{'}-\theta_{yd}\right)+\overline{\rho I_y}\omega^2\theta_{yd}=0 \\ \overline{GD_\omega}\left(\phi_d^{''}-\theta_d^{'}\right)+\overline{GJ}\phi_d^{''}-N_{cr}I_1\phi_d^{''}+\overline{\rho I_s}\omega^2\phi_d=0 \\ \overline{EC_w}\theta_d^{''}-\overline{GD_\omega}\left(\phi_d^{'}-\theta_d\right)+\overline{\rho C_w}\omega^2\theta_d=0 \end{cases} \tag{25}$$

4 NUMERICAL EXAMPLES

The post-buckling and vibration of a clamped-hinged I-beam is considered ($u_0=v_s=w_s=\theta_y=\theta_z=\phi=0$ at the left end, and $u=u_L, v_s=w_s=\phi=M_y=M_z=B=0$ at the right end). It has a length L=6 m and the following cross-sectional dimensions (Fig. 2): b=0.2 m, h=0.4 m y t=0.01 m, and is made of a graphite-epoxy (AS4/3501-6) composite laminate $\left[0_\alpha/90_\beta/\pm 45_\gamma\right]_s$ with the following characteristics: $E_1=142GPa$, $E_2=10.2GPa$, $G_{12}=7.2GPa$, $\upsilon_{12}=0.27$, $\rho=1580N/m^3$, $\alpha=.45, \beta=.45$ and $\gamma=.05$.

In order to solve numerically the equations presented in section 3 the Finite Element Method is used (FlexPde, 2015).

The critical loads have been obtained for torsional mode and *x-y*, *x-z* bending modes. The corresponding values along with the critical axial displacement u_L are given in Table 1. As observed, the buckling is produced in the *x-y* plane.

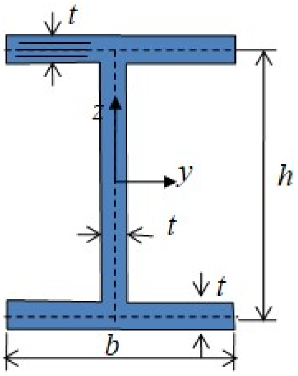

Figure 2. Cross section of the beam

Table 1. Critical loads and their corresponding critical axial displacement

	N_{cr} (N)	u_{cr} (m)
x-y bending	-5.3896E+05	-5.5969E-03
x-z bending	-6.9163E+06	-7.1825E-02
torsional	-8.5765E+05	-8.9065E-03

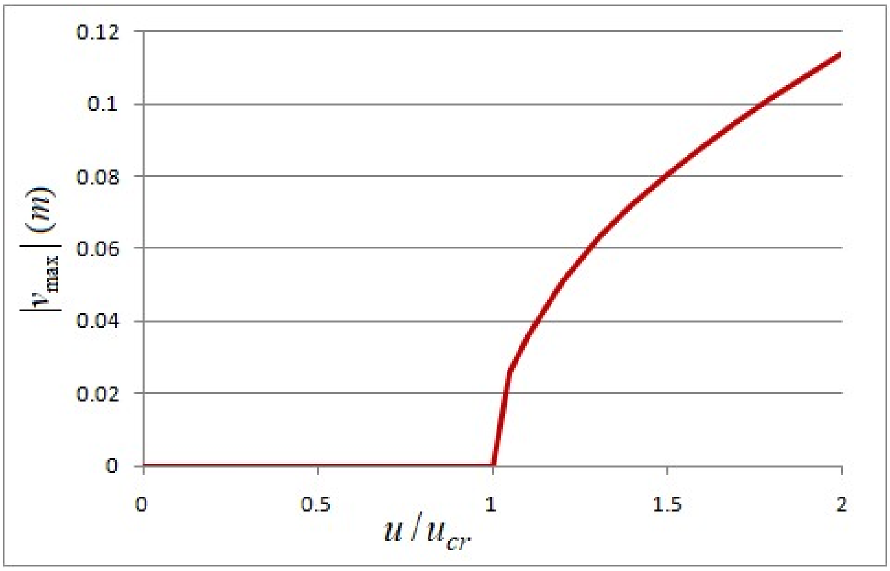

Figure 3. Post-buckling path

The post-buckling path (*x-y* plane) is shown in Fig. 3. The buckling (post-buckling) shape is shown in Fig. 4.

Natural Frequencies of vibration for different values of the right axial displacement are given in Fig. 5 (first frequencies) and Fig. 6 (second frequencies).

In Fig. 5, it is possible to see that the lowest frequency corresponding to the *x-y* plane decreases up to zero, when buckling is produced. Overpassing this point, this frequency shows an increasing behavior. The corresponding lowest frequency for the *x-z* bending mode (torsional mode) decreases reaching a minimum value when $u=u_{cr}$. Of course, this minimum value is not zero because buckling is produced in *x-y* plane. For displacements of higher absolute values ($\left|u_L\right|$), frequencies corresponding to *x-z* bending and torsional modes become independent of u.

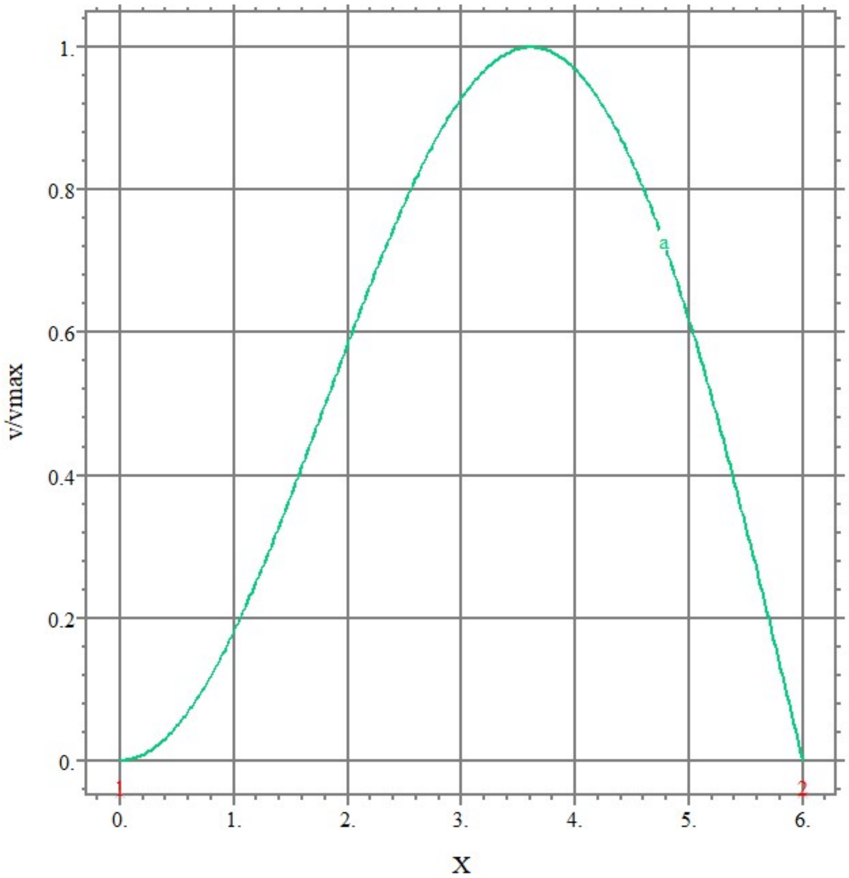

Figure 4. Buckling *x-y* mode

The second frequency corresponding to *x-y* plane decreases reaching a minimum value when $u=u_{cr}$. On the other hand, the second frequencies for the other modes decrease in the pre-buckling range and become independent of u in the post-buckling range (Fig. 6).

Finally, in Fig. 7, the amplitude of the post-buckled displacement as a function of the frequency corresponding to *x-y* plane is shown. This kind of figure shows, conceptually, the fact that the post-buckling amplitude could be estimated by measuring vibration natural frequencies. This fact may be useful for detecting accidental post-buckling events for beams installed in non accessible places, such as heat exchanges, nuclear reactors, etc. (Cortínez et al., 2017).

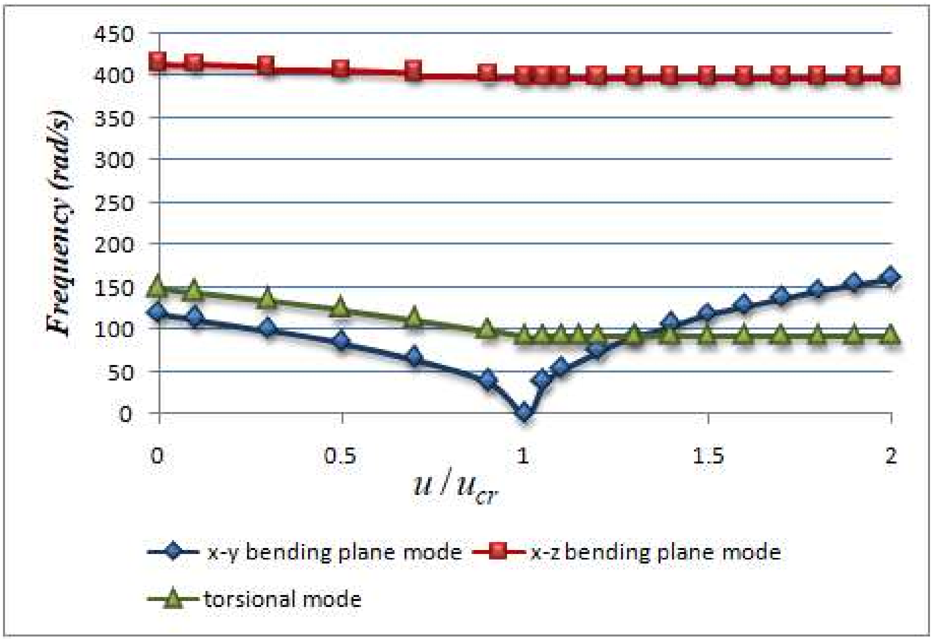

Figure 5. First natural Frequencies

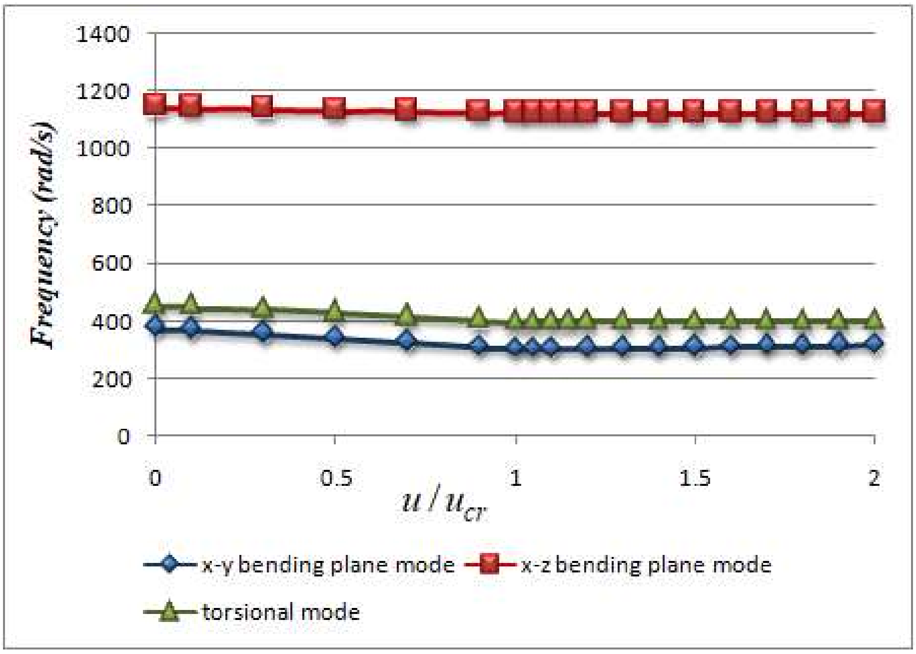

Figure 6. Second natural frequencies

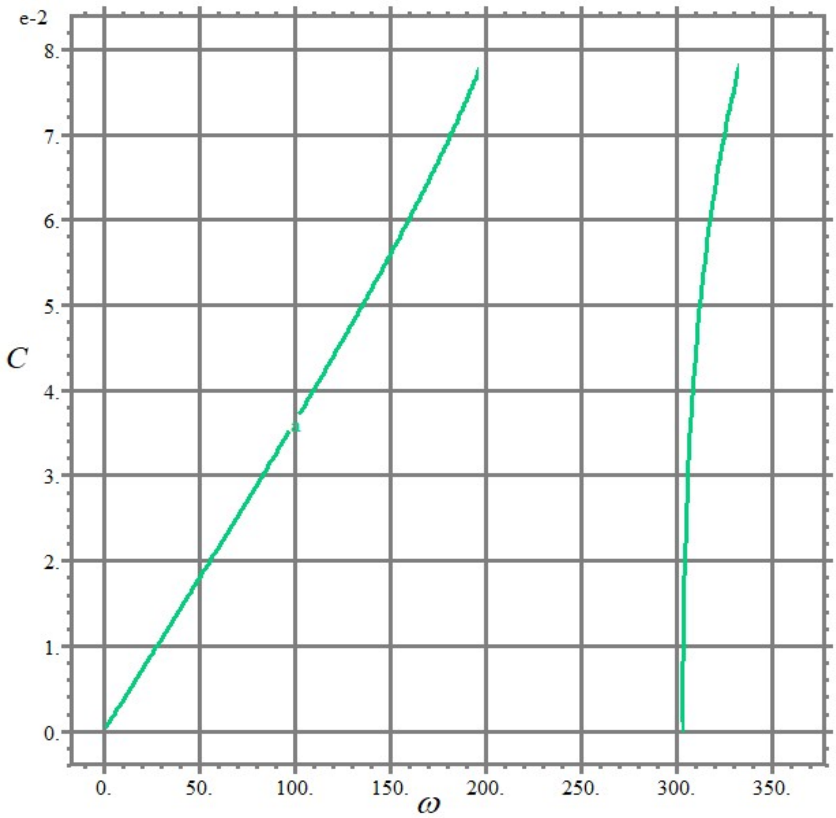

Figure 7. Amplitude of the post-buckled displacement corresponding to x-y plane

5 CONCLUSIONS

A theoretical model to analyze the small vibrations of composite beams around a post-buckled equilibrium form was developed. The model is based on a geometric non-linear formulation taking into account shear and warping effects for completely, arbitrary cross sectional shapes. The approach consists in solving the non-linear equilibrium problem in a first step and then, in employing the corresponding displacements for deriving the governing equations for the vibrations of the buckled beams.

In this paper, a simple numerical example was given in order to show the possibilities of the theory. In a future work, a general numerical approach will be explained in order to deal with more complex situations.

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