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ANALYSIS OF DISCONTINUOUS GALERKIN METHOD OF NONLINEAR PROBLEMS

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Abstract. *We investigated a model for the study of numerical aspects in the treatment of a nonlinear problem. The nonlinearity is related to the couplings of the coefficients. We consider the variation of potential u and temperature θ in a given medium. The squared of the charge flow is a nonlinearity in the heat source term, which is known as the Joule effect. A numerical analysis of the method in hybridized mixed form was performed considering elements of Raviart-Thomas. Computational experiments validate the convergence of the method.*

Keywords: *finite elements, Mixed method, Discontinuous Galerkin method, Joule effect, thermistor problem.*

1 INTRODUCTION

In this paper we consider stationary diffusion problems of the form

$$\begin{cases} -\text{div}(\mu(\theta)\nabla u) = f \text{ in } \Omega \\ -\text{div}(k(\theta)\nabla\theta) = \mu(\theta)|\nabla u|^2, \text{ in } \Omega \\ u = 0, \text{ on } \Omega \\ \theta = 0, \text{ on } \partial\Omega \end{cases} \tag{1}$$

where suppose conductivities $0 < \kappa_1 \leq \kappa(\theta) \leq \kappa_2$ e $0 < \mu_1 < \mu(\theta) \leq \mu_2$, temperature θ and potential u. O domain $\Omega \subset I\!R^3$, *with regular contour*; definition seen in (Meyers, 1963) or (Loula & Zhu, 2001), in the latter, existence, uniqueness and regularity are established for the problem (1). A numerical analysis of the hybrid mixed form was performed considering Raviart-Thomas elements of order 1 . Appropriate norms are defined and we proof inf-sup that is the condition of method stability. The numerical results was according with the expected in terms of convergence order to scalar variable and flux of electric charge. The hybridization is introducted by lagrange multipliers, that determined the flux continuity. These multipliers have convergence order bigger than the expected.

1.1 FORMULATION OF THE PROBLEM AND PRELIMINARIES

Let $H^1(\Omega)$ the Sobolev space with norm $\|.\|_{H^1}$, L^2 Hilbert space with norm $\|.\|_{L^2}$, H_0^1 is the clousure of the space C_0^∞ for norm $\|.\|_{H^1}$ and domain Ω is subset Euclidean space in the 2 or 3 dimension. We solved problem (2-3) employing the point fix iterative method.

For an arbitrary θ_0 , and $n = 1, 2, ...,$ we calculate $(\boldsymbol{\sigma^n}, u^n, \boldsymbol{q}^n, \theta^n)$.

$$\begin{cases} (a(\theta^{n-1})\boldsymbol{\sigma}^n, \boldsymbol{\tau}) + (\nabla.\boldsymbol{\tau}, u^n) = 0 & \forall \boldsymbol{\tau} \in \boldsymbol{V} \\ (\nabla.\boldsymbol{\sigma}^n, v) = (f, v) & v \in L^2(\Omega) \end{cases} \tag{2}$$

$$\begin{cases} (\boldsymbol{q}^n, \boldsymbol{\tau}) + (\nabla.\boldsymbol{\tau}, \theta^n) = 0 & \forall \boldsymbol{\tau} \in \boldsymbol{V} \\ (\boldsymbol{q}^n, \nabla w) = (a(\theta^{n-1})|\boldsymbol{\sigma}^n|^2, w), & w \in L^2(\Omega) \end{cases} \tag{3}$$

where $a(\theta^{n-1}) = \mu(\theta^n)^{-1}$, $\boldsymbol{\sigma}^n = -\mu(\theta^{n-1})\nabla u^n$ and $\boldsymbol{q}^n = -\nabla\theta^n$ is charge flux and heat respectively, than is seek in

$$\boldsymbol{V} = H(div;\Omega) = \{\boldsymbol{v} \in [L^2(\Omega)]^N : \nabla.\boldsymbol{v} \in L^2(\Omega)\},\ N = 2, 3.$$

We assume that $f \in L^2(\Omega)$ and the to heating source suppose $\boldsymbol{\sigma}^n$ belongs $H^1(\Omega) \cap L^\infty(\Omega)$, (Zhu & Loula, 2006) established existence, uniqueness and regularity of the test function in $H^1(\Omega)$ and (Zhu J. et al., 2011) concluded the following estimates for regularity:

$$\|\boldsymbol{\sigma}^n\|_{H^1(\Omega)} + \|u^n\|_{H^2(\Omega)} \leq \begin{cases} C\{1 + \|\nabla\theta^0\|_{L^\infty(\Omega)}\}\|f\|_{L^2(\Omega)},\ n = 1 \\ C\{1 + \|f\|^2_{L^2(\Omega)}\}\|f\|_{L^2(\Omega)},\ n \geq 2 \end{cases} \tag{4}$$

e

$$\|\boldsymbol{q}^n\|_{H^1(\Omega)} + \|\theta^n\|_{H^2(\Omega)} \leq \begin{cases} C\{1 + \|\nabla\theta^0\|_{L^\infty(\Omega)}\}\|f\|_{L^2(\Omega)},\ n = 1 \\ C\{1 + \|f\|^2_{L^2(\Omega)}\}\|f\|_{L^2(\Omega)},\ n \geq 2 \end{cases} \tag{5}$$

and the convergence of this algorithm was also established, so the following convergence estimates follow;

$$\|\boldsymbol{\sigma} - \boldsymbol{\sigma}^n\|_{L^2(\Omega)} + \|\nabla(u - u^n)\|_{L^2(\Omega)} \leq C\|f\|_{L^2(\Omega)} M(f)^{n-1}\|\nabla(\theta - \theta^0)\|_{L^2(\Omega)}, \tag{6}$$

$$\|\boldsymbol{q} - \boldsymbol{q}^n\|_{L^2(\Omega)} + \|\nabla(\theta - \theta^n)\|_{L^2(\Omega)} \leq M(f)^n\|\nabla(\theta - \theta^0)\|_{L^2(\Omega)}, \tag{7}$$

where $M(f) = C\|f\|_{L^2(\Omega)}$. In the present work, we solve problem (2-3) by a mixed discontinuous Galerkin (MHDG) method based on (Brezzi, et al., 1985) and (Egger, 2010). Besides the usual advantages and disadvantages of discontinuous Galerkin (DG) methods (Arnold, 1982), for exemple, increase of degrees of freedom in relation to the mixed method, but this problem was circumvented by the hybridization. Observe than in estimatives appears the gradient of scalar variables u and θ respectively. These estimate applied in our case no sense, leading into account the gradient in side left in the expressions (6) and (7). However we apploying only the flux norm in $\|.\|_{L^2}$, ignoring the gradient, than no is calculated these method. In this sense we make a hypothesis of regularity $|\boldsymbol{\sigma}|_\infty < \infty$ that ensures not be necessary the gradient the scalar variables in ours analysis. In the following, in sections two, we make analysis the mixed hybrid Galerkin discontinuous method (MHGD) and we present numerical results respectively (Oliveira, 2017).

2 FINITE ELEMENTS APPROXIMATION

Suppose that Ω is a domain polyhedral and $\partial\Omega_D = \partial\Omega$. Let $\mathcal{T}_h$ is the contour with Dirichlet conditions. $\mathcal{T}_h$ is a partition of Ω, where T are the parts or elements satisfying basic conditions for finite element (Ciarlet,. 2002). Let h_T be the diameter of T, then we denoting $h = \max_{T\in\mathcal{T}_h} h_T$ the side maximum of T. We define $\mathcal{E}_h$ face set (or edge) called E; ∂T is T contour, being $\boldsymbol{\nu}$ normal vector to contour. For the interfaces, we consider the following Hilbert space

$$L^2(\partial\mathcal{T}_h) := \{v : v \in L^2(\partial T),\ \forall\, T \in \mathcal{T}_h\};$$

where $u, v \in L^2(\partial\mathcal{T}_h)$, and $\langle u, v\rangle_{\partial T} := \int_{\partial T} uvds$, $\langle u, v\rangle_{\partial\mathcal{T}_h} := \sum_T \langle u, v\rangle_{\partial T}$ integrals on the interfaces, as with the norm

$$|v|_{L^2(\partial\mathcal{T}_h)} = \sqrt{\langle v, v\rangle_{\partial\mathcal{T}_h}}.$$

We call the sets $\mathcal{P}_k(T)$, $\mathcal{P}_k(E)$ of polynomials of order less than or equal to k on T and ∂T respectively. Raviart-Thomas elements is pairs $RT_k(T) := \mathcal{P}_k(T) \oplus \boldsymbol{x}.\mathcal{P}_k(T)$ with $T \in \mathcal{T}_h$. This way we can defined the functions spaces

$$\boldsymbol{\Sigma}_h := \left\{\boldsymbol{\tau}_h \in (L^2(\mathcal{T}_h))^N : \boldsymbol{\tau}_h|_T \in RT_k(T),\ \forall T \in \mathcal{T}_h\right\};$$

$$\mathcal{V}_h := \{v_h \in L^2(\mathcal{T}_h) : v_h|_T \in \mathcal{P}_k(T),\ \forall T \in \mathcal{T}_h\};$$

$$\mathcal{M}_h := \{m_h \in L^2(\mathcal{E}_h) : m|_E \in \mathcal{P}_k(E),\ \forall E \in \mathcal{E}_h) \text{ and } m|_E = 0 \text{ se } E \subset \partial\Omega,\ \forall\, E \in \mathcal{E}_h\}$$

where $L^2(\mathcal{E}_h)$ Hilbert's space on set of edges. Thus we can writing the cartesian product $\mathcal{W}_h := \boldsymbol{\Sigma}_h \times \mathcal{V}_h \times \mathcal{M}_h$.

Find $(\boldsymbol{\sigma}^n, u^n, \boldsymbol{q}^n, \theta^n) \in \mathcal{W}_h$, such that:

$$\begin{cases} \left(a(\theta_h^{n-1})\boldsymbol{\sigma}_h^n, \boldsymbol{\tau}_h\right)_{\mathcal{T}_h} - (u_h^n, \nabla.\boldsymbol{\tau}_h)_{\mathcal{T}_h} + \langle \lambda_h^n, \boldsymbol{\tau}_h\boldsymbol{\nu}\rangle_{\partial\mathcal{T}} = 0, & \forall\, \boldsymbol{\tau}_h \in \sum_h \\ (\nabla.\boldsymbol{\sigma}_h^n, v_h)_{\mathcal{T}_h} = (f, v_h)_{\mathcal{T}_h}, & \forall\, v_h \in \mathcal{V}_h \\ \langle \boldsymbol{\sigma}_h\boldsymbol{\nu}, m_h\rangle_{\partial\mathcal{T}_h} = 0, & \forall\, m_h \in \mathcal{M}_h \end{cases} \tag{8}$$

and

$$\begin{cases} (\boldsymbol{q}_h^n, \boldsymbol{\tau}_h)_{\mathcal{T}_h} - (\theta_h^n, \nabla.\boldsymbol{\tau}_h)_{\mathcal{T}_h} + \langle \gamma_h^n, \boldsymbol{\tau}_h\boldsymbol{\nu}\rangle_{\partial\mathcal{T}} = 0, & \forall\, \boldsymbol{\tau}_h \in \sum_h \\ (\nabla.\boldsymbol{q}_h^n, v_h)_{\mathcal{T}_h} = \left(a(\theta_h^{n-1})|\boldsymbol{\sigma}_h^n|^2, v_h\right)_{\mathcal{T}_h}, & \forall\, v_h \in \mathcal{V}_h \\ \langle \boldsymbol{q}_h\boldsymbol{\nu}, m_h\rangle_{\partial\mathcal{T}_h} = 0, & \forall\, m_h \in \mathcal{M}_h \end{cases} \tag{9}$$

Applying integrations by parts into the equations (8) e (9) we rewrote the method of compact form. Find $(\boldsymbol{\sigma}_h^n, u_h^n, \lambda_h^n, \boldsymbol{q}_h^n, \theta_h^n, \gamma_h^n) \in \mathcal{W}_h \times \mathcal{W}_h$, such that

$$\begin{aligned} \mathcal{B}\left(\theta^{n-1}; (\boldsymbol{\sigma}_h^n, u_h^n, \lambda_h^n), (\boldsymbol{\tau}_h, v_h, m_h)\right) &= \mathcal{F}_{\mathcal{B}}(\boldsymbol{\tau}_h, v_h, m_h) \\ \mathcal{A}\left((\boldsymbol{q}_h^n, \theta_h^n, \gamma_h^n), (\boldsymbol{\tau}_h, v_h, m_h)\right) &= \mathcal{F}_{\mathcal{A}}\left(\theta^{n-1}; (\boldsymbol{\tau}_h, v_h, m_h)\right) \end{aligned} \tag{10}$$

$\forall\, \boldsymbol{\tau}_h \in \sum_h,\ v_h \in \mathcal{V}_h$ e $m_h \in \mathcal{M}_h,$ where $\mathcal{B}$, $\mathcal{F}_{\mathcal{B}}$, $\mathcal{A}_\theta$ and $\mathcal{F}_{\mathcal{A}}$ are definided as:

$$\begin{aligned} \mathcal{B} :=& \left(a(\theta_h^{n-1})\boldsymbol{\sigma}_h^n, \boldsymbol{\tau}_h\right)_{\mathcal{T}_h} + (\nabla u_h^n, \boldsymbol{\tau}_h)_{\mathcal{T}_h} + \langle \lambda_h^n - u_h^n, \boldsymbol{\tau}_h\boldsymbol{\nu}\rangle_{\partial\mathcal{T}_h} + \\ & (\boldsymbol{\sigma}_h^n, \nabla v_h)_{\mathcal{T}_h} + \langle \boldsymbol{\sigma}_h^n\boldsymbol{\nu}, m_h - v_h\rangle_{\partial\mathcal{T}_h} \end{aligned} \tag{11}$$

$$\begin{aligned} \mathcal{A} :=& (\boldsymbol{q}_h^n, \boldsymbol{\tau}_h)_{\mathcal{T}_h} + (\nabla\theta_h^n, \boldsymbol{\tau}_h)_{\mathcal{T}_h} + \langle \gamma_h^n - \theta_h^m, \boldsymbol{\tau}_h\boldsymbol{\nu}\rangle_{\partial\mathcal{T}_h} + (\boldsymbol{q}_h^m, \nabla v_h)_{\mathcal{T}_h} \\ & + \langle \boldsymbol{q}_h^n\boldsymbol{\nu}, m_h - v_h\rangle_{\partial\mathcal{T}_h} \end{aligned} \tag{12}$$

and

$$\mathcal{F}_{\mathcal{B}} := (f, v_h)_{\mathcal{T}_h}, \quad \mathcal{F}_{\mathcal{A}} := (a(\theta_h^{n-1})|\boldsymbol{\sigma}_h^n|^2, v_h). \tag{13}$$

Consistency and conservation are gives in (Oliveira, 2015) well as the norms fowlling are defined in (Egger, 2010):

$$|||(\boldsymbol{\tau}, v, m)|||_{\mathcal{B}} := \left(\frac{1}{\mu_1}\|\boldsymbol{\tau}\|_{\mathcal{T}_h} + \mu_1\|\nabla v\|^2_{\mathcal{T}_h} + \frac{\mu_1}{h}|\lambda - u|_{\partial\mathcal{T}_h}\right)^{1/2} \tag{14}$$

$$|||(\boldsymbol{\tau}, v, m)|||_{\mathcal{A}} := \left(\frac{1}{\kappa_1}\|\boldsymbol{\tau}\|_{\mathcal{T}_h} + \kappa_1\|\nabla v\|^2_{\mathcal{T}_h} + \frac{\kappa_1}{h}|\lambda - \theta|_{\partial\mathcal{T}_h}\right)^{1/2} \tag{15}$$

and

$$|||(\boldsymbol{\tau}, v, m)|||_{\mathcal{B},*} := \left(|||(\boldsymbol{\tau}, v, m)|||_{\mathcal{B}}^2 + \frac{h}{\mu_1} |\boldsymbol{\tau}\boldsymbol{\nu}|_{\partial\mathcal{T}_h}^2 \right)^{1/2} \tag{16}$$

$$|||(\boldsymbol{\tau}, v, m)|||_{\mathcal{A},*} := \left(|||(\boldsymbol{\tau}, w, m)|||_{\mathcal{A}}^2 + \frac{h}{\kappa_1} |\boldsymbol{\tau}\boldsymbol{\nu}|_{\partial\mathcal{T}_h}^2 \right)^{1/2} \tag{17}$$

Following the text we can write $|||(\cdot,\cdot,\cdot)|||$ instead of $|||(\cdot,\cdot,\cdot)|||_{\mathcal{B}}$ whenever there is no risk of confusion, the same for $|||(\cdot,\cdot,\cdot)|||_*$ instead of $|||(\cdot,\cdot,\cdot)|||_{\mathcal{B},*}$. Analogy to (Egger, 2010) the demonstration of the stabilities were also proved in (Oliveira, 2015). There are a positive constant $C_{\mathcal{B}}$, independent of h such that the estimate

$$\sup_{(\boldsymbol{\tau}_h, v_h, m_h)} \frac{\mathcal{B}(a(\theta_h^{n-1}); \boldsymbol{\sigma}_h^n, u_h^n, \lambda_h^n; \boldsymbol{\tau}_h, v_h, m_h)}{|||(\boldsymbol{\tau}_h, v_h, m_h)|||_{\mathcal{B}}} \geq C_{\mathcal{B}} |||(\boldsymbol{\sigma}_h^n, u_h^n, \lambda_h^n)|||_{\mathcal{B}} \tag{18}$$

$$\sup_{(\boldsymbol{\tau}_h, v_h, m_h)} \frac{\mathcal{A}(\boldsymbol{q}_h^n, \theta_h^n, \gamma_h^n; \boldsymbol{\tau}_h, v_h, m_h)}{|||(\boldsymbol{\tau}_h, v_h, m_h)|||_{\mathcal{A}}} \geq C_{\mathcal{A}} |||(\boldsymbol{q}_h^n, \theta_h^n, \gamma_h^n)|||_{\mathcal{A}} \tag{19}$$

holds for all triples $(\boldsymbol{\sigma}_h^n, u_h^n, \lambda_h^n) \in \sum_h \times \mathcal{V}_h \times \mathcal{M}_h$ e $(\boldsymbol{\tau}_h, v_h, m_h) \in \mathcal{W}_h$. the bilinear forms (11) (12) are continuous; that is, there are constants $c_{\mathcal{B}}$ e $c_{\mathcal{A}}$ independent h, such that tha estimates

$$|\mathcal{B}\left(a(\theta^{n-1}); (\boldsymbol{\sigma}^n, u^n, \lambda^n), (\boldsymbol{\tau}_h, v_h, m_h)\right)| \leq c_{\mathcal{B}} |||(\boldsymbol{\sigma}^n, u^n, \lambda^n)|||_{\mathcal{B},*} |||(\boldsymbol{\tau}_h, v_h, m_h)|||_{\mathcal{B}} \tag{20}$$

$$|\mathcal{A}(\boldsymbol{q}^n, \theta^n, \gamma^n; \boldsymbol{\tau}_h, v_h, m_h)| \leq c_{\mathcal{A}} |||(\boldsymbol{q}^n, u^n, \gamma^n)|||_{\mathcal{A},*} |||(\boldsymbol{\tau}_h, v_h, m_h)|||_{\mathcal{A}} \tag{21}$$

holds for all triples $(\boldsymbol{\sigma}^n, u^n, \lambda^n) \in \mathcal{W} \oplus \mathcal{W}_h$ e $(\boldsymbol{\tau}_h, v_h, m_h) \in \mathcal{W}_h$.

Consider definition of interpolators for approximation, specifically, interpolation Raviart-Thomas and their estimates (Egger, 2010) is proved the following estimate;
Let $u \in H^1(\Omega) \cap H^{3/2+\delta}(\mathcal{T}_h)$ e $\boldsymbol{\sigma} := -\mu(\theta)\nabla u$. Then

$$|||(\boldsymbol{\sigma} - \Pi_k^{RT}\boldsymbol{\sigma}, u - \Pi_k^T u, \lambda - \Pi_k^T u)|||_* \leq C h^s \mu_1^{-1/2} \mu_2 |u|_{s+1,\mathcal{T}_h}, \;\; 1/2 < s \leq k \tag{22}$$

$$|||(\boldsymbol{q} - \Pi_k^{RT}\boldsymbol{\sigma}, \theta - \Pi_k^T \theta, \gamma - \Pi_k^T u)|||_* \leq C h^s |\theta|_{s+1,\mathcal{T}_h}, \;\; 1/2 < s \leq k \tag{23}$$

this result of projections, Raviart-Thomas order k, is fundamental for we obter the error estimatites a priori.

2.1 A priori estimates for potential variable

Lets $(\boldsymbol{\sigma}_h, u_h, \lambda_h)$ discrete solution of (10), and u exact solution of of problem, we define $\boldsymbol{\sigma} = \mu(\theta)\nabla u$. then we have

$$|||(\boldsymbol{\sigma}^n - \boldsymbol{\sigma}_h^n, u^n - u_h^n, u - \lambda_h^n)||| \leq |||(\boldsymbol{\sigma}^n - \Pi_k^{RT}\boldsymbol{\sigma}^n, u^n - \Pi_k^T u^n, \lambda^n - \Pi_k^E u)|||+$$

$$|||(\Pi_k^{RT}\boldsymbol{\sigma}^n - \boldsymbol{\sigma}_h^n, \Pi_k^T u^n - u_h^n, \Pi_k^E u - \lambda^n)||| \tag{24}$$

using *Galerkin orthogonality*, the estimate (18) and the continuity (20), we proved (24):

$$\begin{aligned}
0 &= \mathcal{B}\left(\theta^{n-1}; (\boldsymbol{\sigma}^n, u^n, \lambda^n), (\boldsymbol{\tau}_h, v_h, m_h)\right) - \mathcal{B}\left(\theta_h^{n-1}; (\boldsymbol{\sigma}_h^n, u_h^n, \lambda_h^n), (\boldsymbol{\tau}_h, v_h, m_h)\right) \\
&= \mathcal{B}\left(\theta^{n-1}; (\boldsymbol{\sigma}^n, u^n, \lambda^n), (\boldsymbol{\tau}_h, v_h, m_h)\right) - \mathcal{B}\left(\theta^{n-1}; (\boldsymbol{\sigma}_h^n, u_h^n, \lambda_h^n), (\boldsymbol{\tau}_h, v_h, m_h)\right) \\
&\quad + \left([a(\theta^{n-1}) - a(\theta_h^{n-1})]\boldsymbol{\sigma}^n, \boldsymbol{\tau}_h\right)_{\mathcal{T}_h} \\
&= \mathcal{B}\left(\theta^{n-1}; (\boldsymbol{\sigma}^n - \Pi_k^{RT}\boldsymbol{\sigma}^n, u^n - \Pi_k^T u^n, \lambda^n - \Pi_k^E \lambda^n), (\boldsymbol{\tau}_h, v_h, m_h)\right) \\
&\quad + \mathcal{B}\left(\theta^{n-1}; (\Pi_k^{RT}\boldsymbol{\sigma}^n - \boldsymbol{\sigma}_h^n, \Pi_k^T u^n - u_h^n, \Pi_k^E \lambda^n - \lambda_h^n), (\boldsymbol{\tau}_h, v_h, m_h)\right) \\
&\quad + \left([a(\theta^{n-1}) - a(\theta_h^{n-1})]\boldsymbol{\sigma}^n, \boldsymbol{\tau}_h\right)_{\mathcal{T}_h}
\end{aligned}$$

that is

$$\begin{aligned}
&\mathcal{B}\left(\theta^{n-1}; (\Pi_k^{RT}\boldsymbol{\sigma}^n - \boldsymbol{\sigma}_h^n, \Pi_k^T u^n - u_h^n, \Pi_k^E \lambda^n - \lambda_h^n), (\boldsymbol{\tau}_h, v_h, m_h)\right) \\
&= \mathcal{B}\left(\theta^{n-1}; (\Pi_k^{RT}\boldsymbol{\sigma}^n - \boldsymbol{\sigma}^n, \Pi_k^T u^n - u^n, \Pi_k^E \lambda^n - \lambda^n), (\boldsymbol{\tau}_h, v_h, m_h)\right) \\
&\quad + \left([a(\theta^{n-1}) - a(\theta_h^{n-1})]\boldsymbol{\sigma}^n, \boldsymbol{\tau}_h\right)_{\mathcal{T}_h}
\end{aligned}$$

Applying (18) on the left side and (20) On the left side, we have:

$$|||(\Pi_k^{RT}\boldsymbol{\sigma}^n - \boldsymbol{\sigma}_h^n, \Pi_k^T u^n - u_h^n, \Pi_k^E \lambda^n - \lambda_h^n)|||_{\mathcal{B}} \leq \frac{c_{\mathcal{B}}}{C_{\mathcal{B}}}|||(\Pi_k^{RT}\boldsymbol{\sigma}^n - \boldsymbol{\sigma}^n, \Pi_k^T u^n - u^n, \Pi_k^E \lambda^n - \lambda^n)|||_*$$

$$+||[a(\theta^{n-1}) - a(\theta_h^{n-1})]\boldsymbol{\sigma}^n||_{L^2(\Omega)}$$

$$\leq Ch^s|u^n|_{s+1,\mathcal{T}_h} + L\|\boldsymbol{\sigma}\|_{L^\infty(\Omega)}||\theta^{n-1} - \theta_h^{n-1}||_{L^2(\Omega)},\ 1/2 < s \leq k.$$

where L is a Lipschtiz constant, holds

$$\begin{aligned}
&|||(\Pi_k^{RT}\boldsymbol{\sigma}^n - \boldsymbol{\sigma}_h^n, \Pi_k^T u^n - u_h^n, \Pi_k^E \lambda^n - \lambda_h^n)|||_{\mathcal{B}} \\
&\leq C\{\|\boldsymbol{\sigma}\|_{L^\infty(\Omega)}||\theta^{n-1} - \theta_h^{n-1}||_{L^2(\Omega)} + h^s|u^n|_{s+1,\mathcal{T}_h}\}
\end{aligned} \tag{25}$$

It was considering the affirmation on the regularity $\|\boldsymbol{\sigma}^n\|_{L^\infty(\Omega)} \leq C' < \infty$, there are a constant C' dependent only of Ω, μ_1, μ_2 e L such that;
$|||(\boldsymbol{\sigma}^n - \boldsymbol{\sigma}_h^n, u^n - u_h^n, \lambda^n - \lambda_h^n)|||$

$$\leq C\{||\boldsymbol{\sigma}||_{L^\infty(\Omega)}||\theta^{n-1} - \theta_h^{n-1}||_{L^2(\Omega)} + h^s|u^n|_{s+1,\mathcal{T}_h}\},\ 1/2 < s \leq k. \tag{26}$$

2.2 A priori estimate for temperature

Suppose $(\boldsymbol{q}_h, \theta_h, \gamma_h)$ a discrete solution of (10), and θ a exact solution of problem, we define $\boldsymbol{q}^n = -\mu(\theta^n)\nabla\theta^n$. Then we have

$$|||(\boldsymbol{q}^n - \boldsymbol{q}_h^n, \theta^n - \theta_h^n, \theta^n - \gamma_h^n)||| \le |||(\boldsymbol{q}^n - \Pi_k^{RT}\boldsymbol{q}^n, \theta^n - \Pi_k^T\theta^n, \gamma_h^n - \Pi_k^E\theta^n)|||+$$
$$|||(\Pi_k^{RT}\boldsymbol{q}^n - \boldsymbol{q}_h^n, \Pi_k^T\theta^n - \theta_h^n, \Pi_k^E\theta^n - \gamma_h^n)||| \tag{27}$$

By (19), we have;

$$C_{\mathcal{A}}|||(\Pi_k^{RT}\boldsymbol{q}^n - \boldsymbol{q}_h^n, \Pi_k^T\theta^n - \theta_h^n, \Pi_k^E\gamma^n - \gamma_h^n)|||_{\mathcal{A}} \le$$
$$\sup_{(\boldsymbol{\tau}_h, v_h, m_h)} \frac{\mathcal{A}((\Pi_k^{RT}\boldsymbol{q}^n - \boldsymbol{q}_h^n, \Pi_k^T\theta^n - \theta_h^n, \Pi_k^E\gamma^n - \gamma_h^n); \boldsymbol{\tau}_h, v_h, m_h)}{|||(\boldsymbol{\tau}_h, v_h, m_h)|||_{\mathcal{A}}}. \tag{28}$$

the method is consistent, follow that:

$$(a(\theta^{n-1})|\boldsymbol{\sigma}^n|^2 - a(\theta_h^{n-1})|\boldsymbol{\sigma}_h^n|^2, v_h)_{\mathcal{T}_h} =$$
$$\mathcal{A}\left((\boldsymbol{q}^n, \theta^n, \gamma^n), (\boldsymbol{\tau}_h, v_h, m_h)\right) - \mathcal{A}\left((\boldsymbol{q}_h^n, \theta_h^n, \gamma_h^n), (\boldsymbol{\tau}_h, v_h, m_h)\right)$$
$$\Rightarrow$$
$$\mathcal{A}\left((\boldsymbol{\sigma}_h^n, u_h^n, \gamma_h^n), (\boldsymbol{\tau}_h, v_h, m_h)\right) =$$
$$\mathcal{A}\left((\boldsymbol{q}^n, \theta^n, \gamma^n), (\boldsymbol{\tau}_h, v_h, m_h)\right) + \underbrace{(a(\theta^{n-1})|\boldsymbol{\sigma}^n|^2 - a(\theta_h^{n-1})|\boldsymbol{\sigma}_h^n|^2, v_h)_{\mathcal{T}_h}}_{R}$$

The term R was dealt with in (Loula & Zhu, 2001), we have;

$$|R| \le$$
$$C\left(||\boldsymbol{\sigma}||_{L^\infty(\Omega)}\|(\theta^{n-1} - \theta_h^{n-1})\|_{L^2(\Omega)} + \|\boldsymbol{\sigma}^n - \boldsymbol{\sigma}_h^n\|_{L^2(\Omega)}(\|\boldsymbol{\sigma}^n - \boldsymbol{\sigma}_h^n\|_{L^2(\Omega)} + \|f\|_{L^2(\Omega)})\right)\|\nabla v\|_{L^2(\Omega)}$$

$$|||(\Pi_k^{RT}\boldsymbol{q}^n - \boldsymbol{q}_h^n, \Pi_k^T\theta^n - \theta_h^n, \Pi_k^E\gamma^n - \gamma_h^n)||| \le \frac{c_{\mathcal{A}}}{C_{\mathcal{A}}}|||(\Pi_k^{RT}\boldsymbol{q}^n - \boldsymbol{q}^n, \Pi_k^T\theta^n - \theta^n, \Pi_k^E\gamma^n - \gamma^n)|||+$$
$$C\left(||\boldsymbol{\sigma}||_{L^\infty(\Omega)}\|(\theta^{n-1} - \theta_h^{n-1})\|_{L^2(\Omega)} + \|\boldsymbol{\sigma}^n - \boldsymbol{\sigma}_h^n\|_{L^2(\Omega)}(\|\boldsymbol{\sigma}^n - \boldsymbol{\sigma}_h^n\|_{L^2(\Omega)} + \|f\|_{L^2(\Omega)})\right) \tag{29}$$

by (23); (29) holds:

$$|||(\Pi_k^{RT}\boldsymbol{q}^n - \boldsymbol{q}_h^n, \Pi_k^T\theta^n - \theta_h^n, \Pi_k^E\gamma^n - \gamma_h^n)||| \le$$
$$C\left(||\boldsymbol{\sigma}||_{L^\infty(\Omega)}\|(\theta^{n-1} - \theta_h^{n-1})\|_{L^2(\Omega)} + \|\boldsymbol{\sigma}^n - \boldsymbol{\sigma}_h^n\|_{L^2(\Omega)}(\|\boldsymbol{\sigma}^n - \boldsymbol{\sigma}_h^n\|_{L^2(\Omega)} + \|f\|_{L^2(\Omega)}) + \right.$$
$$\left. h^s|\theta^n|_{s+1,\mathcal{T}_h}\right) \tag{30}$$

Considering the affirmation on the regularity for $\boldsymbol{\sigma}^n$, there are a constant $C > 0$ dependent only Ω, μ_1, μ_2 e L such that;

$$|||(\boldsymbol{q}^n - \boldsymbol{q}_h^n, \theta^n - \theta_h^n, \gamma^n - \gamma_h^n)|||\le C\{\|\boldsymbol{\sigma}^n\|^2_{L^\infty(\Omega)}\|\theta^{n-1} - \theta_h^{n-1}\|_{L^2(\Omega)} + \|\boldsymbol{\sigma}_h^n - \boldsymbol{\sigma}^n\|^2_{L^2(\Omega)}$$
$$+\|\boldsymbol{\sigma}^n\|_{L^\infty(\Omega)}\|\boldsymbol{\sigma}_h^n - \boldsymbol{\sigma}^n\|_{L^2(\Omega)} + h^s|\theta^n|_{s+1,\mathcal{T}_h}\} \tag{31}$$

where $M(h)$ in (29) is a constant calculate in work (Loula & Zhu, 2001). The expressions (26) and (31 are the estimates of errors a-priori.

3 NUMERICAL RESULTS

Was implemented in two dimensions, the finite element method (MHDG). The computacional experiments were carried out in the following way: Solving the problem in five triangular and uniform meshes consecutively refined (see fig. 1), that is, in each mesh the interactions are executed until reaching a given tolerance, and then it goes to the next more finer mesh and so on. The last (fifth) mesh on the order of h is 10^{-2}. For the iterative method, the tolerance was $tol \leq 10^{-9}$, with $tol = \|u_h^{n+1} - u_h^n\|_{L^2} + \|\theta_h^{n+1} - \theta_h^n\|_{L^2}$. We compare the numerical solution with exact solution calculating the errors $\|u - u_h\|_{L^2}$ and $\|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_{L^2}$, where u and $\boldsymbol{\sigma}$ are exact solutions of problem; u_h $\boldsymbol{\sigma}_h^n$ are approximate solutions for scalar variable and vector, respectively. Be nonlinear problem (32);

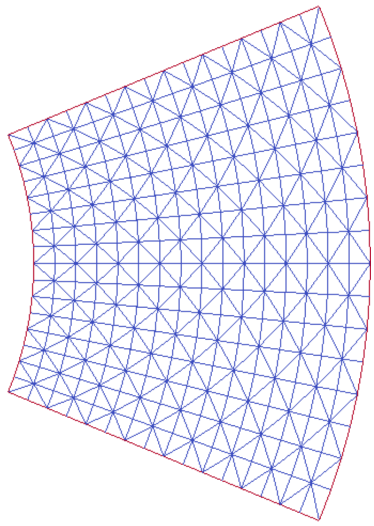

Figure 1: Meshes after of the fourth refinement in polar coordinate.

$$
\begin{array}{ll}
(i) & \frac{1}{r}\frac{d}{dr}.(r\mu(\theta)\frac{du}{dr}) = 0 \qquad r_0 \leq r \leq r_1 \\
(ii) & -\frac{1}{r}\frac{d}{dr}.(r\frac{d\theta}{dr}) = \mu(\theta)|\frac{du}{dr}|^2, \quad r_0 \leq r \leq r_1 \\
(iii) & u(r_0) = 0, \;\; u(r_1) = u_1, \\
(iv) & \theta(r_0) = \theta(r_1) = 0,
\end{array}
\tag{32}
$$

with exact solution:

$$
\begin{array}{l}
\theta(x,y) = -\ln\left[1 - \frac{1}{2}u(r)\,(u(r) - u_1)\right] \\
u(x,y) = \frac{1}{2}u_1 + \frac{1}{2}\sqrt{8+u_1^2}\tan\left[\arctan\left(\frac{u_1}{\sqrt{8+u_1^2}}\right)\left(\frac{2\ln(\frac{r}{r_0})}{\ln(\frac{r_1}{r_0})} - 1\right)\right]
\end{array}
\tag{33}
$$

where $u_1 = 1$, $r_0 = 1$ and $r_1 = 2$ and a total angle $\pi/4$.

The tables 1 and 2, showing of convergence to scalar variables θ and u and their flows respectively, as we using Raviart-Thomas of 1-order. If k If k is the degree Raviart-Thomas, then the convergence order is $k+1$. In fig. 2a, 2b, 2c and 2d the level curves compared the exact solutions with of the approximate solutions and again the convergence are show in fig. 3, now the convergence order is 1.

Table 1: Convergence order in norm L^2; error to approximate temperature θ and flux q Raviart-Thomas elements, order 1

RT1						
h	$\|\theta - \theta_h\|_{L^2}$	order	$\|\boldsymbol{q} - \boldsymbol{q}_h\|_{L^2}$	order	$\|\gamma - \gamma_h\|_{L^2}$	order
0.1189E+01	0.69663E-01	0.00	0.90457E-01	0.00	0.75446E-01	0.00
0.6277E+00	0.17785E-01	1.97	0.24502E-01	1.88	0.19416E-01	1.96
0.3217E+00	0.44697E-02	1.99	0.63261E-02	1.95	0.49311E-02	1.98
0.1627E+00	0.11189E-02	2.00	0.16038E-02	1.98	0.12416E-02	1.99
0.8183E-01	0.27981E-03	2.00	0.40355E-03	1.99	0.31144E-03	2.00

Table 2: Convergence order in norm L^2; error to approximate potential u and flux $\boldsymbol{\sigma}$. Raviart-Thomas elements, order 1

RT1						
h	$\|u - u_h\|_{L^2}$	order	$\|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_{L^2}$	order	$\|\lambda - \lambda_h\|_{L^2}$	order
0.1189E+01	0.97783E-02	0.00	0.21936E-01	0.00	0.82675E-02	0.00
0.6277E+00	0.23450E-02	2.06	0.58303E-02	1.91	0.20740E-02	2.00
0.3217E+00	0.57976E-03	2.02	0.14954E-02	1.96	0.51947E-03	2.00
0.1627E+00	0.14453E-03	2.00	0.37814E-03	1.98	0.13015E-03	2.00
0.8183E-01	0.36105E-04	2.00	0.95037E-04	1.99	0.32585E-04	2.00

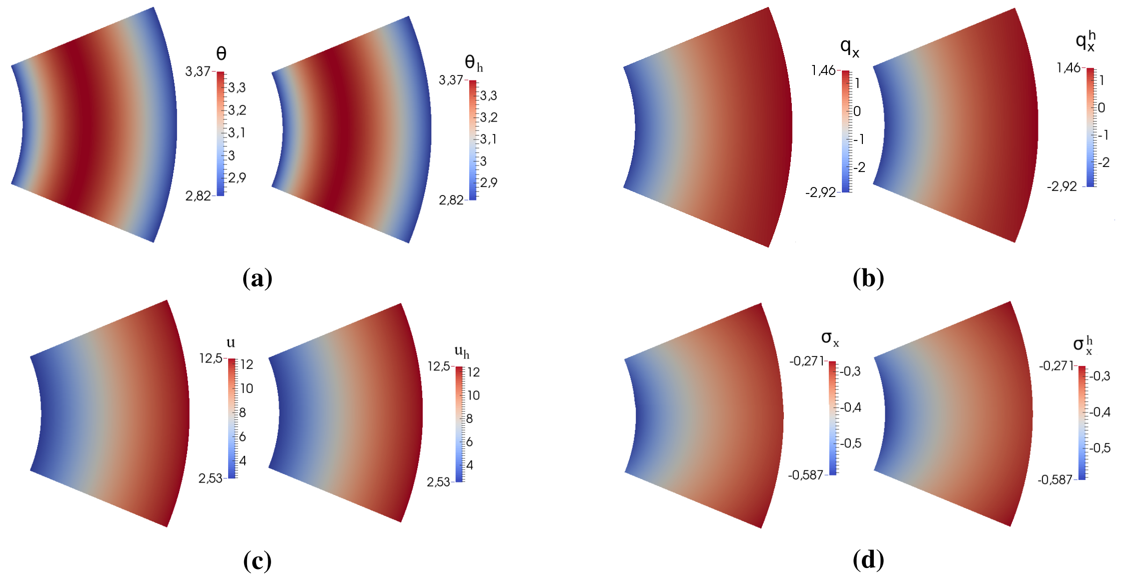

Figure 2: Level curves related to heights of the surface, nonlinear problem.

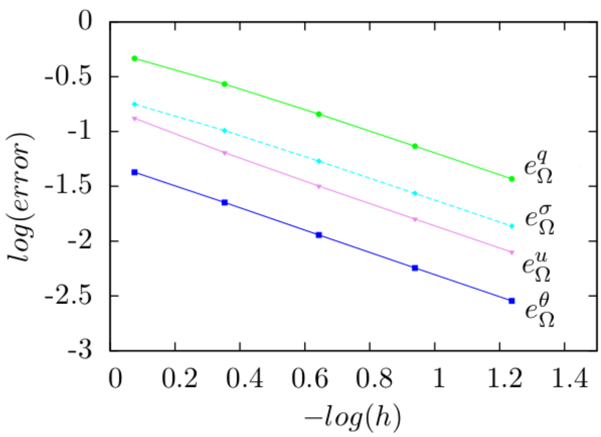

Figure 3: Convergence error analysis: e_Ω^θ is convergence error in temperature scalar variable and e_Ω^σ convergence error to potential flux.

4 CONCLUSIONS

This work present a numerical analysis of a coupled problem. A regularity hipothesis was made for the charge flux $\boldsymbol{\sigma}$. It was employed a iterative algorithm, due to the fact that it is a nonlinear problem. Since the estimates depend on $\|(\theta^{n-1} - \theta_h^{n-1})\|_{L^2(\Omega)}$, a convergence proof of the fixed-point algorithm is required. The numerical results suggest a super convergence for the Lagrange multipliers. However, the scalar and vector variables presented results according to the expected.

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