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Control problem applied to the Helmholtz problem.

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Abstract. *In this work, the optimal control problem for pointwise sources is studied. In particular, the control is given by a finite linear combination of Dirac mass and the state is a solution of an Helmholtz equation. The purpose is thus to minimize a functional, which measures the distance between the state equation and the target function, relative to the amount, intensities, and locations of pointwise loads that constitute the source term of the boundary value problem. To this end, applies a method that consists in calculating the sensitivity of functional with respect to the set of admissible solutions. The result is used to devise a non-iterative second order method, independent of any initial guess and without introducing regularization techniques. The optimization problem is used to solve the problem of reconstruction pointwise sources from the reading data inside a subset of the domain. Finally, the devised reconstruction algorithm is verified numerically.*

Keywords: *Control problem, Inverse problem, Helmholtz equation, Sensitivity analysis.*

1 Introduction

In this work the optimal control problem for pointwise sources is studied. The control is given by a finite linear combination of Dirac mass and the state is solution of an Helmholtz equation with Robin condition. The purpose is thus to minimize a functional, which measures the distance between state and a target function, relative to the amount, intensities and locations of pointwise loads that constitute the source term of the boundary value problem. To this end, applies a method that consists in calculate the sensitivity of functional with respect to the set of admissible solutions.The result is then used to devise a non-iterative second order method, independent of any initial guess and without introducing regularization techniques. The optimization problem is used to solve the problem of reconstruction pointwise sources from the reading data inside a subset of the domain. The devised reconstruction algorithm is verified numerically. In the following paragraphs is introduced a basic notion about control problems and inverse problems.

According to [3], in a control problem we find the following basic elements:

(i) A control that we can handle according to our interests;

(ii) The state of the system to be controlled, which is directly related to the control;

(iii) A state equation that establishes the dependence between the control and the state;

(iv) A function to be minimized, called the objective function or the cost function.

For details on the study of optimal control problems governed by partial differential equations see [13], a reference book in the area. Because of the specificity of the control problem to be studied, we present some works in which the control is in measure space, denoted by $\mathcal{M}(\Omega)$ and described in [7]. In addition to this we should cite the works of [18] and [4] for elliptic equations, in which it is demonstrated that a measure in Ω can be efficiently approximated by a combination of Dirac measures. We can also emphasize the work of [5], [6] and [11], for parabolic equations, and [12] for wave equation.

We call two problems inverses of one another if the formulation of each involves all or part of the solution of the other. Often, for historical reasons, one of the two problems has been studied extensively for some time, while the other is newer and not so well understood. In such cases, the former is called the direct problem, while the latter is called the inverse problem. This definition, formulated in [9], gives us an initial idea of what is an inverse problem. In many cases, one of the problems mentioned above is not well-posed. According to [8], a problem is called well-posed if:

(i) There exists a solution to the problem;

(ii) There is at most one solution to the problem;

(iii) the solution depends continuously on the data.

A problem which is not well-posed is called ill-posed. Among the works on inverse problem of reconstruction of pointwise sources, we can cite [1], [15] and [10] for Poisson equation and [2] for Helmholtz equation. The approach used in this work is based on the method proposed in [14] and was inspired by definition of topological derivative [20] and [17].

This work is organized as follows. In Section 2 we formulate the control problem. In Section 3 the functional is minimized with respect to a set of admissible pointwise sources. In Section 4 we devise a reconstruction algorithm. Finally, in Section 5 we present several numerical examples.

2 Problem Formulation

Let $\Omega \subset \mathbb{R}^2$ be an open domain with Lipschitz boundary, denoted by $\partial\Omega$. Let $\Omega_o \subseteq \Omega$ and $\Omega_c \subseteq \Omega$ be denominated , respectively, control domain and observation domain, as illustrated in Figure 1.

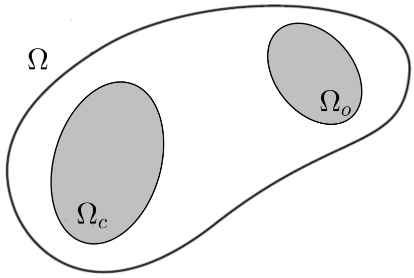


Figure 1: Control domain and observation domain

In this work we consider the following optimal control problem

$$\underset{b \in C_\delta(\Omega_c)}{\text{Minimize}} \ \mathcal{J}(u) = \frac{1}{2}\|u - z\|^2_{L^2(\Omega_o)}. \tag{1}$$

The state variable u is the solution of the state equation

$$\left\{ \begin{array}{rcll} -\Delta u - k^2 u & = & b & \text{in} \quad \Omega, \\ \partial_n u + iku & = & 0 & \text{on} \quad \partial\Omega, \end{array} \right. \tag{2}$$

where $i = \sqrt{-1}$ and $k > 0$ is the wave number. The target function z, may be associated with an unknown source b^*, relating the control problem to a problem of sources reconstruction. The control, source b of equation (2), is a finite combination of Dirac mass. Therefore in this work we consider the following set $C_\delta(\Omega_c)$ of admissible solutions

$$C_\delta(\Omega_c) = \left\{ \varphi : \Omega_c \to \mathbb{R};\ \varphi(x) = \sum_{i=1}^{M} \alpha_i \delta(x - x_i) \right\}, \tag{3}$$

where $M \in \mathbb{N}$, $\alpha \in \mathbb{R}$, $x_i \in \Omega_c$. Moreover, the distribution $\delta : \mathbb{R}^2 \to \mathbb{R}$ is defined by

$$\int_{\mathbb{R}^2} \delta(x - y)\phi(y)dy = \phi(x) \qquad \forall \phi \in C_c^\infty(\mathbb{R}^2). \tag{4}$$

Therefore, the goal of this work is to determine the source term $b^\star$ given by

$$b^\star = \underset{b \in C_\delta(\Omega_c)}{\operatorname{argmin}} \mathcal{J}(u) = \frac{1}{2}\|u - z\|^2_{L^2(\Omega_o)}, \tag{5}$$

subject to (2). Therefore, solving the control problem means to

$M^\star \in \mathbb{N}$, $\alpha_i^\star \in \mathbb{R}$ e $x_i^\star \in \Omega_c$, which denote the number, intensities and locations of the unknown pointwise sources, respectively.

To prove the existence and uniqueness of (2), we propose the decomposition $u = v+w$, from which the following boundary value problems arise

$$\begin{cases} -\Delta v &= b \quad \text{in} \quad \Omega, \\ \quad v &= 0 \quad \text{on} \quad \partial\Omega, \end{cases} \tag{6}$$

and

$$\begin{cases} -\Delta w - k^2 w &= k^2 v \quad \text{in} \quad \Omega, \\ \partial_n w + ikw &= -\partial_n v \quad \text{on} \quad \partial\Omega. \end{cases} \tag{7}$$

According to [18] and [7], the equation (6) has unique solution for $v \in W_0^{1,s}(\Omega)$, for $1 \leq s < 2$, where $W_0^{1,s}(\Omega)$ is compactly embedded in $L^2(\Omega)$. According to [16] and [19], the solution w of equation (7) satisfies

$$\|w\|^2_{\mathcal{H}(\Omega)} \leq C\left(\|v\|_{L^2(\Omega)} + \|\partial_n v\|_{L^2(\partial\Omega)}\right),$$

with C constant , and the norm $\|\psi\|^2_{\mathcal{H}(\Omega)}$, equivalent to the usual $H^1(\Omega)$-norm, given by

$$\|\psi\|^2_{\mathcal{H}(\Omega)} := \int_\Omega \|\nabla\psi\|^2 + \int_\Omega k^2|\psi|^2.$$

The proposed optimization problem will be used to solve the inverse problem of reconstruction of pointwise sources for the Helmholtz equation. For this, we consider z as a solution of

$$\begin{cases} -\Delta z - k^2 z &= b^* \quad \text{in} \quad \Omega, \\ \partial_n z + ikz &= 0 \quad \text{on} \quad \partial\Omega. \end{cases} \tag{8}$$

3 Sensitivity Analysis

In this section, let us minimize the functional (1) with respect to the set of admissible solutions (3). Therefore, the idea is to perturb the source b by introducing a number m of pointwise sources with arbitrary locations and intensities. The perturbed source $b_\delta \in C_\delta(\Omega_c)$ is defined as follows

$$b_\delta(x) = b(x) + \sum_{i=1}^{m} \alpha_i \delta(x - x_i). \tag{9}$$

The perturbation imposed in the source term causes a perturbation in the satate equation and, consequently, in the functional. The perturbed functional $\mathcal{J}(u_\delta)$ is given by

$$\mathcal{J}(u_\delta) = \frac{1}{2}\|u_\delta - z\|^2_{L^2(\Omega_o)}. \tag{10}$$

Calculating the variation of functional , we obtain

$$\mathcal{J}(u_\delta) - \mathcal{J}(u) = \frac{1}{2}\|u_\delta - z\|^2_{L^2(\Omega_o)} - \frac{1}{2}\|u - z\|^2_{L^2(\Omega_o)}. \tag{11}$$

The variation of the functional (11) does not have a direct relation with the parameters that define the source term. Therefore, we introduce the following expansion

$$u_\delta(x) = u(x) + \sum_{i=1}^{m} \alpha_i v_i(x). \tag{12}$$

The perturbed helmholtz equation problem is given by

$$\mathcal{J}(u_\delta) = \frac{1}{2}\int_{\Omega_o} (u_\delta - z)\overline{(u - z)} dx, \tag{13}$$

subject to

$$\left\{ \begin{array}{rcll} -\Delta u_\delta - k^2 u_\delta & = & b_\delta & \text{in} \quad \Omega, \\ \partial_n u_\delta + iku_\delta & = & 0 & \text{on} \quad \partial\Omega. \end{array} \right. \tag{14}$$

subtracting the equation (2) from equation (14), it follows that v_i is solution of the following boundary value problem

$$\left\{ \begin{array}{rcll} -\Delta v_i - k^2 v_i & = & \delta_i & \text{in} \quad \Omega, \\ \partial_n v_i + ikv_i & = & 0 & \text{on} \quad \partial\Omega, \end{array} \right. \tag{15}$$

where $\delta_i := \delta(x - x_i)$.

Using the expansion (12), the perturbed functional (13) is rewritten as

$$\mathcal{J}(u_\delta) = \frac{1}{2}\int_{\Omega_o} \left((u - z) + \sum_{i=1}^{m} \alpha_i v_i \right) \left(\overline{(u - z)} + \sum_{i=1}^{m} \alpha_i \overline{v_i} \right) dx. \tag{16}$$

Considering $\Psi(m, \alpha, \xi) = \mathcal{J}(u_\delta) - \mathcal{J}(u)$, we have that

$$\begin{array}{rcl} \Psi(m, \alpha, \xi) & = & \displaystyle\frac{1}{2}\int_{\Omega_o} \left[(u - z)\sum_{i=1}^{m} \alpha_i \overline{v_i} + \overline{(u - z)}\sum_{i=1}^{m} \alpha_i v_i \right] dx \\ & + & \displaystyle\frac{1}{2}\int_{\Omega_o} \left(\sum_{i=1}^{m} \alpha_i v_i \right) \left(\sum_{i=1}^{m} \alpha_i \overline{v_i} \right) dx. \end{array} \tag{17}$$

For any complexes, w_1 and w_2, the relation

$$w_1\overline{w_2} + \overline{w_1}w_2 = 2\Re(w_1\overline{w_2}).$$

is valid. In addition to this consider the matrix $H \in \mathbb{R}^{m\times m}$ and the vector $d \in \mathbb{R}^m$ whose coefficients are

$$H_{ji} := A(i,j) = \int_{\Omega_o} \Re\left(v_j\overline{v_i}\right) dx \quad \text{e} \quad d_j := f(j) = \int_{\Omega_o} \Re\left((u-z)\overline{v_j}\right) dx. \tag{18}$$

Thus, we can rewrite the equation (17) as follows

$$\Psi(m,\alpha,\xi) = d\cdot\alpha + \frac{1}{2}H\alpha\cdot\alpha. \tag{19}$$

The matrix H assures us the second order method. In addition, we observe that the functional (19) is strictly convex with respect to the variable α, so that there exists a global minimum denoted by $\alpha^\star$.

4 Reconstruction Algorithm

For each m and ξ fixed, there is a global minimum for Ψ. Let us minimize $\Psi(m,\alpha,\xi)$ with respect to α, namely

$$\langle D_\alpha\Psi(m,\alpha,\xi),\beta\rangle = 0, \quad \forall\beta\in\mathbb{R}^m. \tag{20}$$

which leads to the following linear system

$$(H\alpha + d)\cdot\beta = 0, \quad \forall\beta\in\mathbb{R}^m, \tag{21}$$

that is

$$H\alpha = -d. \tag{22}$$

In addition, the solution of (22), α, becomes a function of the locations ξ, namely $\alpha = \alpha(\xi)$. Let us now replace the solution of (22) into $\Psi(m,\alpha,\xi)$. Therefore, the optimal locations $\xi^\star$ can be trivially obtained from a combinatorial search over the domain Ω, solution to the following minimization problem

$$\xi^\star = \underset{\xi\in X}{\operatorname{argmin}}\left\{\Psi(m,\alpha,\xi) = \frac{1}{2}d\cdot\alpha(\xi)\right\}, \tag{23}$$

where X is the set of admissible source locations. Finally, the optimal intensities are given by $\alpha^\star = \alpha(\xi^\star)$. In summary, our method is able to find in one step the optimal intensities $\alpha^\star$ and locations $\xi^\star$ for a given m. The problem on how to find the optimal number of pointwise sources $m^\star$ will be discussed in the numerical experiments section.

In order to summarize the calculations presented in this section we introduce now the resulting reconstruction algorithm. It describes the process of obtaining the optimal parameters $\alpha^\star$ and $\xi^\star$ from the computational point of view. The entries of the algorithm are listed below

- The quantity m of pointwise sources to be reconstructed;
- The n-points on which the systems (22) are solved;
- The vector d and the matrix H whose entries are given by (18).

The reconstruction algorithm returns optimal intensities $\alpha^\star$ and locations $\xi^\star$. The reconstruction procedure written in pseudo-code format is shown in Algorithm 1.

Algorithm 1: Reconstruction Algorithm

Data: m, n, d, H
Result: $S^\star$, $\alpha^\star$, $\xi^\star$

Initialization: $S^\star \leftarrow \infty$; $\alpha^\star \leftarrow 0$; $\xi^\star \leftarrow 0$;
for $i_1 \leftarrow 1$ **to** n **do**
    for $i_2 \leftarrow i_1 + 1$ **to** n **do**
        $\vdots$
        for $i_m \leftarrow i_{m-1} + 1$ **to** n **do**

$$d \leftarrow \begin{bmatrix} b(i_1) \\ b(i_2) \\ \vdots \\ b(i_m) \end{bmatrix}; \quad H \leftarrow \begin{bmatrix} A(i_1,i_1) & A(i_1,i_2) & \cdots & A(i_1,i_m) \\ A(i_2,i_1) & A(i_2,i_2) & \cdots & A(i_2,i_m) \\ \vdots & \vdots & \ddots & \vdots \\ A(i_m,i_1) & A(i_m,i_2) & \cdots & A(i_m,i_m) \end{bmatrix};$$

            $I \leftarrow (i_1, i_2, ..., i_m)$; $\xi \leftarrow \Pi(I)$; $\alpha \leftarrow H^{-1}d$; $S \leftarrow -\frac{1}{2} d \cdot \alpha$;
            if $S < S^\star$ **then**
                $\xi^\star \leftarrow \xi$;
                $\alpha^\star \leftarrow \alpha$;
                $S^\star \leftarrow S$;
            end if
        end for
    end for
end for
return $S^\star$, $\alpha^\star$, $\xi^\star$

In the above algorithm Π maps the vector of nodal indexes $I = (i_1, i_2, ..., i_m)$ into the corresponding vector of nodal coordinates ξ. As can be noted in the reconstruction Algorithm 1, the optimal solution $(\xi^\star, \alpha^\star)$ is obtained through an exhaustive and combinatorial search over the n-points. According to[15], the complexity $\mathcal{C}(n,m)$ of the algorithm can be evaluated by the formula

$$\mathcal{C}(n,m) = \begin{pmatrix} n \\ m \end{pmatrix} m^3 = \frac{n!}{m!(n-m)!} m^3. \tag{24}$$

In Fig. 2 the graphics of $m \times \log_{10}(\mathcal{C}(n,m))$ for $n = 100$ and $n = 400$ are presented in blue and red, respectively.

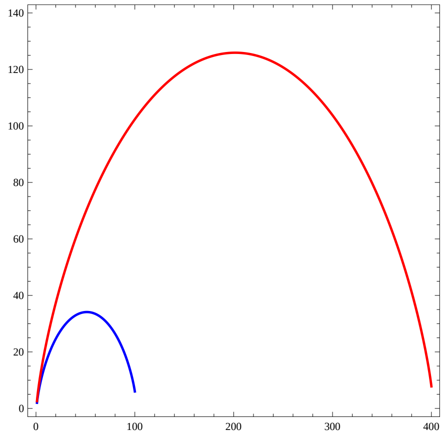


Figure 2: Complexity order: $m \times \log_{10}(\mathcal{C}(m, 100))$ **e** $m \times \log_{10}(\mathcal{C}(m, 400))$**.**

5 Numerical results

In this section, we present some numerical examples in order to demonstrate the efficacy of the proposed method to solve the optimization problem in this work.The boundary value problems are solved using the Finite Element Method. The computational implementation is performed using MATLAB. In all examples, we consider the domain $\Omega = (-0.5, 0.5) \times (-0.5, 0.5)$.

Example 1: Sensitivity of the reconstruction with respect to the sub-grid size.

The purpose of this example is to analyze the sensitivity of the reconstruction with respect to the sub-grid size . For this, different mesh configurations are considered so that the location of the source to be reconstructed does not coincide with one of its nodal points. We consider a target given by one pointwise source, $b^*(x) = \alpha^* \delta(x - x^*)$, located at $x^* = (0.2375, 0.1625)$, with intensity $\alpha^* = 1$. The wave number considered is $k = 5$. Applying the Algorithm 1 and considering sub-grids with 16,81 , 361, 1521 and 6241 points. The solution obtained is described in Table 1. The observation domain Ω_o is represented in Figure 3.

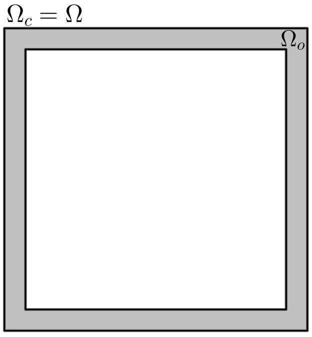


Figure 3: Example 1. Control domain Ω_c and observation domain Ω_o

Table 1: Example 1: Results for different sizes of the sub-grid

Points sub-grid	Intensity $\alpha^\star$	Location $x^\star$
16	0.8303	$(0.30, 0.10)$
81	1.0037	$(0.20, 0.20)$
361	0.9798	$(0.25, 0.15)$
1521	1.0247	$(0.225, 0.15)$
6241	1	$(0.2375, 0.1625)$

The analysis of the Table 1 indicates that the optimal solution returned by the Algorithm 1 approaches the source to be reconstructed with the increase in the number of nodes of the grid. In the case where the location of b^* coincides with a sub-grid node, the reconstruction is exact, that is, $b^\star(x) = b^*(x)$.

Due to the algorithm complexity order, it is adopted for the next examples a sub-grid with 361 nodal points. We assume from now on that the locations x^* belongs to the sub-grid. The representation of the pointwise source will be made by circles, with center corresponding to the location and the radius representing the intensity.

Example 2: Seeking for the number of pointwise sources.

In this example the target consists of three pointwise sources

$$b^*(x) = \alpha_1^*\,\delta(x - x_1^*) + \alpha_2^*\,\delta(x - x_2^*) + \alpha_3^*\,\delta(x - x_3^*). \tag{25}$$

The intensities and the locations are represented, respectively, by the vectors $\alpha^* = (\alpha_1^*, \alpha_2^*, \alpha_3^*)$ e $\xi^* = (x_1^*, x_2^*, x_3^*)$, with $\alpha_1^* = 2$, $\alpha_2^* = 3$, $\alpha_3^* = 1$, $x_1^* = (-0.25, -0.25)$, $x_2^* = (0.15, 0)$ and $x_3^* = (-0.2, 0.2)$. The wave number considered is $k = 5$. The target is represented in the Figure 4. The observation domain Ω_o is represented in Figure 5.

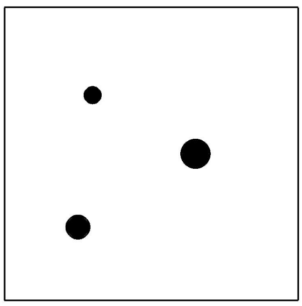


Figure 4: Example 2. Target

The objective of this experiment is to present a method to obtain the number of pointwise sources. We start the reconstruction algorithm by taking $m = 1$ and proceed to increasing the value of m until that the vector of optimal intensities $\alpha^\star$ contains one

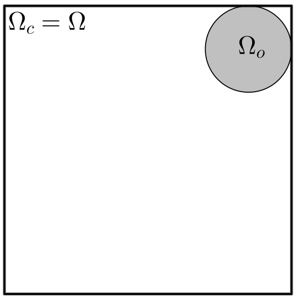
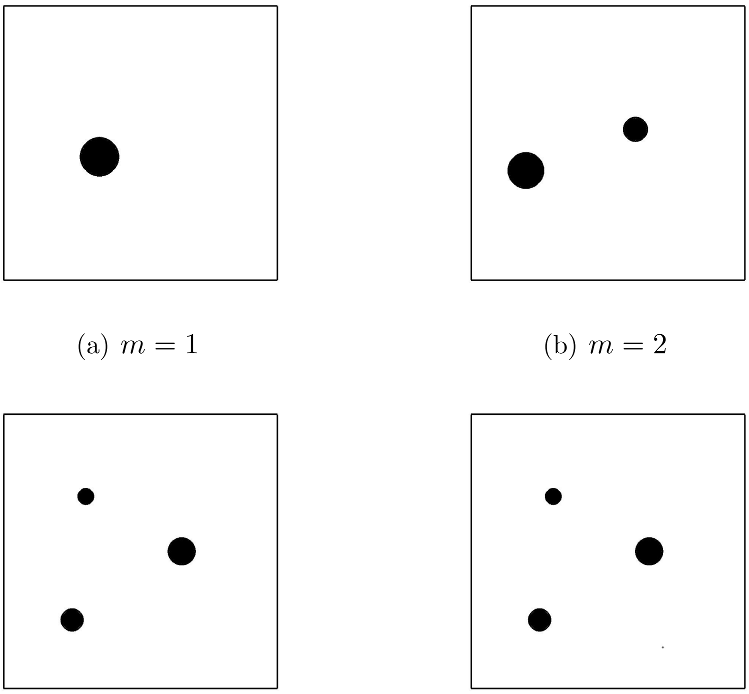


Figure 5: Example 2. Control domain Ω_c and observation domain Ω_o

Table 2: Example 2. Results

m	$\alpha^\star$	$\xi^\star$
1	6.06	$(-0.15, -0.05)$
2	$(2.36, 5.23)$	$((-0.10, 0.05), (-0.30, -0.10))$
3	$(2, 3, 1)$	$((-0.25, -0.25), (0.15, 0), (-0.20, 0.20))$
4	$(2, 3, 1, 10^{-14})$	$((-0.25, 0.25), (0.15, 0), (-0.20, 0.20), (0.25, -0.35))$



(a) $m = 1$ (b) $m = 2$

(c) $m = 3$ (d) $m = 4$

Figure 6: Example 2. Results

entry with negligible value. The algorithm is applied considering $m = 1$, $m = 2$, $m = 3$ and $m = 4$. The results are given in the Table 2 and represented by Figure 6.

For $m = 1$ and $m = 2$ the solutions are clearly far from the target. For $m = 3$ the locations and intensities are perfectly reconstructed. Finally, for $m = 4$ there is an additional pointwise source with negligible intensity, namely $\alpha_4^\star = 10^{-14}$. Therefore we can conclude that the correct quantity of pointwise sources is $m = 3$. Note that this procedure is not iterative, since there is no relation between the results for two consecutive values of m. In addition, we can start the Algorithm 1 based on the assumption that there exists $m > m^*$ and find a number $(m - m^*)$ of trial balls with negligible sizes in just one shot.

Example 3: Noise imposition

We are interested in investigating the robustness of the method with respect to imposition of noise in the target function z. Representing by r a random value located in the range $(0, 1)$ and considering the parameter η as noise level, the corrupted target function z_r it is given by:

$$z_r^* = z^*(1 + \eta r). \tag{26}$$

The target contains three pointwise sources located at $x_1^* = (-0.25, 0.3)$, $x_2^* = (0.3, 0.0)$ e $x_3^* = (-0.15, -0.2)$, with same intensities $\alpha_i^* = 2$. considering $k = 5$, $m = 3$, $\eta = 5\%, \eta = 10\%, \eta = 15\%$ and $\eta = 20\%$, the obtained results are presented in Table 3 and in the Figure 8 . The observation domain $\Omega_o = \Omega_1 \cup \Omega_2$ is represented in Figure 7.

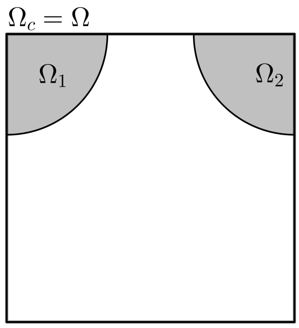
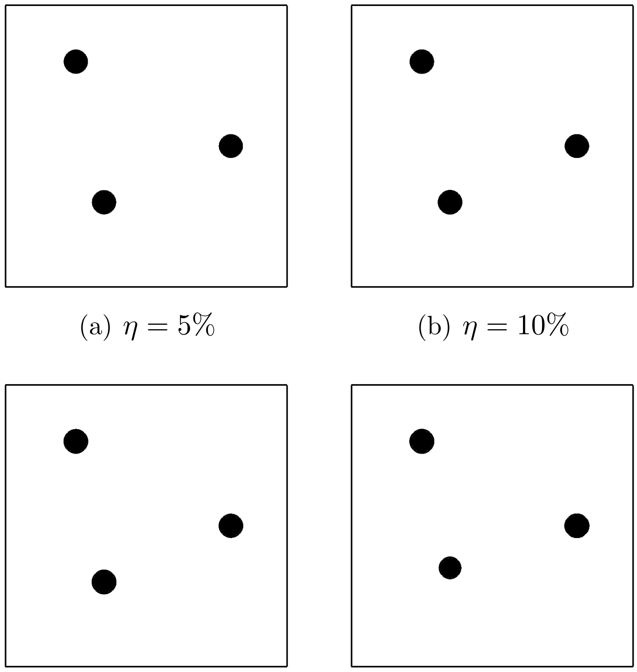


Figure 7: Example 3. Control domain Ω_c and observation domain Ω_o

Table 3: Result obtained for different noise levels

	$\eta = 5\%$	$\eta = 10\%$	$\eta = 15\%$	$\eta = 20\%$
$\alpha_1^\star$	2.05	2.09	2.14	2.19
$x_1^\star$	$(-0.25, 0.30)$	$(-0.25, 0.30)$	$(-0.25, 0.30)$	$(-0.25, 0.30)$
$\alpha_2^\star$	2.05	2.09	2.14	2.23
$x_2^\star$	$(0.30, 0)$	$(0.30, 0)$	$(0.30, 0)$	$(0.30, 0)$
$\alpha_3^\star$	2.06	2.12	2.17	1.78
$x_3^\star$	$(-0.15, -0.20)$	$(-0.15, -0.20)$	$(-0.15, -0.20)$	$(-0.15, -0.15)$

Analyzing the data, it is verified that only from $eta = 20\%$ the results obtained become discrepant of the term to be reconstructed and in only one pointwise sources. This



(a) $\eta = 5\%$ (b) $\eta = 10\%$

(c) $\eta = 15\%$ (d) $\eta = 20\%$

Figure 8: Example 3. Results

fact indicates that the method is robust to noise. This conclusion is partly a consequence of the proposed second order method not being iterative, that is, the solution is obtained in a single step. Another fact that contributes to the robustness of the method is the expansion given by the equation (12) doesn't produce residue.

Example 4: Sensitivity to wave number

Now we are interested in investigating the sensitivity of the method in relation to the wave number. The source $b^\star$ consists of three pointwise sources whose intensities and locations are given, respectively, by the vectors $\alpha^* = (2, 3, 1)$ and $\xi^* = ((-0.25, -0.25), (0.15, 0), (-0.2, 0.2))$. The observation domain Ω_o is represented in Figure 9. Considering $m = 4$, $\eta = 10$, and applying the algorithm 1 for $k = 5$, $k = 10$ and

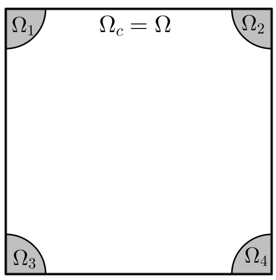


Figure 9: Example 4. Control domain Ω_c and observation domain Ω_o

$k = 20$, the results obtained are presented in the Table 4. The data of the fourth mass, of negligible intensity, were disregarded.

Table 4: Results for different wave numbers

	$k = 5$	$k = 10$	$k = 20$
$\alpha_1^\star$	2.20	2.20	2.19
$x_1^\star$	$(-0.25, -0.25)$	$(-0.25, -0.25)$	$(-0.25, -0.25)$
$\alpha_2^\star$	3.30	3.30	3.29
$x_2^\star$	$(0.15, 0)$	$(0.15, 0)$	$(0.15, 0)$
$\alpha_3^\star$	1.10	1.10	1.10
$x_3^\star$	$(-0.20, 0.20)$	$(-0.20, 0.20)$	$(-0.20, 0.20)$

The data obtained indicate that the wave number does not influence the result. It should be noted that the mesh, on which the Helmholtz equation is solved, it is sufficiently refined so as to maintain the stability of the numerical approximation.

The experiments performed in this section prove the efficiency of the proposed method for numerical resolution of the inverse problems of reconstruction of concentrated sources. But because of the complexity of the algorithm, all experiments were carried out considering, at most, $m = 4$. To solve this problem, we describe in the next section a new numerical method of selective search to contemplate a larger number of pointwise sources.

5.1 Selective search

In this section we describe the numerical method that allows the realization of experiments with a greater number of pointwise sources.This method is based on the joint analysis of the complexity of the algorithm, and the numerical results obtained in Example 1. The basic idea is to start the exhaustive and combinatorial search with few points in the sub-grid. Once the first approximation to the solution is obtained, the mesh is refined and the new search set is the neighborhood of the solution found. The described procedure is repeated until the desired result is obtained.To illustrate the application of the method, consider the following problem: rebuild the source term b^* of the equation (8), with wave number $k = 5$ and considering

$$b^*(x) = \sum_{i=1}^{8} \alpha_i^* \, \delta(x - x_i^*),$$

with $\alpha_i^* = 3$ e $x_i \in \{(\pm 0.15, \pm 0.15), (\pm 0.35, \pm 0.35)\}$. The Figure 10 illustrates the source term to be reconstructed.

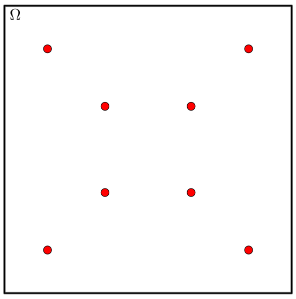
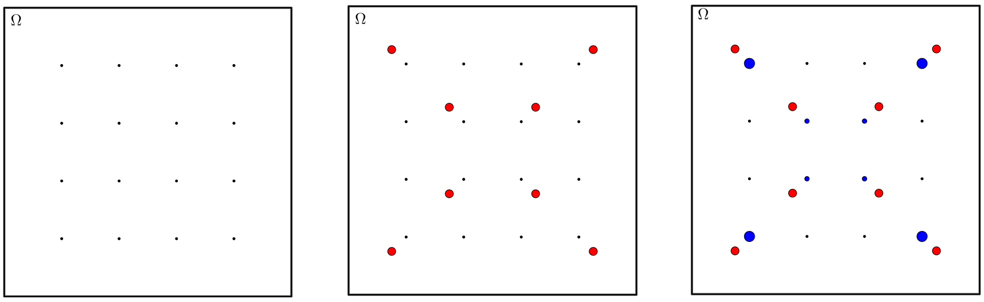


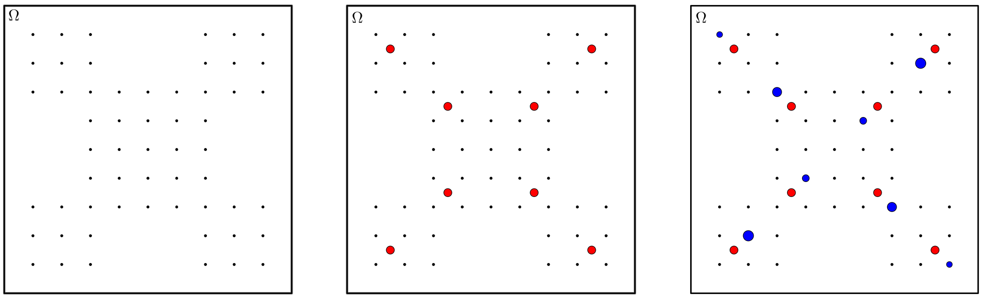
Figure 10: Target

For purposes of comparison with the exhaustive and combinatorial search method, the goal is for the rebuild to be done in a 20×20 mesh, or 361 nodal points. For this, the procedure is started with a 5×5 mesh and two refinements are performed. the steps 1, 2 e 3 are represented, respectively, in the figures 11, 12 and 13



(a) Grid (b) Grid with target (c) Solution

Figure 11: Selective search - Step I

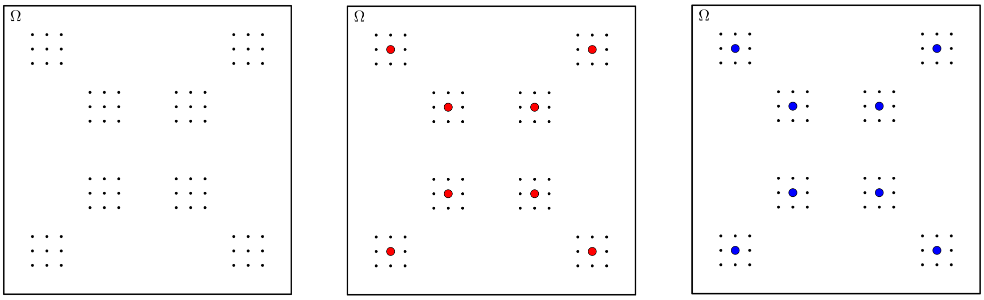


(a) Grid (b) Grid with target (c) Solution

Figure 12: Selective search - Step II

For purposes of comparison, the Table 5 presents the processing time, in seconds, for each method proposed in this work.

The analysis of the Table 5 demonstrates the efficiency of one method compared to



(a) Grid (b) Grid with target (c) Solution

Figure 13: Selective search - Step III

Table 5: Processing time (s)

Pointwise sources	Selective search	Exhaustive and combinatorial search
1	0.014	0.092
2	0.021	0.117
3	0.032	3.264
4	0.138	309.42
5	2.834	25914.5
6	84.12	1817592
7	3893.4	− − −
8	56859.2	− − −

another. Finally, we present the comparative graph, Figure 14, Which relates the number of loads with the processing time, in logarithmic scale, for each method.

6 Conclusion

In this work, the optimal control problem for concentrated sources was studied. We used the optimization problem to solve the problem of reconstruction of concentrated sources for the Helmholtz equation, from the reading data inside a subset of the domain. To this end, we applied a method that consists in calculating the sensitivity of functional with respect to the set of admissible solutions. The result was used to devise a non-iterative second order method, independent of any initial guess and without introducing regularization technique. the devised reconstruction algorithm was verified numerically.

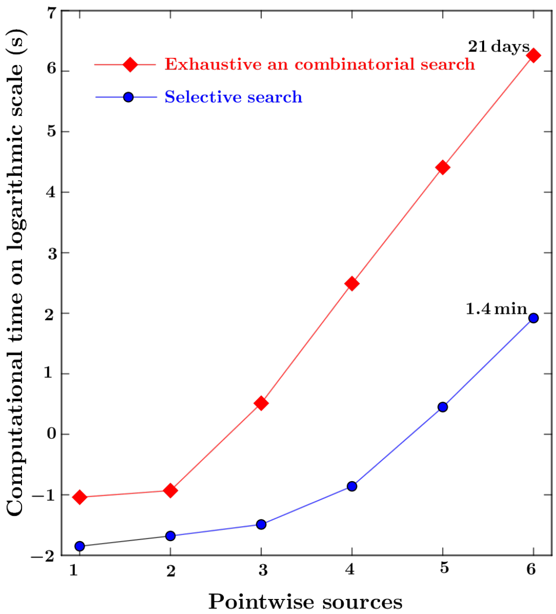


Figure 14: Comparison between the methods

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