# REVISITING THE INDIRECT-BEM'S FULFILLMENT OF EQUILIBRIUM CONDITIONS IN INTERNAL PROBLEMS

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***Abstract.*** *This paper reassesses the suitability of the Indirect Boundary Element Method (I-BEM) for solving closed-domain problems. The I-BEM differs from its Direct-BEM (D-BEM) counterpart in that it relates tractions and displacements at the discretized boundary through a set of fictitious stresses, rather than physical displacements and tractions. Previous investigations seemed to show that the solution of closed-domain problems by the I-BEM failed to satisfy elementary equilibrium criteria, which was then attributed to the formulation's inability to comply with criteria on the spectral properties of the influence matrices. This article proposes that the issue of equilibrium in the application of the I-BEM to solve close-domain problems arose from incorrect interpretations of a) traction discontinuities at the boundary of the problem and b) the meaning of equilibrium in the I-BEM formulation. The results show that the I-BEM does satisfy elementary equilibrium conditions in internal problems, provided a sufficiently fine discretization is chosen.*

***Keywords:*** *Indirect-BEM, equilibrium, Green's functions, internal problems.*

# 1 INTRODUCTION

The Indirect-Boundary Element Method (I-BEM) is a numerical method based on the superposition of Green's functions to approximate the solution of differential equations (Vable and Ammons, 1995). The I-BEM resembles its Direct-BEM (D-BEM) counterpart in that it involves the integration of an influence function for the medium that is being treated, in that it requires only the discretization of the domain boundary, provided a corresponding influence function is available, and that it is especially well-suitable for the study of unbound domains. Unlike in the D-BEM, however, in which a final set of algebraic equations relates physical displacements and tractions at the domain boundary, in the I-BEM those equations relate tractions and displacements through a set of fictitious stresses.

Mesquita, Rapajakse and Labaki (2010) showed that the I-BEM presents a much slower convergence rate with respect to the number of boundary elements than the D-BEM, and claimed that the discontinuity observed in some of the traction components of its stress influence matrix keeps it from satisfying equilibrium conditions in internal problems.

The present article looks more thoroughly into the solution of closed domains (internal problems) through the I-BEM. Special attention is given to the fulfillment of elementary equilibrium conditions in such problems.

# 2 INDIRECT BOUNDARY ELEMENT METHOD

Consider the external and internal problems shown in Fig. 1.

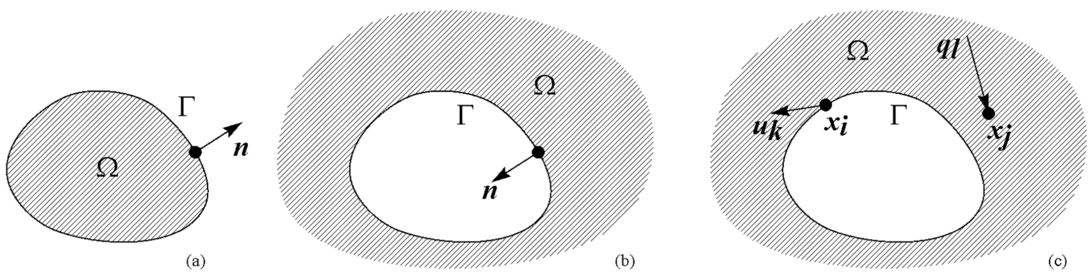


**Figure 1. Internal (a) and external (b) problems and application of fictitious load (c).**

The displacements $u_{kl}$ at a field point $\boldsymbol{x_i}=(x_i,y_i,z_i)$ in the $k$-direction due to a fictitious load $q^1{}_l$ applied in a point $\boldsymbol{x_j}=(x_j,y_j,z_j)$ in the $l$-direction is given by (Labaki, Mesquita and Adolph, 2008; Guerra, Barros and Pavanello, 2017)

$$u_{kl}\left(\boldsymbol{x}_i\right)=G_{kl}\left(\boldsymbol{x}_i,\boldsymbol{x}_j\right)q_l^1\left(\boldsymbol{x}_j\right) \tag{1}$$

while the tractions $t_{kl}$ due to a fictitious load $q^2{}_l$ in the $l$-direction is given by

$$t_{kl}\left(\boldsymbol{x}_i\right)=H_{ksl}\left(\boldsymbol{x}_i,\boldsymbol{x}_j\right)n_s\left(\boldsymbol{x}_i\right)q_l^2\left(\boldsymbol{x}_j\right)=S_{kl}\left(\boldsymbol{x}_i,\boldsymbol{x}_j\right)q_l^2\left(\boldsymbol{x}_j\right) \tag{2}$$

in which $n_s(\boldsymbol{x_i})$ the $s$-component of the normal vector at point $\boldsymbol{x_i}$ of $\Gamma$, and $G_{kl}(x_i,x_j)$ and $H_{rsl}(x_i,x_j)$ are respectively the displacement of point $\boldsymbol{x_i}$ in the $k$-direction due to a unit load applied at $\boldsymbol{x_j}$ in the $l$-direction, and the stress component $\sigma_{rs}$ at point $\boldsymbol{x_i}$ due to a unit load applied at $\boldsymbol{x_j}$ in the $l$-direction.

$G_{kl}(x_i,x_j)$ and $H_{rsl}(x_i,x_j)$ are non-singular Green's functions – influence functions – corresponding to the continuum in which the particular $u_{kl}$–$q^1{}_l$ and $t_{kl}$–$q^2{}_l$ relations take place. Influence functions may be obtained by the integration of classical Green's functions, such as the one presented by Kitahara (1985) for elastodynamics of isotropic full-spaces. In this case, the non-singular influence function is obtained by integrating the point load considered in the solution over an area. Alternatively, influence functions may be derived directly in the form of non-singular loadings of particular shapes within particular media. Mesquita, Romanini and Labaki (2012) and Labaki, Romanini and Mesquita (2012) derived an influence function for the time-harmonic response of homogeneous, isotropic full-spaces due to unit loads applied in a rectangular area. Wang and Rajapakse (1993) derived an influence function for time-harmonic response of homogeneous, transversely isotropic half-spaces due to axisymmetric ring, disc, or annular loads.

While the relations expressed in Eqs. (1) and (2) are independent, one may establish the case in which a fictitious load $q_l$=$q^1{}_l$=$q^2{}_l$ results in the displacement and traction field of a certain problem. In that situation,

$$q_l^1\left(\boldsymbol{x}_j\right)=G_{kl}^{-1}\left(\boldsymbol{x}_i,\boldsymbol{x}_j\right)u_{kl}\left(\boldsymbol{x}_i\right)=q_l^2\left(\boldsymbol{x}_j\right) \text{ and} \tag{3}$$

$$t_k\left(\boldsymbol{x}_i\right)=S_{kl}\left(\boldsymbol{x}_i,\boldsymbol{x}_j\right)G_{lm}^{-1}\left(\boldsymbol{x}_i,\boldsymbol{x}_j\right)u_m\left(\boldsymbol{x}_i\right)=K_{km}\left(\boldsymbol{x}_i,\boldsymbol{x}_j\right)u_m\left(\boldsymbol{x}_i\right) \tag{4}$$

Equation (4) cannot be solved directly because the traction distribution along the boundary of the problem is not known. The I-BEM consists in discretizing the boundary of the problem into a number of elements, in each of which the applied fictitious load is assumed to have a known distribution. In the present work, fictitious loads are assumed to be constant over each element (Fig. 2).

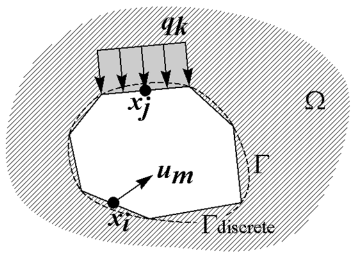


**Figure 2. Discretized boundary and constant fictitious load distribution.**

In this discretization with N constant elements (boundary elements with constant traction distribution), the middle point of the elements may be taken to represent $\boldsymbol{x_i}$ and $\boldsymbol{x_j}$. The discretized version of Eq. (4) becomes

$$\{t\}=[S][G]^{-1}\{u\}=[K]\{u\} \tag{5}$$

in which $\{t\}=\{t_{x1}\ t_{y1}\ t_{z1}\ \ldots\ t_{xN}\ t_{yN}\ t_{zN}\}^T$, $\{u\}=\{u_{x1}\ u_{y1}\ u_{z1}\ \ldots\ u_{xN}\ u_{yN}\ u_{zN}\}^T$, and $[S]$ and $[G]$ contain the influence functions of stresses and displacements. The influence functions of



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stresses and displacements in *[S]* and *[G]* are measured at each element of the discretized boundary in response to constant unit loads applied at each element of the boundary.

The solution of Eq. (5) yields the traction-displacement relation at the discretized boundary.

# 3 INTERNAL VERSUS EXTERNAL PROBLEMS

Consider the external and internal problems shown in Fig. 1, in which the normal vector $\boldsymbol{n}$ points outward from the domain Ω. One can see from the application of Cauchy's equation in Eq. (2), $S_{kl}=H_{ksl}\boldsymbol{n}_s$, that the terms in the stress influence matrix *[S]* depend on the direction of the normal vector, therefore depending on the problem being external or internal. This dependency is illustrated in Fig. 3. Figures 3a and 3b illustrate an internal problem and its external counterpart. It does not matter which is which; only that the normal vector points outward from the domain in both cases. In both cases the same fictitious load $q_z$ is applied in the positive $z$-direction, resulting in a displacement $u_z$ in the positive $z$-direction and a traction $t_z$ in the positive $z$-direction. The resulting normal stress, however, has different signs in cases *a* and *b*, depending on the normal vector having the same or the opposite direction of $t_z$.

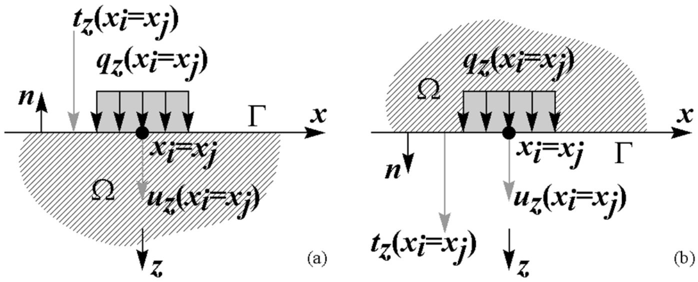


**Figure 3. Application of fictitious loads in internal and external problems.**

The discontinuity between the values of the stress component at the boundary in the internal and external problems corresponds to the applied load (in Fig. 3, $q_z$). In the I-BEM, since the influence functions correspond to unit loads, this discontinuity is also unit. This means that, in the discretized boundary, $[S_2]=-([S_1]-[I])$, in which $[S_1]$ and $[S_2]$ are the stress influence matrix for the internal problem and its corresponding external problem, and *[I]* is the identity matrix of compatible dimensions. Conversely, the displacement influence matrices are the same for both cases, $[G_2]=[G_1]$. This fundamental characteristic was not considered by Mesquita, Rapajakse and Labaki (2010) in their solution of internal problems with the I-BEM.

# 4 EQUILIBRIUM AND CONVERGENCE

The fictitious load–displacement relation in Eq. (1) is independent from the fictitious load–traction relation in Eq. (2). In the passage from Eqs. (1) and (2) to Eqs. (3) and (4), one



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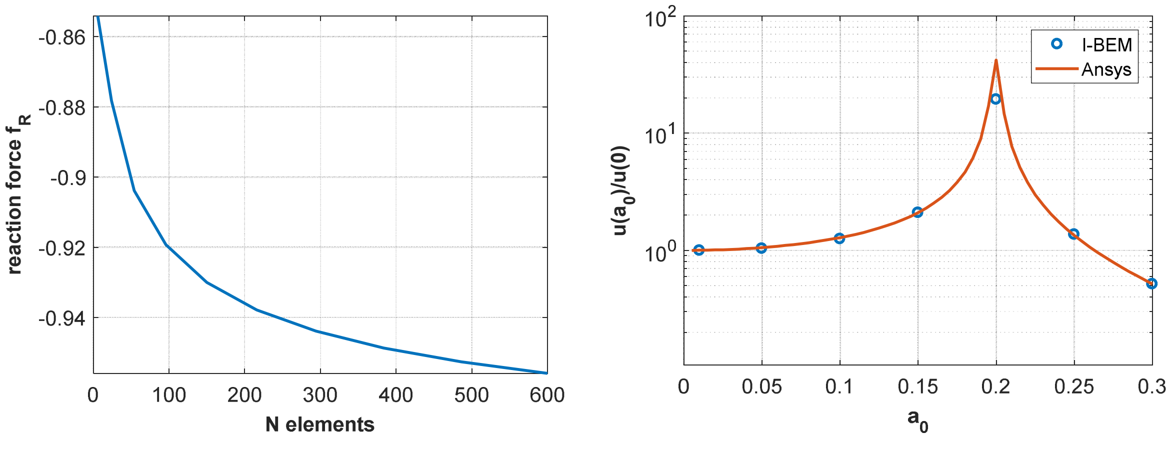
finds a (potentially unique) solution of fictitious loads in which a traction-displacement relation is obtained. While Eq. (4) describes a problem that is in equilibrium, there is potentially only one case of fictitious loads that satisfies it. In the discretization of Eq. (4) leading to the I-BEM, one replaces this one solution that satisfies equilibrium by a piece-wise approximation of it, which most likely does not. The system will tend to show that the equilibrium is satisfied as the discretization of the boundary – the piece-wise approximation of the fictitious loads – tend to the continuous boundary.

This is fundamentally different than in the finite element method (FEM), for example, in which the equilibrium of the system is satisfied by definition in the formulation of each element. The equilibrium of the system is satisfied regardless of the coarseness of the discretization. The same happens in the D-BEM, in which the traction-displacement relation comes from an integral equation in equilibrium. Because of the I-BEM's inability to satisfy the equilibrium in internal problems with coarse discretizations, Mesquita, Rajapakse and Labaki (2010) were misled into deeming it unsuitable to solve internal problems. They did, however, show that the method slowly converges to the equilibrium condition as the discretization gets finer.

# 5 NUMERICAL RESULTS

Figure 4 shows the solution of two internal problems with the I-BEM. Figure 4a considers a cube of size a=1, discretized by N constant, square boundary elements. The I-BEM discretization for this cube uses influence functions obtained by the integration of the Green's function for elastic, isotropic full-spaces (Kitahara, 1985) over a square area. One face of the cube is clamped in all directions, while a uniformly distributed, unit normal force is applied on the opposite face. In this situation, the clamped face should present a traction distribution whose integral (reaction force $f_R$) is minus unit. Figure 4a shows the convergence of the reaction force with increasing number of elements.




(a) (b)

**Figure 4. (a) Reaction forces in clamped cube and (b) frequency response of a clamped cylinder.**

Figure 4b considers the case of a short cylinder clamped in one end and subjected to a uniformly distributed unit force in the other. The cylinder has radius *a* and length *l* such that $l/a=2$, and material properties $\mu$, $\rho$, and $\nu=0.25$, and the frequency of excitation is $\omega$. The results are shown in terms of nondimensional frequency $a_0=a\omega(\mu/\rho)^{0.5}$ and normalized displacement $u(a_0)/u(0)$, in which $u$ is the displacement of the center of the free end of the cylinder in the direction of the loading.

# 6 CONCLUDING REMARKS

This article reconsidered the issue of equilibrium of internal problems solved with the Indirect Boundary Element Method. It discussed that the traction discontinuity at the boundary must be interpreted according to whether the problem is an internal or external one. The article proposes a new interpretation of the meaning of equilibrium in the formulation of I-BEM, and argues that the equilibrium condition can be satisfied by the I-BEM, provided an adequately fine discretization is selected.

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# REFERENCES

Adolph, M., 2006. *Síntese de Funções de Green e Estados Auxiliares Viscoelastodinâmicos em Meios Tridimensionais Ilimitados com Auxílio de Transformada de Radon*. PhD thesis, University of Campinas.

Barros, P. L. A. and Mesquita, E., 2000. Singular-ended spline interpolation for two-dimensional boundary element analysis. *International Journal for Numerical Methods in Engineering*, vol. 47, n. 5.

Dumont, N. A., 1998. An assessment of the spectral properties of the matrix G used in the boundary element methods. *Computational Mechanics*, vol. 22.

Guerra, C. S., Barros, P. L. A. and Pavanello, R., 2016. Dynamic Behavior of Flexible Plates Supported by a Transversely Isotropic Half-Space. In: *XXXVII Iberian Latin American Congress on Computational Methods in Engineering*.

Kitahara, M., 1985. *Boundary Integral Equation Methods in Eigenvalueproblems of Elastodynamics and Thin Plates*. Elsevier, Amsterdam.

Labaki, J., Mesquita, E. and Adolph, M., 2008. 3D Dynamic Structure Analysis Based on an Indirect Version of the Boundary Element Method Using a Non-Singular Auxiliary State. In: *XXIX CILAMCE - Iberian Latin American Congress on Computational Methods in Engineering*.

Labaki, J., Romanini, E. and Mesquita, E., 2012. Stationary Dynamic Displacement Solutions for a Rectangular Load Applied within a 3D Viscoelastic Isotropic Full Space Part II: Implementation, Validation, and Numerical Results. *Mathematical Problems in Engineering (Print)*, vol. 2012.

Mesquita, E., Romanini, E. and Labaki, J., 2012. Stationary Dynamic Displacement Solutions for a Rectangular Load Applied within a 3D Viscoelastic Isotropic Full Space Part I: Formulation. *Mathematical Problems in Engineering (Print)*, vol. 2012.

Mesquita, E., Rajapakse, R. K. N. D. and Labaki, J., 2010. An Investigation of the Indirect-BEM for 3D Elastostatic and Elastodynamic Problems. In: *ASCE EMI*.

Rajapakse, R. K. N. D and Wang. Y.,1993. Green's Functions for Transversely Isotropic Elastic Half Space. *Journal of Engineering Mechanics*, vol. 119, n. 9, pp. 1724-1746.

Vable, M. and Ammons, M. B., 1995. A Study of the Direct and Indirect BEM. *Transactions on Modelling and Simulation*, vol. 10.