ANALYSIS OF HIGH ORDER APPROXIMATIONS BY SPECTRAL INTERPOLATION APPLIED TO THE FINITE ELEMENT METHOD IN ELASTOSTATICS

Luís Philipe Ribeiro Almeida

Fábio Carlos da Rocha

lpralmeida@academico.ufs.br

fcrocha@ufs.br

Department of Civil Engineering, Federal University of Sergipe - UFS

Av. Marechal Rondom, S/n Jardim Rosa Elze, 49100-000, Sergipe, São Cristovão, Brasil

Abstract. *Structural modeling requires the use of tools that guarantee the efficiency and precision in the reproduction of geometry and loading, with variable complexity. The quality of the physical solution is associated with numerical errors found in the formulation of the finite element method, which depends on both the interpolating polynomial, the bases of the nodal points and the degree of approximation. In order to overcome this difficulty, the high order approximations, consisting of both equidistant bases and orthogonal bases of Lobatto (spectral base), is used. A comparative study between these bases applied to one-dimensional and two-dimensional elastotatic problems is analyzed. To evaluate the performance of the approximations, the Lebesgue constant and the condition number are used. From these parameters, a convergence and efficiency analysis of the bases under study is performed, in the minimization or disappearance of the Runge phenomenon, as the order of the approximation is increased. Examples are evaluated and a significant improvement of results is observed when spectral interpolation is used in detriment to the equidistant base interpolation.*

Keywords: *Finite Element Method, Spectral interpolation, Orthogonal polynomials, Lobatto base, Runge phenomenon.*

1 INTRODUCTION

The computational simulation has been gaining significant advances in the modeling of large and complex physical systems. The implementation of high-order (spectral) approximations associated to the finite element method arises as a way to overcome difficulties encountered in the numerical analysis of these types of problems, whose solution is difficult to reproduce. The methods using spectral elements combine geometric flexibility of the classical finite element techniques with the ability to ensure uniform convergence, which generally requires the error estimation in problem analysis (Zienkiewicz and Rank, 1987). One way to estimate this error is the adoption of quantifying parameters, such as condition number and the Lebesgue constant. This latter is defined as the sum of the absolute maximum values of the cardinal functions queried on all interpolation nodes (Blyth and Pozrikidis, 2005).

In this work, high-order approximations are used in the analysis of convergence for one-dimensional and two-dimensional elastostatic problems by FEM. The bases of equally spaced nodes and Lobatto nodes are used for construction of analyzed approximations. This latter base (Lobatto nodes) is constructed from the position of the zeros of Lobatto orthogonal polynomials, which constitutes a so-called spectral base. The present work uses the condition number and a Lebesgue constant, as quantifying parameters for the estimation of error. In this way, the distribution of points that minimize the interpolation error becomes of paramount importance.

In view of the explained above, this work is divided as follows. Section 2 presents a study of high order one-dimensional interpolation as well as the nodal bases used. Section 3 shows the high order interpolation for triangular domains, the error associated with interpolation and matrix conditioning. In section 4 the formulation for the one-dimensional and two-dimensional FEM is presented and finally examples are analyzed to illustrate the interpolation strategies applied to the FEM.

2 ONE-DIMENSIONAL HIGH-ORDER APPROXIMATION

In this section, the theoretical basis (spectral and non-spectral) for the development of high order elements through the construction of the generalized Vandermond matrix is presented.

2.1 Generalized Vandermond Matrix

In this work, the construction of the Vandermond matrix was used to increase the polynomial order. This procedure is employed due to easy generalization in the generation of shape functions for one- and two-dimensional elements of any order (Pozrikidis, 2005).

Let be the set of points $\{\xi_k \in \mathbb{R}; -1 \le \xi_k \le +1\}$, which depend on the degree of approximation (m) and are generated from the base in dimensionless space. The general term of the interpolator polynomial by imposing the Kronecker delta property on the nodal base, is determined as (Rocha and Kzam, 2013):

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$$\varphi_i\left(\xi=\xi_k\right)=\sum_{j=0}^{n}a_{ij}\xi^{j}=a_{i0}+a_{i1}\xi_k+\ldots+a_{im}\xi_k{}^{m}=\delta_{im}, \tag{1}$$

where φ_i are shape function evaluated at points $i=1,...,(m+1)$, a_{ij} the coefficients of the polynomials and δ_{ik} the Kronecker delta.

Equation (1) can be written in matrix notation (Eq. 2)

$$\begin{bmatrix} a_{10} & \cdots & a_{1m} \\ \vdots & \ddots & \vdots \\ a_{k0} & \cdots & a_{km} \end{bmatrix}\begin{bmatrix} 1 & \cdots & 1 \\ \vdots & \ddots & \vdots \\ \xi_1^m & \cdots & \xi_k^m \end{bmatrix}=\begin{bmatrix} 1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 1 \end{bmatrix}. \tag{2}$$

The symbolic representation can be expressed as:

$$MV=I, \tag{3}$$

in that V is the matrix constituted by the powers of the dimensionless coordinates, called the generalized Vandermond matrix.

2.2 Nodal bases

In the attempt to construct an approximation function for the resolution of the physical problem, the interpolation base used is of the utmost importance. Since the choice of the base interferes directly in the efficiency and precision of the numerical analysis, whether due to errors in the geometric representation or errors obtained in the mechanical response of the problem. The nodal bases studied during this work are presented below.

2.3 Evenly spaced nodal base

For the construction of the set of equally spaced points, the interval of points $\left\{\xi_k \in \mathbb{R}; -1\le\xi_k\le+1\right\}$ is used in the mapping of the base of the dimensionless space. This set is obtained through the following arithmetic progression (Rocha and Kzam, 2013):

$$\xi_k=\xi_1+r\left(k-1\right), \tag{4}$$

where $k=1,\ldots,(m+1)$ is the partition of the interval, $\xi_1=-1$ the starting point, $r=\dfrac{2}{m}$ the rate of progression, and m the degree of interpolation.

2.4 Spectral nodal base

For the construction of the spectral nodal base, the orthogonal polynomials are of fundamental importance because the zeros of these polynomials determine the position of the points of this base. Among the main orthogonal polynomials, the Lobatto's polynomial is discussed in this paper.

2.5 Lobatto's nodal base

Lobatto's polynomials are mapped in the range of $[-1,1]$, and their recurrence equation is defined by (Pozrikidis, 2015):

$$Lo_m(\xi) = \frac{1}{\xi^2 - 1}[(m+1)\xi L_{(m+1)}(\xi) - (m+1)L_m(\xi)]. \tag{5}$$

From the nodal basis of Lobatto's interpolation, the $m+1$ interpolation nodes in the element are distributed as follows (Rocha and Kzam, 2013):

$$\xi_1 = -1, \quad \xi_2 = t_1, \ldots, \quad \xi_m = t_{m-1}, \quad \xi_{m+1} = -1, \tag{6}$$

where $t_1, t_2, \ldots, t_{m-1}$ are the zeros of the Lobatto's orthogonal polynomial of degree $m-1$.

3 HIGH-ORDER INTERPOLATION ON TRIANGULAR DOMAIN

The interpolation of a $f(\xi_1, \xi_2)$ function is carried out through a polynomial expansion of degree m over the area of the triangle contained in the dimensionless space given by:

$$f(\xi_1, \xi_2) = \sum_{j=1}^{N} f(\xi_1^j, \xi_2^j)\psi(\xi_1^j, \xi_2^j), \tag{7}$$

with $N = \frac{(m+1)(m+2)}{2}$ linearly independent basis functions, $f(\xi_1^j, \xi_2^j)$ is the nodal values specified, $\psi_i(\xi_1^j, \xi_2^j) = \delta_{ij}$ is the cardinal interpolation function for node i and δ_{ij} is the Kronecker delta (Rocha, 2015).

In a more general approach, the cardinal interpolation function is written in the form:

$$\psi(\xi_1, \xi_2) = \phi(\xi_1, \xi_2).\left[V_\phi\right]^{-1}, \tag{8}$$

where V_ϕ the generalized Vandermond matrix, with components $V_{\phi_i} = \phi_i(\xi_1^j, \xi_2^j)$.

3.1 Proriol's polynomial base

According to Blyth and Pozrikidis (2005), the most desirable base consists of the Proriol Polynomial (Proriol, 1957) that is full orthogonal over the triangle area and it provides less interpolation error. The Proriol's polynomial, PR_{kl}, involves monomials of the form $\xi_1^p \xi_2^{(j-p+q)}$ with order $k+q$ combined, where $p = 1, 2, \ldots, k$ and $q = 1, 2, \ldots, l$, such that:

$$PR_{kl} = \left[\sum_{i=0}^{k} \binom{k}{i}\binom{k}{k-i}\left(\frac{\xi_1' - 1}{2}\right)^{k-i}\left(\frac{\xi_1' + 1}{2}\right)^{i}\left(\frac{1-\xi_2'}{2}\right)^{k}\right].\left[\sum_{j=0}^{l}\binom{l+2k+1}{j}\binom{l}{l-j}\left(\frac{\xi_2' - 1}{2}\right)^{l-j}\left(\frac{\xi_2' - 1}{2}\right)^{j}\right]. \tag{9}$$

3.2 Uniform nodal base in triangular domain

To obtain the interpolating polynomial of degree m, the distribution with equally spaced points is given by (Pozrikidis, 2005):

$$\xi_1^i = v_i, \quad \xi_2^j = 1 - v_{m+2-j}, \tag{10}$$

with $v_i = (i-1)/m$ the dimensionless points for $i = 1, 2, ..., m+1$ and $j = 1, ..., m+2-i$.

3.3 Lobatto's base in triangular domain

Similarly, the one-dimensional nodal distribution (Eq. 6) is used to generate the two-dimensional base shown below:

$$\xi_1^i = \tfrac{1}{3}\left(1 + 2v_i - v_j - v_k\right), \quad \xi_2^j = \tfrac{1}{3}\left(1 + 2v_j - v_i - v_k\right), \tag{11}$$

with $i = 1, \ldots, m+1$, $j = 1, \ldots, m+2-i$, $k = m+3-i-j$ and $v_1 = 0$, $v_i = \left(1 + t_{i-1}\right)$ and $\mathrm{v}_{m+1} = 1$ are dimensionless points, where t_i for $i = 2, 3, ..., m$ are the zeros of the polynomial of degree $m-1$ distributed in the interval of $[-1,1]$.

3.4 Error and Conditioning in interpolation

Consider a function $f(x)$ which is interpolated by polynomials $P_m(x)$, of degree equal to or less than m, the relative error is given by:

$$Error = \frac{\left|P_m(x) - f(x)\right|}{\left|P_m(x)\right|}. \tag{12}$$

In order to quantify the error associated with the use of high-order approximations, the Lebesgue constant (Eq. 13) and condition number (Franco, 2006) of the Vandermond matrix (Eq. 14) are used as quantifying parameters.

$$\Im_m(\xi) \equiv \sum_{i=1}^{n+1} \left|\varphi_i(\xi)\right|, \quad \Lambda_m \equiv Max\left(\Im_m(\xi)\right), \tag{13}$$

$$cond(A) = \|A\|_E \, \|A\|_E^{-1}, \quad \|A_E\| = \sqrt{\left(\sum_{i,j=1}^{n} a_{ij}^2\right)} \ . \tag{14}$$

4 ELASTOSTATIC FINITE ELEMENTS METHOD FORMULATION

The FEM formulations for one-dimensional and two-dimensional structures are presented in this section. In the one-dimensional approach, the FEM formulation for the Euller-Bernoulli beam theory is used. For the two-dimensional approach, the FEM is modeled for plane stress state problems.

4.1 One-dimensional and two-dimensional FEM formulation

The formulation of the one-dimensional FEM is substantiated in the Principle of Virtual Works (PVW) applied to Euler-Bernoulli beam theory, which can be written in variational form (Oñate, 2009):

$$\iint_{A^{(e)}} \delta\varepsilon^T \sigma t dA = \oint \delta u^T \sigma q t dS + \sum_{i=1}^{N} \delta u_{y_i} f_{y_i}, \tag{15}$$

with the nodal forces expressed by f_{y_i}. ε and σ are column vectors formed by the components of strain and stress tensors, respectively, and u the displacement vector.

From the integral formulation (Eq. 15), the discretization of the one-dimensional finite element domain of any order is performed. Thus, this discretization provides a final system of equations:

$$[k]\{d\} = \{f\}, \tag{16}$$

where $[k] = \sum_{e=1}^{N}[k^e]$ is the stiffness matrix of the system, and $\{f\} = \sum_{e=1}^{N}\{f^e\}$ is the global force vector. The stiffness matrix of the element and force vector are expressed by (Oñate, 2009):

$$[K^e] = \int_{-1}^{1} (EI)^e [B^e]^T [B^e] J(\xi) d\xi, \ \{f^e\} = \int_{-1}^{1} [N]^T q(x) J(\xi) d\xi. \tag{17}$$

with $[N]$ the vector of the one-dimensional shape functions, $[B^e]$ is the vector formed by the second derivate of the shape functions and $J(\xi)$ is the Jacobiano of the transformation.

For two-dimensional problems, the analysis of stress plane problems is performed and its variational formulation, via PVW, is described in the form:

$$\iint_{A^{(e)}} \delta\varepsilon^T \sigma t dA - \iint_{A^{(e)}} \delta u^T b t dA - \oint \delta u^T \sigma q t dS = \sum_{i=1}^{N} \delta u_{x_i} f_{x_i} + \sum_{i=1}^{N} \delta u_{y_i} f_{y_i}, \tag{18}$$

where nodal forces are expressed as f_{x_i} e f_{y_i}, t the thickness of the element, b representes the distributed forces acting per unit area (body forces), q the distributed forces acting around the contour of the element, δu is the admissible virtual displacement field and $\delta\varepsilon$ is the compatible virtual strain field compatible.

From the discretization of the problem in triangular finite elements of any order, it is possible to write the stiffness matrix and the load vector as follows (Oñate, 2009):

$$K^{(e)} = \iint_{A^{(e)}} B^T D B t dA; \ f^{(e)} = \oint_{l^{(e)}} \varphi^t q t dS \ , \tag{19}$$

where D is the constitutive matrix and B the matrix containing the derivates of the shape functions.

5 APPLICATIONS

In order to evaluate the interpolation strategies applied to the analysis of one and two-dimensional elastostatic problems by FEM, the error analysis, having as parameters the Lebesgue constant (Eq. 13), condition number (Eq. 14) and relative error (Eq. 12), is performed.

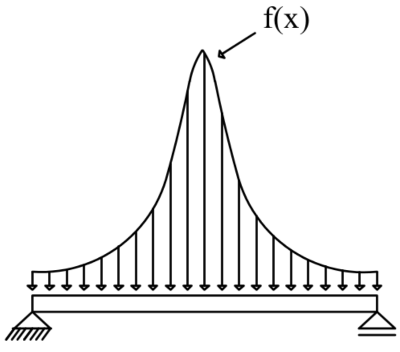
5.1 One-dimensional FEM

In the one-dimensional approach, Euler Bernoulli's beam theory is described by the differential equation:

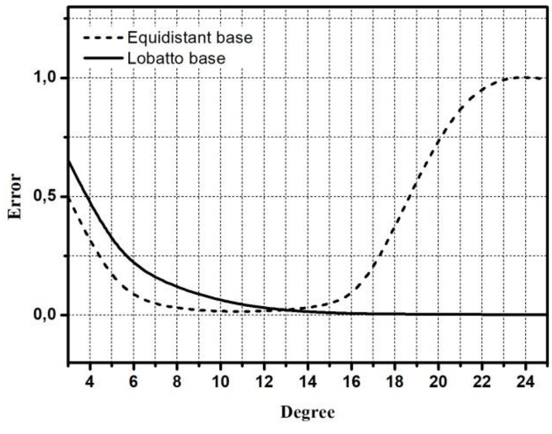
$$EI\frac{d^4y}{dx^4} = -\frac{1}{1+25x^2} \,. \tag{20}$$

In that the complex distributed load is given by the rational function $1/(1+25x^2)$ (Fig. 1 a). The structure is composed of two simple supports with unit length, modulus of elasticity (E) and moment of inertia (I), in their compatible units. For analysis, a single finite element and several degrees of approximation were performed for the equidistant and Lobatto bases.

In the numerical analysis of the problem it is possible to observe, by means of the Lesbegue constant (Fig. 2a) and the condition number (Fig. 2b), the low performance of the equidistant base in recovering the load analyzed here (with peak intensity) when compared to the reference solution $\Lambda_m \le (2/\pi)\ln(m+1) + 0,685$, as the degree of approximation is high. On the other hand, Lobatto's spectral base was able to ensure the convergence (see Fig. 2a-b) for the reference solution as the polynomial order was increased. With respect to the mechanical response (displacement), the relative error of the equidistant and Lobatto bases is shown in Fig. 1b with reference to the analytical solution obtained from Eq. (20). In Figure 1b, the stability and low error in the Lobatto approximation can be observed, and on the other hand, the instability and high error of the equidistant approximation as the order of the interpolation is increased.



(a)



(b)

Figure 1. Simply supported beam subjected to a rational loading (a) maximum displacement error (b).

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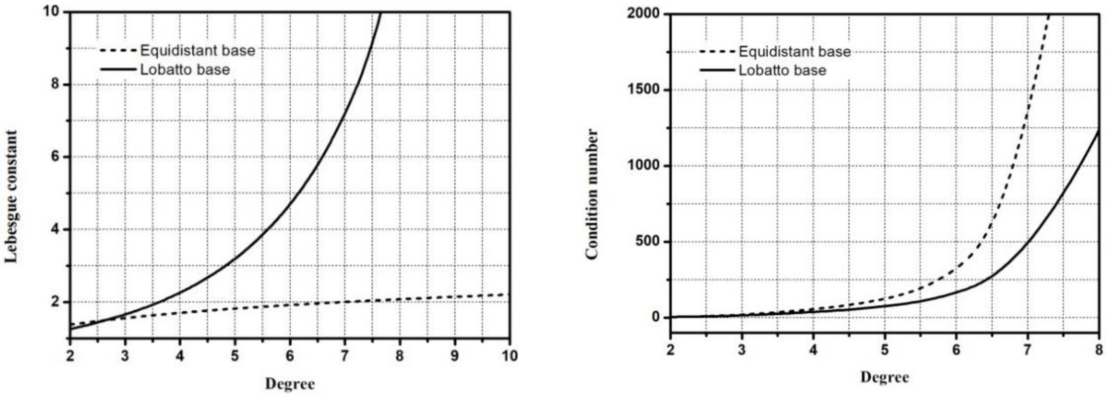
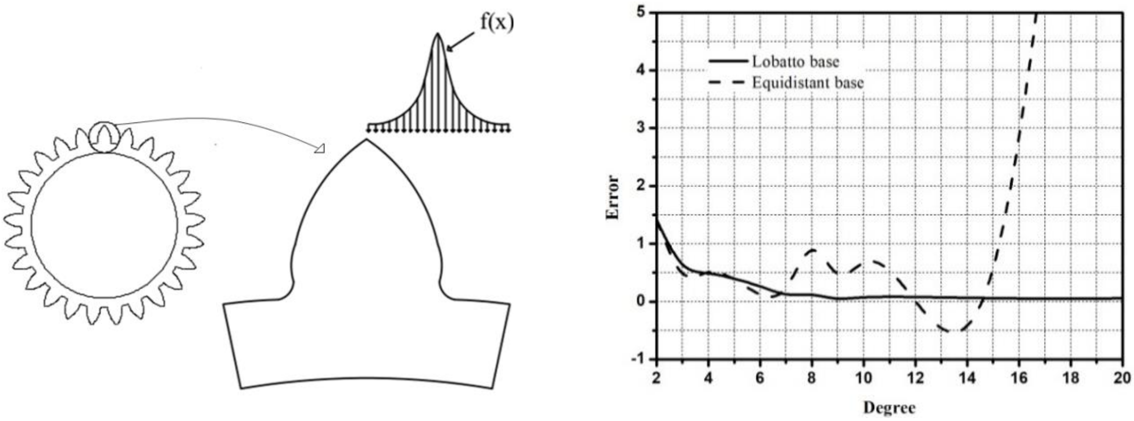


(a) (b)

Figure 2. Lebesgue constant (a). Condition number (b).

5.2 Two-dimensional FEM

The proposed example refers to the mechanical analysis of a gear subjected to a rational loading identical to that used in the previous example (see Fig. 3a). For numerical analysis, the solution obtained by a discretization of 22000 triangular elements of grade 2 for the domain and 112 one-dimensional elements of grade 2 for the approximation of the load was adopted as reference solution. For the performance analysis between the equidistant and Lobatto triangular bases, the following discretizations were adopted: 2 elements for the domain and only a single element for the complex loading (one-dimensional approach), with all elements varying the degree of approximation of 2 to 25.



(a) (b)

Figure 3. Geometric representation of the gear (a) . Analysis of the displacement error of the gear subjected to a rational loading (b).

In Figures 3b, 4 and 5 both stability and convergence for the reference solution are shown when the Lobatto base is used, while adverse effect is presented by the evenly spaced base with increasing order of approximation.

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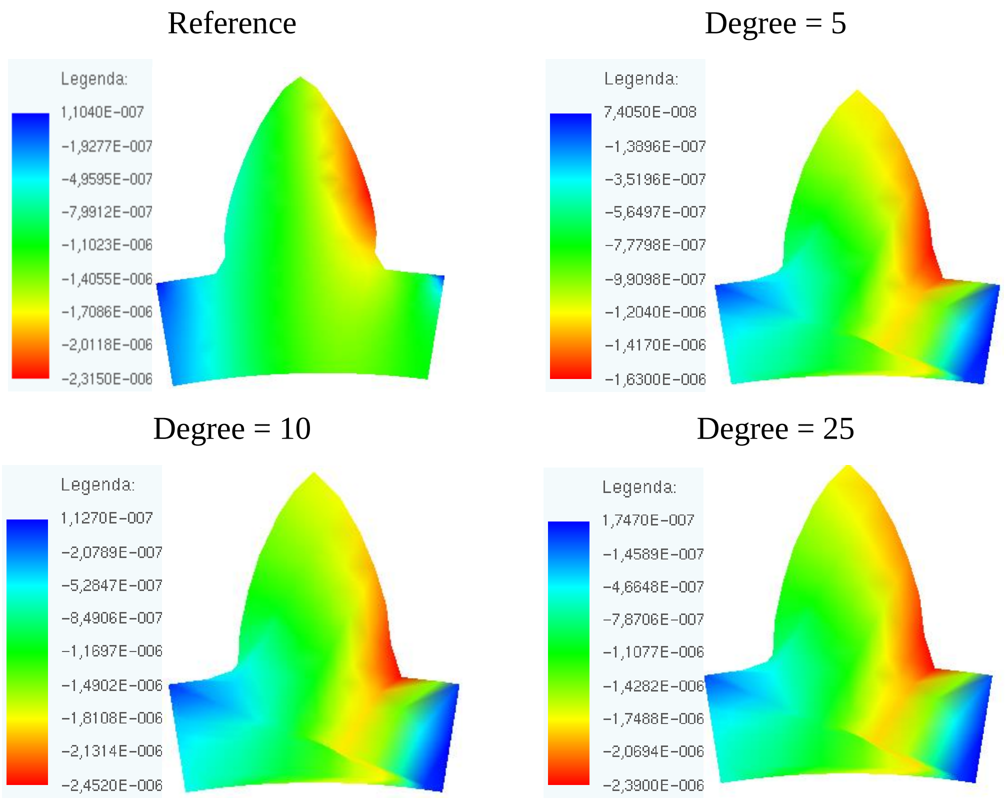

Figure 4. Vertical displacement (m) of the gear for the Lobato's base.

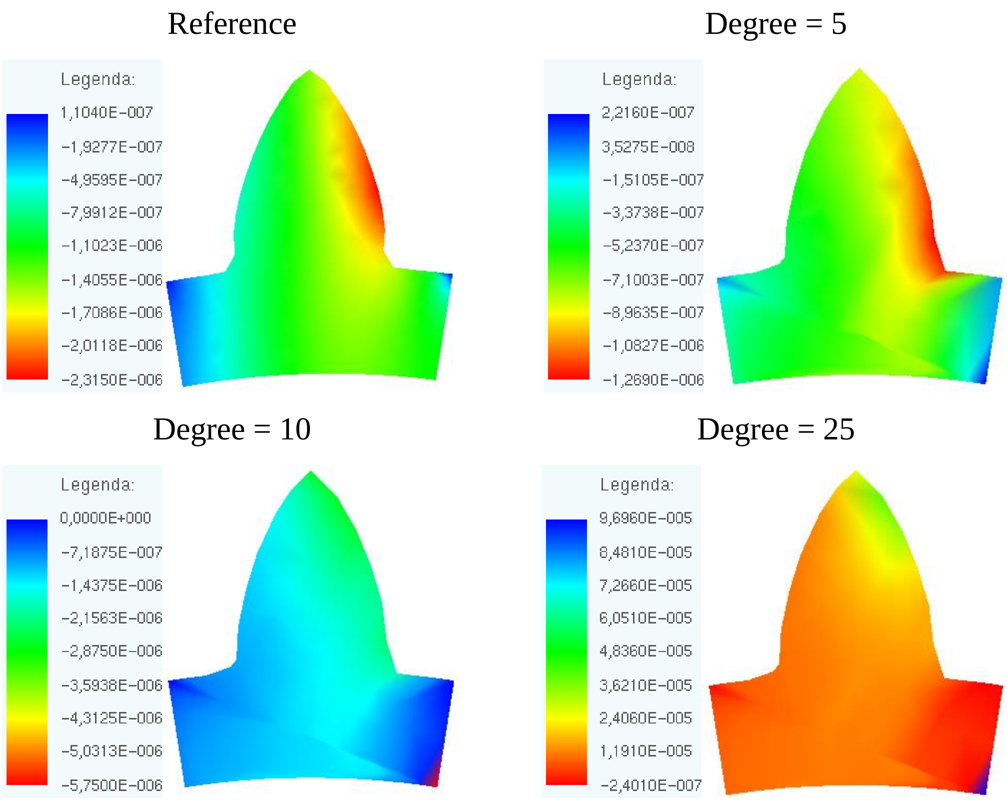

Figure 5. Vertical displacement (m) of the gear for the equidistant base.

6 CONCLUSION

Based on the presented approach, it is verified that the error is directly connected to the choice of the base of the nodal points, being this, decisive in the generation of interpolating functions. In this work, significant efficiency was observed for the Lobatto base when compared to the equidistant base, both to represent complex loading and to represent fields of convergent displacements when the number of elements is fixed and the degree of approximation is high.

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