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APPLICATION OF MONTE CARLO λ-NEUMANN TO THE STOCHASTIC EULER-BERNOULLI BEAM BENDING PROBLEM

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Abstract. *Reliability theory had its origins in the mid-1940s and has been used to predict, for example, the expected life or average life of electronic and mechanical systems or components. It has been developed using concepts and definitions related to areas of probability, statistics, and mathematical optimization, among others. On the other hand, the technological evolution witnessed in recent decades is partly due to the development of mathematical and computational models that are similar to real systems in engineering. The present work aims at studying and proposing new numerical methodologies and strategies to quantify uncertainty in structural problems. Although widely used, the Monte Carlo simulation generates high computational costs making its application prohibitive regarding some problems. Recent works propose methodologies based on Newmann series that have presented a satisfactory performance in problems to which they have been applied. Among those methodologies, the Monte Carlo λ-Neumann introduces the λ-convergence parameter, which, besides reducing computational time, presents non-intrusive characteristics, that is, repetitive iteractions with restart of the programming routine. Examples of this evolution of methods are presented at the end of the current article together with further considerations and hypotheses associated to each of them.*

Keywords: *Numerical Methods, Stochastic Computation, Monte Carlo Method, Newmann Expansion.*

INTRODUCTION

The technological evolution witnessed in recent decades present relevant results regarding the reliability of structural systems that are very often complex and whose functionality depends heavily on the ability to predict their performance, in some cases, still subject to not-totally-controlled conditions, that is, seismic loads, noises or random loads.

Recent literature provides works in which the formulation of mathematical and numerical models associate such uncertainties to the material and geometry of the structural elements as well as to external forces acting on them. In those models, such physical magnitudes are identified as random variables.

On the other hand, structural reliability consists of the ability of a system to meet a required function when submitted to certain conditions and during a given period of time. In statistical terms, reliability is interpreted as the probability of a system to attain success with regard to a performance measure.

Stochastic mechanics seeks to quantify the variability of responses associated to the random input data based on the understanding that the process is a result of spatial variability. This variation forms a multivariate stochastic field, where one seeks to evaluate the first- and second-order statistical moments of responses. In this sense, Benjamin and Cornell (1970), Vanmarcke (1983), and Wang *et. al.* (2013), among other authors, state that, to formulate the structural problem, first the properties of the system must be adequately modeled with a probability distribution and, at a second stage, one must solve the differential equation that governs the problem aiming at obtaining the statistic moments associated to the variability of responses, such as stress and strain.

Ghanem and Spanos (1991) also present a methodology for evaluating random variables in which the random parameters of the system are modeled within a second-order scholastic process defined by its averages and covariance function. When applied to finite elements, the differential equation solution is obtained by functionals that act as a linear filter on the process. This concept can also be generalized to nonlinear functional representations.

In the variational formulation, the numerical solution of those differential equations requires the inversion of the stochastic stiffness matrix, which is usually performed by the Monte Carlo simulation. In this method, the random variables are multiple and, in order to account for the result, each iteration has the sum of the random values associated to those variables. The method transforms the problem of evaluating a definite integral into the statistical problem of estimating an average.

Another matrix inversion numerical model is presented by Shinozuka and Nomoto (1980), Adomian and Malakian (1980), and Shinozuka (1987), who introduce Newmann expansion as an iterative process for solving linear algebraic equations assuming that the spectral radius of the iterative matrix is smaller than 1.

Yamazaki *et. al.* (1988) apply Newmann expansion to the stochastic finite elements where Young’s modulus and Poisson’s ratio are random space-dependent variables with a null first-order moment and a unitary second-order moment. The correlation function is formed by a spectral density function based on the wave number vector. In each finite element, covariance is obtained from the geometric distance between the centroids of each element. When the finite-element method is used, each element must be orthogonally positioned in relation to the previous one. This prerogative constitutes a system of equations that must ensure the orthogonalization and normalization of those vectors. In general, the inversion of the stiffness matrix is obtained by applying Cholesky decomposition, in which the components of that factorization comply with a normal distribution.

In Newmann series model, the matrix inversion is centered on its eigenvalues, which satisfy a convergence criterion based on the admissible error. In Shinozuka and Deodatis (1986), the method is also applied for solving the problem involving plates and beams, dealing with response variability as a result of a spatial variability of the material properties along the structure when it is submitted to static and deterministic loadings. Multi-dimensional and multivariate fields used in the digital generation of its sampling functions are presented by Shinozuka and Astill (1972).

Ávila and Beck (2015-April) propose an introduction to λ-parameter, which accelerates the Newmann series convergence optimizing the set of solutions through finite-dimensional linear operator norms such as the Euclidean, infinity, Frobenius, and maximum norms. The accuracy and efficiency of the method are demonstrated by applying it to beam bending problems. The efficiency of the proposed method is its non-intrusive characteristics and also in the substantial reduction in data processing. The same authors - Ávila and Beck (2015-June) – obtain numerical results for linear and non-linear stochastic systems referencing the method with regard to the single Monte Carlo simulation.

The scope of the present work is the basis for the stochastic Euler-Bernoulli beam bending problem and, subsequently, proposes a solution using the Galerkin method. In association with this model, the Monte Carlo simulation is used to determine the first- and second-order moments for random displacement fields. The following sections present the matrix inversion model by Newmann expansion and the proposal of an introduction to the λ-convergence parameter. The model is discussed and compared for beams with various boundary and loading conditions.

1. THE STOCHASTIC BENDING PROBLEM

In the present work, the classical Euler-Bernoulli theory – or pure bending theory – is applied to uniform prismatic beams of constant cross-section and with longitudinal length as a predominant dimension. Ávila and Beck (2010-May) present the differences between Euler-Bernoulli and Timoshenko beam theories. They establish uncertainty results associated to the elasticity modulus and cross-section height. Despites the sharp disagreement between the uncertainty results, the authors point out that the deterministic values are very similar for beams with intermediate length within the limits of each application.

In the light of Lax-Milgram's - which ensures the existence, unicity, and continuity of the solution -, it becomes necessary that stiffness to bending (*EI*) be strictly positive as a compact support so that $\exists\, \underline{\alpha}, \overline{\alpha} \in R^+ \setminus \{0\}, |\underline{\alpha}, \overline{\alpha}| < \infty, P(\{\omega \in \Omega : EI(x,\omega) \in [\underline{\alpha}, \overline{\alpha}], \forall\, x \in (0,l)\} = 1)$. Besides, loading must have a finite variance, that is, $f \in L^2(\Omega, f, P; L^2(0,l))$.

The relations between random variables are described using the triple $(\Omega, \mathcal{F}, \mathcal{P})$ where Ω is the space of events, $\mathcal{F}$ is an σ-algebra, and $\mathcal{P}$ is a probability measure. In this context, the strong form of a cantilever Euler-Bernoulli beam bending problem is written by:

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$$(\text{P.1})\begin{cases}\dfrac{d^2}{dx^2}\left[E(x,\omega)I(x,\omega)\dfrac{d^2u}{dx^2}\right]=f(x,\omega),\ \forall x,\omega\in(0,l)\times(\Omega,F,P);\textit{such that}:\\[2ex] u(0,\omega)=u(l,\omega)=0,\ \dfrac{d^2u}{dx^2}(0,\omega)=\dfrac{d^2u}{dx^2}(l,\omega)=0,\forall\,\omega\in(\Omega,F,P),\end{cases}$$

where $u(x,\omega)$ is the displacement vector defined in $x\in(0.l)$, ω_k is the $k^{\text{-}th}$ single event, or the actualization of the process. The elasticity modulus and the second-order inertia are random variables of $\{EI(x,\omega_k),K_\omega(x,\omega_k)\}$.

Galerkin method is used to obtain numerical solutions of stochastic differential equations from the weak formulation corresponding to the PVC strong formulation. Such transformation allows reducing the required regularities for the numerical solution.

2. GALERKIN'S STOCHASTIC FINITE ELEMENT

After establishing $\omega\in(\Omega,\ F,\ P)$, the model is used to solve the stochastic problem integrating directly to its expression, that is, by taking $v\in V$ as an approximation function and expanding the derivative product in the solution, one has:

$$\int_\Omega\int_a^b-\frac{d}{dx}\left[\left(k\frac{du}{dx}\right)(x,\omega)v(x,\omega)\right]dxdP=\int_\Omega\int_a^b\left(k\frac{du}{dx}\frac{dv}{dx}\right)(x,\omega)dxdP-\left[\left(k\frac{du}{dx}v\right)(x,\omega)\right]_a^b. \tag{1}$$

The deterministic formulation of the variational problem may be found in Reddy (1984), Zienkiewic *et al.*, 2013, and Hugles, 2000. The essential boundary conditions are in terms of cross-section displacement and cross-section rotation. Yet, the natural boundary conditions are given in terms of shear stress and bending moment.

Based on those authors and defined in the sampling space of the equation, the expression of the problem in its bi-linear form becomes:

$$(\text{P.2})\begin{cases}a_{\omega_k}(v,u)=\displaystyle\int_{x_e}^{x_{e+1}}k\frac{d^2v}{dx^2}\frac{d^2u}{dx^2}dx,\\[2ex] l(v)=-\displaystyle\int_{x_e}^{x_{e+1}}(v\,f)dx,\end{cases}$$

where operator $a_{\omega_k}(v,u)$ is symmetric and positive-definite for a beam, x_e, the position of element e, and index k referring to the $k^{\text{-th}}$ actualization of the stochastic process.

In order for the integration of the problem to be meaningful, the interpolation functions must have a continuous second derivative or a piecewise continuous, that is, it must belong to the $C^2(0.l)$ class. In addition, the residue is orthogonal to any basic function. By imposing orthonormality and neglecting the terms relating to the boundary conditions, one has:

$$a_{\omega_k}(\phi_i(x),\phi_j(x))=\int_a^b\left(\mu_k\frac{d^2\phi_i}{dx^2}\frac{d^2\phi_j}{dx^2}\right)dx+\sum_{q=1}^N\xi_q(\omega_k)\int_a^b\left(\varphi_q\frac{d^2\phi_i}{dx^2}\frac{d^2\phi_j}{dx^2}\right)(x)dx. \tag{2}$$

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And,

$$f_i^k = -\int_a^b \phi_i f \, dx, \tag{3}$$

where $\phi_i(x), \phi_j(x)$ are the approximation functions, $\xi_q(\omega_k)$ are the orthogonal random variables associated to the k-th actualization, and φ_q is the standard deviation defined for actualization in $[\xi_1, \xi_1]$.

Considering the finite element of the beam, each node has cross-section displacement and rotation as unknowns, that is, one has a total of 4 unknowns per element. The present work uses the Hermite polynomials for the form functions of Galerkin finite-elements.

It is worth noting that the notation emphasizes the idea of applying the approximation functions ensuring the continuity condition. The works presented by Ávila and Beck identify the elements of the stiffness matrix as follows:

$$k_{ij}^0 = a_0(\phi_i, \phi_j) = \int_a^b \left(\mu_k \frac{d^2\phi_i}{dx^2} \frac{d^2\phi_j}{dx^2} \right) dx, \tag{4}$$

$$k_{ij}^q = a_q(\phi_i, \phi_j) = \int_a^b \left(\varphi_q \frac{d^2\phi_i}{dx^2} \frac{d^2\phi_j}{dx^2} \right) dx. \tag{5}$$

Based on those new coefficients, one obtains:

$$a_{\omega_k}(u,v) = k_{ij}^0 + \sum_{q=1}^{N} \xi_q(\omega_k) k_{ij}^q. \tag{6}$$

Thus, the bi-linear form becomes:

$$a_{\omega_k}(u,v) = \sum_{i=1}^{n} k_{ij}^0 u_i(\omega_k) + \sum_{q=1}^{N} \xi_q(\omega_k) \sum_{i=1}^{n} k_{ij}^q u_i(\omega_k). \tag{7}$$

This way the PVC can be reformulated as follows: to determine $u(x, \omega_i) \in V$, in such way that:

$$\text{P.3} \begin{cases} a_{\omega_k}(u,v) = f(v), \forall\, v \in V, \, where, \\ \sum_{i=1}^{n} k_{ij}^0 u_i(\omega_k) + \sum_{q=1}^{N} \xi_q(\omega_k) \left[\sum_{i=1}^{n} k_{ij}^q u_i(\omega_k) \right] = f_i(v). \end{cases}$$

The inversion of the stiffness matrix requires considerable computational time and, in this sense, the use of the Newmann series is gradually standing out. The operational cost of this inversion is minimized without compromising the accuracy of the results. The method presented in the following section is used to perform this random operation using the Newmann expansion and, subsequently, the introduction of the numerical optimization solution and the convergence parameter.

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3. THE MONTE CARLO λ-NEUMANN METHOD

Yamazaki *et. al.* (1985), Shinozuka and Nomoto (1980), Adomian and Malakian (1980), Shinozuka (1987), and Ávila and Beck (2015) propose that, in order to materialize this operation, one has to rewrite the equation of problem P.3 in the vector-matrix representation,

$$K^0 U(x,\omega_k) + \sum_{q=1}^{N} \xi_q(\omega_k) K^q U(x,\omega_k) = F_i(x,\omega), \tag{8}$$

where *K* and *U* are the nodal vector of stiffness and displacement, respectively, for the structural sample, such that $\left[K^0 + \sum_{q=1}^{N} \xi_q(x,\omega_k) K^q\right] U(x,\omega_k) = F_i(x,\omega) \Rightarrow (KU)(x,\omega_k) = F_i(x,\omega)$.

Therefore, in terms of the random displacement vector, one has:

$$U(\omega_k) = [K(\omega_k)]^{-1} F_i(\omega). \tag{9}$$

The method uses the Newmann series in the solution of the linear system such that for $U_0 : (\Omega, F, P) \to R^n$, the inverse matrix may be replaced by the following expression:

$$\text{(P.4)} \quad \begin{cases} Find\ U(\xi_k) \in (\Omega, F, P),\ such\ that, \\ U(\xi_k) = (I + P(\xi_k))^{-1} U_0. \end{cases}$$

where *P* is the argument of the series, $U_0 = (K_0)^{-1} F$ is the displacement vector.

Expressing $U_0 = \left[u_1^0, K, u_m^0\right]^T F$ and applying the boundary conditions, the global stiffness matrix admits the following decomposition:

$$K(\xi_{\alpha\beta}) = K_0 + \Delta K(\xi_{\alpha\beta}), \tag{10}$$

where *ΔK* is uncertainty associated to the polynomial function *ξ(ω)*, and index *αβ* is related to the matrix components, given by:

$$\Delta K = \sum_{q=1}^{N} \xi_q(\omega_k) K^q. \tag{11}$$

The inputs of the stiffness matrix K_0 are evaluated in the expected values of the stiffness coefficients, whereas the inputs of matrix $\Delta K(\xi_{\alpha\beta})$, are calculated based on the random variability around the values expected from those coefficients.

Based on its argument, $P(\xi_{\alpha\beta}) = K_0^{-1} \Delta K(\xi_{\alpha\beta})$, one can write the stiffness matrix as follows:

$$K(\xi_{\alpha\beta}) = K_0 (I - P(\xi_{\alpha\beta})), \tag{12}$$

where *I* is the identity matrix.

This way, the displacement vector is obtained from:

$$U(\xi_{\alpha\beta}) = K^{-1}(\xi_{\alpha\beta}) F = (I - P(\xi_{\alpha\beta}))^{-1} U_0. \tag{13}$$

Based on the relative properties of Newmann series, that is: $\|P\| < 1$ and $(I - P(\omega_k))^{-1} = \sum_{s=0}^{\infty} P^s = I + P + P^2 + \ldots$ one can observe that the partial sum of order "*M*" of Newmann series is expressed by:

$$(I - P(\varpi_k))_M^{-1} = \sum_{s=0}^{M} P^S. \tag{14}$$

Equation (14) allows introducing the admitted residue, such that $(I - P)(I - P)_k^{-1} = I - \varepsilon$. Ávila and Beck (2015) demonstrate that the admitted tolerance is a linear approximation defined by distance Norms approaching the problem with the definition of parameter λ with components λ_1 and λ_2, such that:

$$(I - P)(I - P)_{(k)}^{-1} = \lambda_1 I + \lambda_2 P. \tag{15}$$

In general, the Norms are associated to the uncertainty required to the structure and, for the first-order approximation, the following optimization problem can be established:

$$\text{(P.5)} \begin{cases} Find\ (\lambda_1^*, \lambda_2^*) \in R^2 \ \ such\ that, \\ (\lambda_1^*, \lambda_2^*) = \underset{(\lambda_1,\lambda_2)\in R^2}{\arg\min} \left\{ \dfrac{1}{2} \left\| \lambda_1 (I - P(\xi_{kn})) + \lambda_2 P(\xi_{kn})(I - P(\xi_{kn})) \right\|^2 \right\}. \end{cases}$$

The objective function is non-negative and convex, and the optimal global point $(\lambda_1^*, \lambda_2^*)$ is obtained for stationarity conditions, $\nabla_\lambda f(\lambda_1^*, \lambda_2^*) = 0$.

Solving the associated equation system, the convergence parameters are determined by:

$$\begin{cases} \lambda_1^* = \dfrac{\phi - \varphi \lambda_2^*}{\alpha}, \\ \lambda_2^* = \dfrac{\alpha \vartheta - \varphi \phi}{\alpha \gamma - \varphi^2}, \end{cases} \tag{16}$$

where:

$$\begin{aligned} &\alpha(\xi_{kn}) = ((I - P(\xi_{kn}))U_0)^T (I - P(\xi_{kn}))U_0; \\ &\varphi(\xi_{kn}) = ((I - P(\xi_{kn}))U_0)^T P(\xi_{kn})(I - P(\xi_{kn}))U_0; \\ &\phi(\xi_{kn}) = U_0^T (I - P(\xi_{kn}))U_0; \\ &\gamma(\xi_{kn}) = (P(\xi_{kn})(I - P(\xi_{kn}))U_0)^T P(\xi_{kn})(I - P(\xi_{kn}))U_0; \\ &\vartheta(\xi_{kn}) = U_0^T P(\xi_{kn})(I - P(\xi_{kn}))U_0; \end{aligned} \tag{17}$$

Thus, the proposal of the *MC N-*λ method establishes that the k-th actualization of the linear system of Eq. (13) through Newmann expansion with $n = 1$ is given by:

$$U_{(1)}(\lambda^*, \xi_{\alpha\beta}) = (\lambda_1^* I + \lambda_2^* P(\xi_{\alpha\beta}))U_0 \tag{18}$$

These results are applied considering the bending problem of a bi-cantilever with uncertainties associated to material and geometry.

4. NUMERICAL RESULTS

In this section, the accuracy and performance of the *MC-N λ* method are evaluated regarding a bending problem of an $l = 1\ m$ long Euler-Bernoulli beam, fixed at both ends, constant cross-sectional loading regarding the axial displacement, $q = 100x10^3\ GPa/m$.

The random physical quantities are Young's modulus, $E = 400x10^9\ GPa$, with a rectangular cross section $b = 1/30\ m$ e $h = 1/25\ m$. The second-order inertia moment, expressed by $I = b.h^3/12$, and the theoretical scalar stiffness is $k = 3EI/L^3$.

The uncertainty associated to the random vectors are obtained through the estimators of their statistical moments, that is,

$$\hat{\mu}_{u(x)} = \frac{1}{N}\sum_{i=1}^{N} u(x,\omega_i);$$
$$\hat{\sigma}^2_{u(x)} = \left(\frac{1}{N-1}\right)\sum_{i=1}^{N}\left[u(x,\omega_i) - \hat{\mu}_{u(x)}\right]^2. \tag{18}$$

where $\hat{\mu}_{u(x)}$ is the average of a sample of the displacement vector $u(x)$ and $\hat{\sigma}^2_{u(x)}$ is the variance of $u(x)$.

Stiffness bending is modeled by a uniform distribution, $\mu = 0$ and $\sigma = 1$, which ensures orthonormality. The values generated in $1x10^5$ of the process actualizations are, $\mu_{EI} = 71331\ Nm^2$, $\sigma_{EI} = 8198.6\ Nm^2$, $s_{EI} = 6.7216x10^7\ Nm^2$. The present work adopts:

$$EI(x,\omega) = \mu_{EI} + \sqrt{3}.\sigma_{EI}\left[\xi_1(\omega)cos\left(\frac{x}{l}\right) + \xi_2(\omega)sen\left(\frac{x}{l}\right)\right]. \tag{19}$$

Figure 1 presents the vertical displacement (w) and the angular displacement ($\theta = dw/dx$) of the beam along its length. The deterministic results are obtained using theoretical models and Galerkin finite elements.

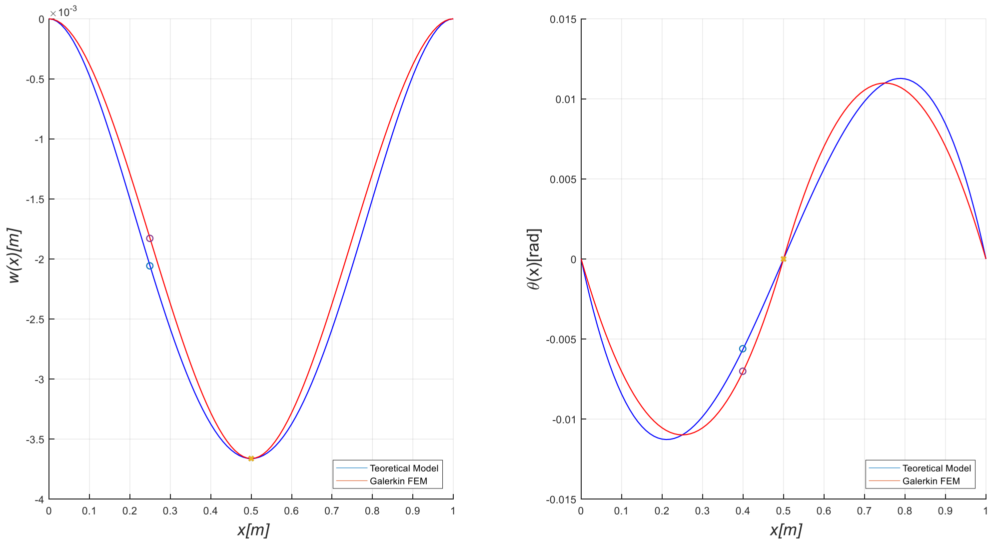

Figure 1: Average value of the vertical (on the right) and angular (on the left) displacement.

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One notices that the biggest difference between the models occur for the vertical displacement in $x = 0.25m$ and for the angular displacement in $x = 0.4\ m$; and that the smallest difference are close to the center of the beam, $x = 0.25m$. Displacements at those points are presented in Table 1.

Table 1. Comparative results of displacements

Displacement	x(m)	Theoretical model	Galerkin-FEM
w (m)	*0.25*	*-0.002059936523437*	*-0.001831054687500*
Θ (rad)	*0.4*	*-0.005625000000000*	*-0.007031250000000*
w (m)	*0.5*	*-0.003662109375000*	*-0.003662109375000*
Θ (rad)		*0*	*0*

Figure 2 illustrates the histogram and adjustment of the vertical and angular displacement, by the theoretical model and by the Galerkin-FEM, for positions $x = 0.25\ m$ and $x = 0.4\ m$ of the beam, and the histogram of the analytical angular displacement obtained for position $x = 0.4\ m$.

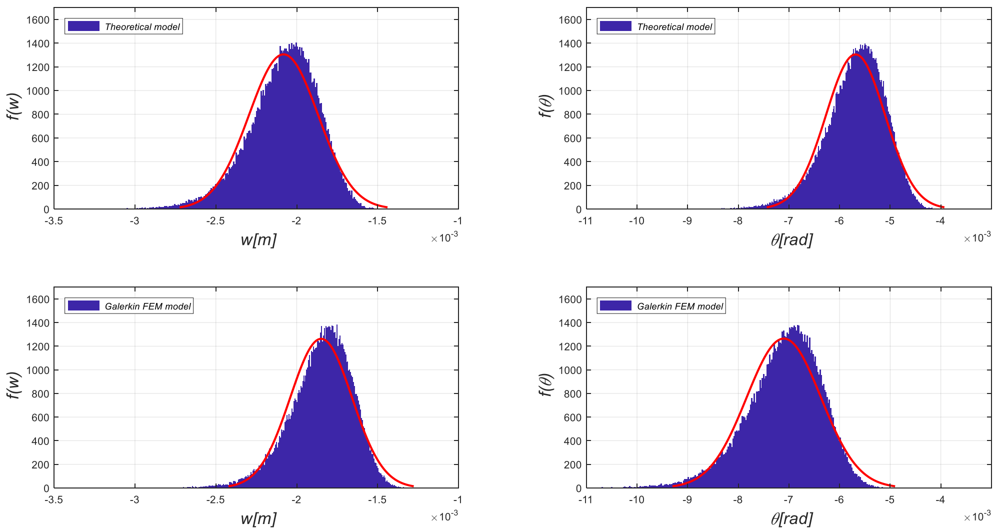

Figure 2. Histogram of vertical (on the right) and angular (on the left) displacements.

In order to apply the Galerkin method, it is necessary to have an explicit representation of uncertainty.

The uncertainty parameters are modeled as a parameterized stochastic process and defined as the linear combination of deterministic functions and random variables, such that, $k(x,\omega) = \sum_{i=1}^{N} g_i(x)\xi_i(\omega)$, where $\{g_i\}_{i=1}^{N}$ are the approximation functions and $\{\xi_i\}_{i=1}^{N}$ are the random actualizations. Table 2 presents the statistical moments obtained for displacements at the same positions referred to in Fig. 2.

Table 2. Statistics on displacements

		Theoretical model		
GL	x(m)	μ	σ	s
w	0.25	-0.002081419523837	2.153943308237751e-04	4.639471775102185e-08
θ	0.4	-0.005683662913092	5.881701193694546e-04	3.459440893190785e-07
		Galerkin FEM Model		
GL	x(m)	μ	σ	s
w	0.25	0.001849602528829	1.903269758110320e-04	3.622435772137317e-08
θ	0.4	0.007102473710702	7.308555871143603e-04	5.341498892162762e-07

Figures 3 and 4 refer to medium, minimum and maximum displacement for different models considering $\delta = (1/10)\mu_{EI}$ and $\delta = (3/10)\mu_{EI}$, respectively.

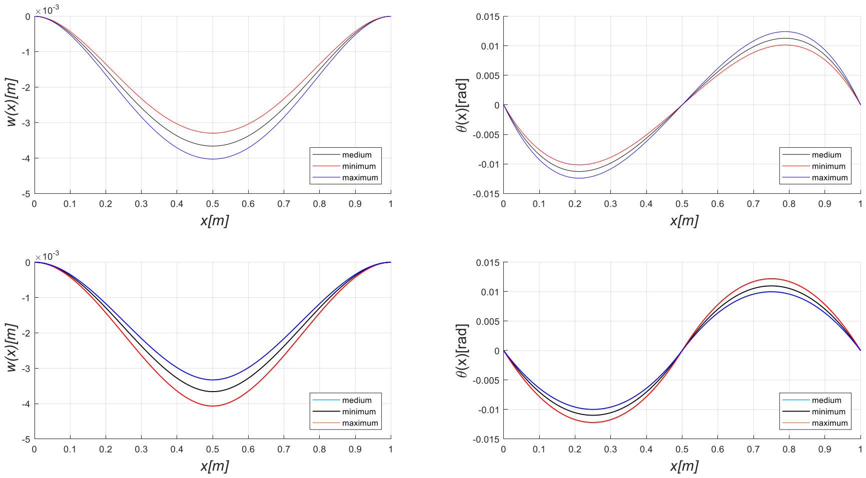

Figure 3. Medium, maximum, and minimum for vertical (on the right) and angular (on the left) displacements obtained for the theoretical model (above) and for the Galerkin-FEM model (below), for $\delta = 1/10\,\mu_{EI}$.

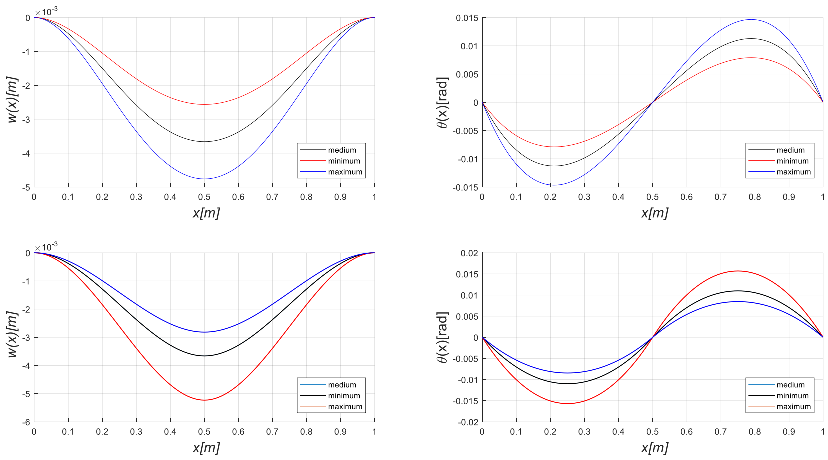

Figure 4. Medium, maximum, and minimum for vertical (on the right) and angular (on the left) displacements obtained for the theoretical model (above) and for the Galerkin-FEM model (below) for $\delta = 3/10\,\mu_{EI}$.

Figure 5 presents the results for the variability of displacements obtained through Newmann expansion method.

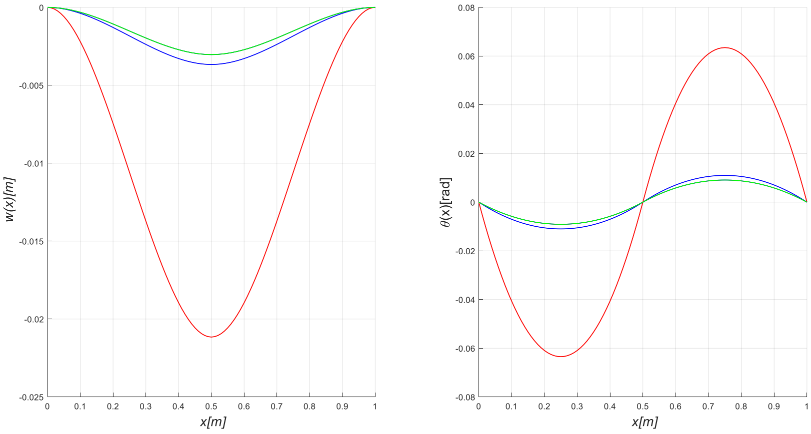

Figure 5. Uncertainties of vertical (on the right) and angular (on the left) displacements obtained by Newmann expansion.

Figure 6 presents the results for displacement variability obtained by the *MCS N-λ* method. One notices that variability is considerably reduced regarding the Monte Carlo method as well as the data processing time.

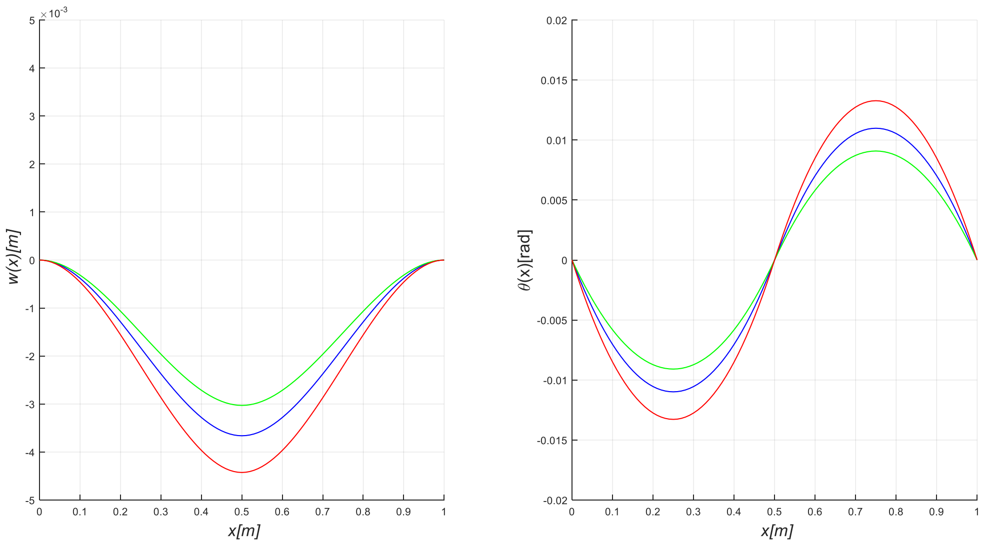

Figure 6. Uncertainties of vertical (on the right) and angular (on the left) displacements obtained by *MCS N-λ*.

CONCLUSION

The present work develops and uses the Newmann Monte Carlo method with convergence parameter λ to calculate the effect of spatial variability of the material properties in solving stochastic systems or uncertainty propagation. Theoretical and Galerkin-FEM results are compared. This method is based on the solution of a distance optimization problem for a low-order approximation of the matrix inversion system using Newmann series.

Regarding computational time and accuracy, effective results are obtained with a first-order approximation of Newmann expansion when compared to the Direct Simulation Monte-Carlo. The series convergence is ensured for the sample having an argument smaller than 1, and deterministic finite-element programs are easily adapted to that end. For future works, it is suggested that the method be implemented for high-order solutions with a resulting increase in the vector λ dimension, and also for solving stochastic systems when combined to Newton's method.

Although the estimated deterministic results of displacements between the theoretical model and the Galerkin-FEM prove to be very close, the statistical moments of uncertainties associated to them present considerable differences. In the Galerkin method, the Hermite polynomials are used as form functions, and it is suggested, for further works, that they be used as Airy stress functions.

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