CILAMCE 2017
Florianópolis - Santa Catarina - Brazil

DYNAMIC ANALYSIS OF TIMOSHENKO BEAMS USING THE GENERALIZED FINITE ELEMENT METHOD

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Abstract. *Structures subjected to time-dependent loadings behave dynamically. Therefore, vibration analysis is an important subject of study in engineering. Many structures may be modelled using Euler-Bernoulli beam elements. However, this theory tends to overestimate the natural frequencies, especially when approximating higher frequencies or with beams with a bigger height. In these cases, the Timoshenko beam theory may be used including the effect of shear and rotational inertia on the dynamic analysis. In this context, this paper presents a dynamic analysis of Timoshenko beam problems with the Generalized Finite Element Method (GFEM), intending to address higher frequencies approximation issue. The results from GFEM are compared to results from exact Timoshenko beam solution and the Finite Element Method (FEM) approximation.*

Keywords: *Generalized Finite Element Method, Timoshenko Beam, Dynamic Analysis*

1 INTRODUCTION

Structures subjected to time-dependent loadings behave dynamically. And dynamic loads are often applied to structures. Therefore, it is important study vibration of structures.

The dynamic response of a structure depends on its properties such as rigidity and mass, which are influenced by the geometry, materials and conditions of connection to the external environment of the structure. These properties influence directly on the natural frequency and vibration mode of the structures, these characteristics are important aspects in the dynamic behavior of structure (Almeida, 2012).

The first significant study on beam vibration was conducted by Jacob Bernoulli. In his studies, in the eighteenth century, he discovered that the curvature of an elastic beam at any point is proportional to the moment of flexion at the same point. Years later, Daniel Bernoulli was the first to formulate the differential equation of motion of a beam. Later, Leonhard Euler accepted this theory and made many advances in the elastic curves for various loading conditions, thus creating the Euler-Bernoulli beam theory. This beam theory is also known as classical beam theory, and in practice was often used to find the natural frequencies of the beams. However, it overestimates the natural frequencies, especially in higher vibration modes and slender beams (Almeida, 2012; Lee & Park, 2013).

Therefore, Rayleigh (1896) introduced the rotational inertia effect, thus correcting the overestimation of the natural frequency for some problems. However it was necessary to manually introduce the shear to improve the accuracy of the natural frequencies for particular problems. In this way, Timoshenko (1921) elaborated a theory of beams that includes the transverse shear deformation and the rotational inertia effect. Later, this theory of beams became widely used in practical engineering problems.

Since engineering problems rarely permit analytical solutions, numerical methods become necessary to obtain solutions. One of the numerical methods most used in the dynamic analysis of structures is the Finite Element Method (FEM). FEM is a general technique for constructing approximate solutions for boundary-value problems. For this purpose, the domain is divided into a finite number of subdomains and then variational concepts are used to construct solution approximation space. The Galerkin Method is applied to the differential equation defined in each subdomains of the problem and it's expressed in the weak form (Becker *et al.*, 1981).

The FEM solution can be improved using two refinement techniques: refinement h and refinement p. Refinement h, consists of a refinement of the element mesh; While refinement p consists of increasing the number of shape functions in the element domain without any change in the mesh. In the context of dynamic modal analysis, h-refinement presents good results for the lower frequencies, however it requires a large computational cost to accurately calculate the higher frequencies. A solution to increase the accuracy of the FEM, would also apply the polynomial p-refinement. However, this still generates a large computational cost, because, every time a refining is done the entire system is recalculated (Bathe, 1996; Arndt *et al.*, 2014).

Several finite elements of Timoshenko beam have been proposed in the literature. Kapur (1966) was one of the firsts to propose a simple element, an element which, for unidirectional bending in a principal plane, has a total of four degrees of freedom, two at each of two nodes. Egle (1969) presented an approximate Timoshenko beam theory which intended to eliminate the coupling between shear deformation and rotary inertia. Davis *et al.* (1972) presented a

Timoshenko beam finite element which is based upon the exact differential equations of an infinitesimal element in static equilibrium, this model satisfied all essential and natural boundary conditions of a Timoshenko beam. Thomas *et al.* (1973) proposed a new complex element which has three degrees of freedom at each of two nodes. Thomas & Abbas (1975), in turn, proposed a finite element model with nodal degrees of freedom which can satisfy all the forced and natural boundary conditions.

Later, To (1979) provided the explicit expressions of the stiffness and mass matrices of two higher order tapered beam elements for vibration analysis. Shastry & Rao (1985) described two-node Timoshenko beam elements with two nodal degrees of freedom to analyze various beams. Laura & Gutierrez (1993) made an analysis on non-uniform Timoshenko beams, using the differential quadrature method. Houmat (1995) used a polynomial and a trigonometric functions as the shape function for the finite element. Lee & Schultz (2004) applied the pseudo-spectral methods to the eigenvalue analysis of Timoshenko beams. Ferreira & Fasshauer (2006) made a new numerical scheme, where collocation by radial basis functions and pseudospectral methods are combined to produce highly accurate results. Xu & Wang (2011) made an analysis of the free vibration of Timoshenko beams with various combinations of boundary conditions by using the discrete singular convolution. The conventional finite element formulation was applied to analyze a cracked Timoshenko beam, using p refine by Stojanović *et al.* (2013). Lee & Park (2013) used an isogeometric approach to study the free vibration of a Timoshenko beam

Given the special nature of certain problems, it is sometimes necessary to improve the response using information known a priori on the problem under study. To solve these problems, enriched methods are used, which use particular problem information to improve approximation characteristics. Among these enriched methods, can be cited the Generalized Finite Element Method (GFEM) (Arndt, 2009; Torii, 2012; Weinhardt, 2016).

The GFEM is based on the ideas of the Partition of Unity Method (PUM), developed by Melenk & Babuška (1996). The approximation ensures accurate local and global approximations (Torii, 2012). Strouboulis *et al.* (2001) and Babuška *et al.* (2005) present the GFEM as an extension of the PUM, where the approximation space is the union of the traditional approximation space of the FEM with the approximation enriched space.

In the context of Dynamic Analysis of Timoshenko Beam, very few works were developed until recently. The most central in this theory is given by Hsu (2016), performing a free vibration analysis using a enriched method approach. In his work only one model, among so many in the literature, was approached and used as the basis for enrichment. In this way, there is a list of several possible applications of enrichment techniques applied in each of the Timoshenko beam modeling strategies. In the present work one of these strategies is presented together with the enrichment proposed by Hsu (2016).

2 TIMOSHENKO FINITE ELEMENT BEAM MODEL

The weak form of Timoshenko beam equation, considering the dynamic equilibrium equation in the free vibration analysis based of virtual work, can be written as:

$$\int_0^L EI\frac{\partial\theta}{\partial x}\delta\left(\frac{\partial\theta}{\partial x}\right)dx + \int_0^L k_sGA\left(\frac{\partial w}{\partial x}-\theta\right)\delta\left(\frac{\partial w}{\partial x}-\theta\right)dx = \int_0^L \delta w\rho A\ddot{w}dx + \int_0^L \delta\theta\rho I\ddot{\theta}dx \quad (1)$$

where w and θ are displacement and rotation, E is Young's modulus, I is the inertia moment, k_s is a shear correction factor, G is the shear modulus, A is the cross sectional area, ρ is the density of material, $\ddot{w}$ and $\ddot{\theta}$ are the transverse and rotary accelerations, respectively, L is the length of the beam and δ denotes that the terms are virtual.

Timoshenko beam theory is adopted to take into account the transverse shear deformation of beams and therefore the deformation of the beam can be illustrated as in Fig. 1.

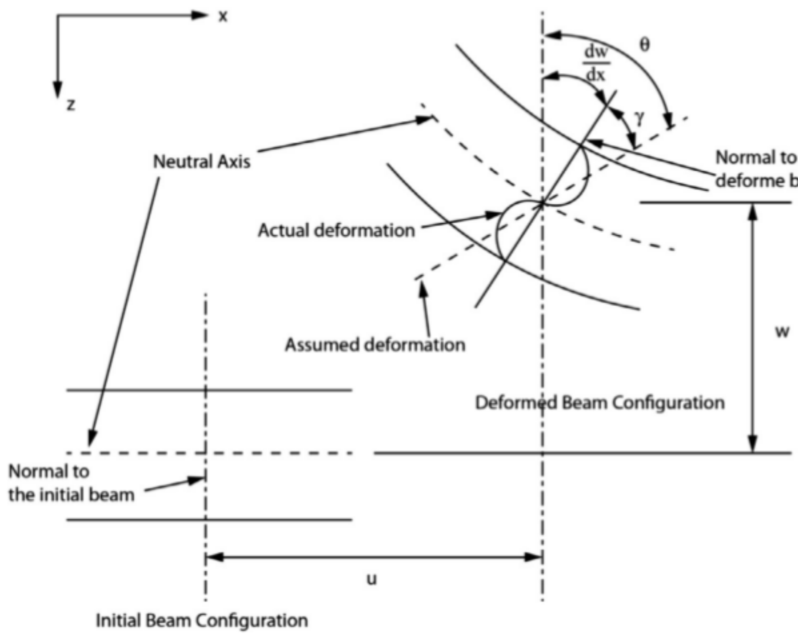

Figure 1: Schematic representation of Timoshenko beam.
SOURCE: Lee & Park (2013).

The Eq. 1 can be discretized by a simple unidirectional finite element, with two nodes and two degrees of freedom at each node, transverse and rotational displacement. Equations 2 and 3 present the approximate solution in the field of displacement, transverse and rotation, respectively:

$$w = \sum_{i=1} \varphi_i w_i \quad (2)$$

$$\theta = \sum_{i=1} \varphi_i \theta_i \quad (3)$$

Using the coordinate $\xi = [-1, 1]$ in the element domain, the Eqs. 2 and 3 are rewritten using the linear form function and are as shown in Eqs. 4 and 5:

$$w(\xi) = \varphi_1(\xi)w_1 + \varphi_2(\xi)w_2 \tag{4}$$

$$\theta(\xi) = \varphi_1(\xi)\theta_1 + \varphi_2(\xi)\theta_2 \tag{5}$$

The Eqs. 2 and 3 can be expressed in matrix form by:

$$\begin{Bmatrix} w(\xi) \\ \theta(\xi) \end{Bmatrix} = [H]\{u\} \tag{6}$$

where $[H]$ is the matrix of shape functions and $\{u\}$ is vector of nodal displacements, represented by:

$$[H] = \begin{bmatrix} \varphi_1 & 0 & \varphi_2 & 0 \\ 0 & \varphi_1 & 0 & \varphi_2 \end{bmatrix} \tag{7}$$

$$\{u\} = \begin{Bmatrix} w_1 \\ \theta_1 \\ w_2 \\ \theta_2 \end{Bmatrix} \tag{8}$$

The Eqs. 2 and 3 are inserted into Eq. 1, the stiffness and mass matrices in the element domain are presented by Eqs. 9 and 10, respectively.

$$[k^e] = \int_{-1}^{1} [B]^T[D][B]|J|d\xi \tag{9}$$

$$[M^e] = \int_{-1}^{1} (\rho I)[H]^T[H]|J|d\xi + \int_{-1}^{1} (\rho A)[H]^T[H]|J|d\xi \tag{10}$$

where $[B]$ is the strain displacement matrix, $[D]$ is the constitutive matrix, and $|J|$ is the Jacobian determinant, presented by Eqs. 11 and 12, respectively:

$$[B] = \begin{bmatrix} 0 & \frac{1}{|J|}\left(\frac{d\varphi_1}{d\xi}\right) & 0 & \frac{1}{|J|}\left(\frac{d\varphi_2}{d\xi}\right) \\ \frac{1}{|J|}\left(\frac{d\varphi_1}{d\xi}\right) & -\varphi_1 & \frac{1}{|J|}\left(\frac{d\varphi_2}{d\xi}\right) & -\varphi_2 \end{bmatrix} \tag{11}$$

$$[D] = \begin{bmatrix} EI & 0 \\ 0 & k_s GA \end{bmatrix} \tag{12}$$

After the appropriate algebraic manipulation, the Eq. 1 can be expressed in the following form:

$$([K]\{u\} - [M]\{\ddot{u}\}) = 0 \tag{13}$$

A general solution of Eq. 13 may be written as

$$u = \phi_k e^{i\omega_k} \tag{14}$$

Substituting Eq. 14 into Eq. 13 yields

$$[K - \omega_n^2 M]\phi_n = 0 \tag{15}$$

where ω_n and ϕ_n represent natural frequency and mode shape.

The discretization of the variable field from the construction of a mathematical space in the domain of the problem is the formulation of the finite element. The shape functions employed must guarantee the continuity of the mathematical space, even if the problem involves higher derivatives. This discretization of the ruling equation of the problem in finite elements using the shape functions gives the unknown variables as the nodal values to be determined from the system of equations. The conventional finite element formulation uses polynomials as shape function (Hsu, 2016).

It will be defined as shape functions:

$$\varphi_1 = \frac{1-\xi}{2} \tag{16}$$

$$\varphi_2 = \frac{1+\xi}{2} \tag{17}$$

where, ξ is natural coordinate of the element, with domain $\xi = [-1, 1]$.

For the mass matrix in FEM the Eq. 18 was used, instead of the the Eq. 10.

$$[M^e] = \int_{-1}^{1} (\rho A)[H]^T [H]|J| d\xi \tag{18}$$

This formulation results in a lumped mass matrix. This matrix has half of the element mass at each translational nodal degree of freedom. And, conserve the mass associated with their translational degrees of freedom (Kwon & Bang, 1997).

3 GENERALIZED FINITE ELEMENT METHOD

The GFEM allows a wide range of choices for local approximation spaces, and this can be used without changing the basic premises of FEM. Examples of Partition of Unity functions are all the conventional finite element shape functions where the clouds are formed by a set of finite elements that contribute to nodal values. This work considers the shape functions of C^0 element, Eqs. 16 and 17, as partition of unity, which serve to enforce the continuity of enriched mathematical space (Shang *et al.*, 2016; Hsu, 2016). The GFEM approximation for enriched C^0 element employed in this work is given as:

$$u(\xi) = \sum_{i=1}^{2} \varphi_i(\xi) \left\{ u_i + \sum_{j=1}^{m} a_j^{(i)} \psi_j^{(i)}(\xi) \right\} = [G]^T \{U\} \tag{19}$$

where $\psi_j^i(\xi)$ is the set of enrichment functions, and u_i is nodal displacement. After algebraic arrangement:

$$\{U\}^T = \begin{bmatrix} u_1 & \theta_1 & u_2 & \theta_2 & a_1^1 & a_1^1 \ldots & a_m^1 & a_m^1 & a_m^2 & a_m^2 \end{bmatrix} \tag{20}$$

$$[G]^T = \begin{bmatrix} \varphi_1 & 0 & \varphi_2 & 0 & \varphi_1\psi_1^1 & 0 \ldots & \varphi_1\psi_m^1 & 0 & \varphi_2\psi_m^2 & 0 \\ 0 & \varphi_1 & 0 & \varphi_2 & 0 & \varphi_1\psi_1^1 \ldots & 0 & \varphi_1\psi_m^1 & 0 & \varphi_2\psi_m^2 \end{bmatrix} \tag{21}$$

Being this work an extension of Hsu (2016) work, the enrichment functions adopted will be the same as those adopted by the author. It is a set of trigonometric sine functions, presented in Eqs. 22 and 23, where $\beta_j = j\pi$, with $j = 1, 2, 3, \ldots, m$ where m is the total quantity of enrichment level employed in the mathematical space. Hsu (2016) chose the sine function due to the facility to increase the enrichment level by changing value of β_j and the similarity between the sine function and the modes of vibration.

$$\psi_j^{(1)} = sin\left(\frac{\beta_j(\xi+1)}{2}\right) \tag{22}$$

$$\psi_j^{(2)} = sin\left(\frac{\beta_j(\xi-1)}{2}\right) \tag{23}$$

4 NUMERICAL RESULTS

In this section you will see two applications. First a low $\frac{h}{L}$ ratio ($\frac{h}{L} = 0.002$), where the shear effect is low and the Timoshenko model must converge to the classic model of Euler Bernoulli.

In a second moment the example with $\frac{h}{L} = 0.1$ is presented, where the shearing effect to be relevant and the model of Timoshenko departs from the classical theory.

For both examples primary unit parameters are used and the others are calculated appropriately. The base finite element mesh is fixed in 50 elements. The natural frequencies are normalized, allowing easy dimensionless comparison according to:

$$\lambda_i^2 = \omega_i L^2 \sqrt{\frac{m}{EI}} \tag{24}$$

where m is mass per unit length of the beam and ω_i is the natural frequency.

The beam model chosen is the clamped-free beam. A schematic representation this beam is presented in Fig. 2.

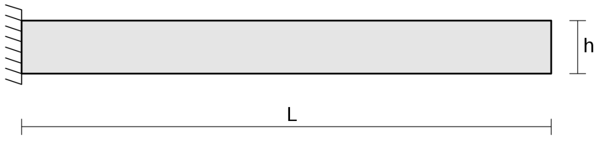

Figure 2: Schematic representation of clamped-free beam.

FEM and GFEM approximations are performed in order to access its gains in accuracy in terms of frequency spectra. As it is difficult to find general solutions in the literature, a reference solution with 10000 degrees of freedom is adopted with elements C^0. The approximated solutions are then normalized with respect to this reference solution for the construction of the frequency spectrum.

The stiffness and mass matrices of FEM are obtained by reduced numerical integration. For GFEM, the numerical integration of these matrices is using Gaussian 30 quadrature points.

4.1 Lower $^h/_L$ ratio

In order to demonstrate the effects of enrichment on the model under study, frequency spectra for Linear FEM and GFEM with up to 4 levels of enrichment are presented. The results can be seen in Fig. 3.

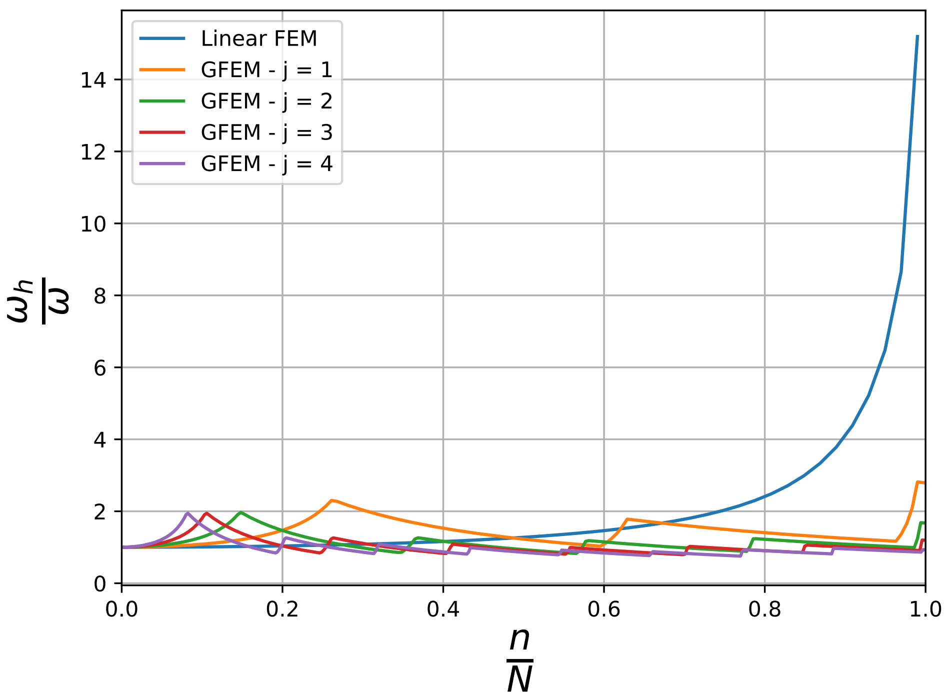

Figure 3: Normalized discrete spectra, $^h/_L = 0.002$**.**

The spectrum of frequencies for the different methods presented in Fig. 3 highlights the differences in intrinsic approach characteristics. It is possible to notice that in the first part of the spectrum the GFEM has a peak, losing to the FEM. However, the enriched strategy is much higher than 40% of the spectrum. An enrichment refining up to grade 4 was performed showing the convergence of the approximation in terms of the spectrum. As expected, the approximation oscillates below $^{w_h}/_w = 1$ at some times due to the mass-grouping strategy, but overall the approximation is very stable and reliable.

Aiming to compare Timoshenko's model with the classical theory model of Euler Bernoulli in this example, Table 1 is presented.

Table 1: Normalized frequencies, $^h/_L = 0.002$.

Modes	Euler Bernoulli	FEM	GFEM (j=1)	GFEM (j=2)	GFEM (j=3)	GFEM (j=4)
1	1.875104	1.875220	1.875220	1.875221	1.875221	1.875221
2	4.694091	4.697631	4.697631	4.697631	4.697631	4.697631
3	7.854757	7.870767	7.870767	7.870767	7.870767	7.870767
4	10.995541	11.039658	11.039658	11.039658	11.039658	11.039658
5	14.137168	14.231252	14.231252	14.231252	14.231252	14.231252
6	17.278760	17.451249	17.451249	17.451249	17.451249	17.451249
7	20.420352	20.706464	20.706464	20.706464	20.706464	20.706463
8	23.561945	24.003963	24.003963	24.003963	24.003963	24.003963
9	26.703538	27.351166	27.351166	2.7351166	27.351166	27.351166
10	29.845130	30.755913	30.755913	30.755913	30.755913	30.755913

4.2 Higher $^h/_L$ ratio

In order to demonstrate the effects of enrichment on the model under study, frequency spectra for Linear FEM and GFEM with up to 4 levels of enrichment are presented. The results can be seen in Fig. 4.

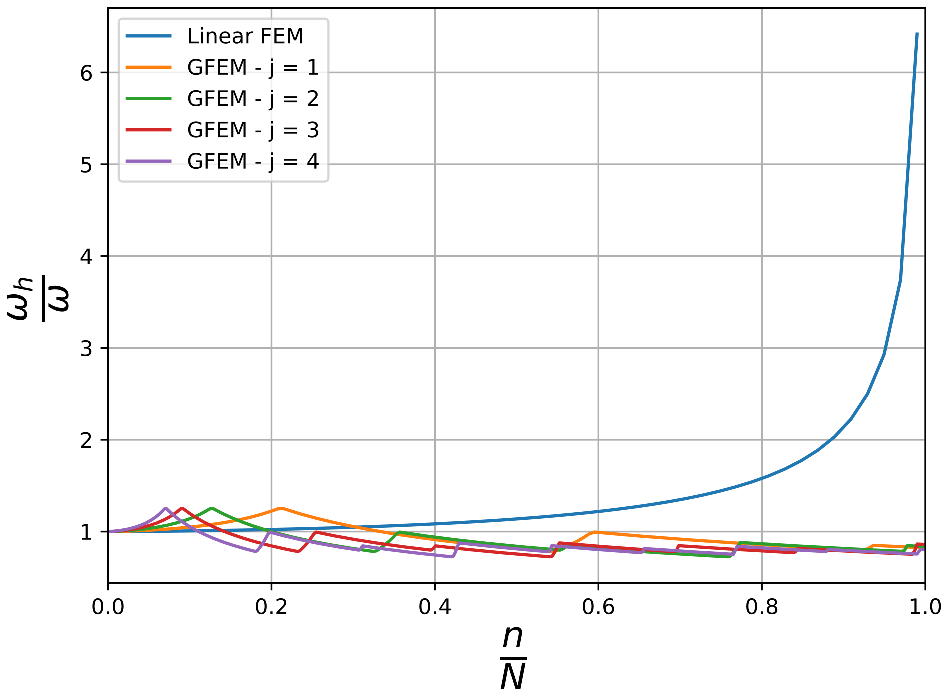

Figure 4: Normalized discrete spectra, $^h/_L = 0.1$.

Analogously to the previous example, the spectrum of frequencies for the different methods presented in Fig. 4 highlights the differences in intrinsic approach characteristics. It is possible to notice that in the first part of the spectrum the GFEM has a peak, losing to the FEM. However, the enriched strategy is much higher than 40% of the spectrum. An enrichment refining up to grade 4 was performed showing the convergence of the approximation in terms of the spectrum. As expected, the approximation oscillates below $w_h/w = 1$ at some times due to the mass-grouping strategy, but overall the approximation is very stable and reliable. However, this time, the dive below the line $w_h/w = 1$ is probably accentuated by the change in the h/L ratio.

Aiming to compare Timoshenko's model with the classical theory model of Euler Bernoulli in this example, Table 2 is presented.

Table 2: Normalized frequencies, $h/L = 0.1$.

Modes	Euler Bernoulli	FEM	GFEM (j=1)	GFEM (j=2)	GFEM (j=3)	GFEM (j=4)
1	1.875104	1.869594	1.869594	1.869594	1.869594	1.869594
2	4.694091	4.603216	4.603216	4.603216	4.603216	4.603216
3	7.854757	7.517542	7.517542	7.517542	7.517542	7.517542
4	10.995541	10.197774	10.197774	10.197774	10.197774	10.197774
5	14.137168	12.659452	12.659452	12.659452	12.659452	12.659452
6	17.278760	14.915473	14.915473	14.915473	14.915473	14.915473
7	20.420352	16.990264	16.990264	16.990264	16.990264	16.990264
8	23.561945	18.909936	18.909936	18.909936	18.909936	18.909936
9	26.703538	20.699104	20.699104	20.699104	20.699104	20.699104
10	29.845130	22.379714	22.379714	22.379714	22.379714	22.379714

5 CONCLUDING REMARKS

The present work presented a brief review of the work in the area of dynamic analysis of Timoshenko beams. It was also presented a brief description of the enrichment process applied to this type of model, with emphasis on trigonometric enrichment. Examples were given illustrating the application of the methodology in the chosen context.

A first example presented in this work was the one with the lowest $\frac{h}{L}$ ratio, where the shear effect is low and the Timoshenko model must converge to the classical Euler Bernoulli model. The obtained results were compared with a reference solution.

A second example using a h / L = 0.1 ratio was presented, in which case the shear effect is relevant, making Timoshenko's theory depart from the result of classical theory. The frequencies obtained are compared with the same reference solution from the previous example. The numerical integrations performed on the stiffness and mass matrices in the two examples were performed in the same way. A reduced numerical integration for FEM was adopted while GFEM integration required 30 Gaussian quadrature points.

There are still very few works developed in the dynamic analysis of Timoshenko beams. However, there is a large list of possible applications of enrichment techniques that can be employed in each of the Timoshenko beam modeling strategies. Therefore, this work had as objective to subsidize new studies of Timoshenko beams with enrichment.

ACKNOWLEDGEMENTS

The authors acknowledge CAPES for the financial support.

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