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DYNAMIC ANALYSIS OF PLANE STEEL FRAMES WITH SEMI-RIGID CONNECTIONS BY THE GENERALIZED FINITE ELEMENT METHOD

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Abstract. *A structural analysis of plane steel frames assumes simplifying assumptions in the structural model, as the behavior of structural connections. They are generally supposed as a fully rigid connection or as an ideally pinned connection. In practice, all connections have relative rotation and connection moment, thus they are classified as semi-rigid connections. For dynamic analysis, only a few structural systems have known analytical solutions, so that approximate computational methods are indicated for the problem solution, like the Finite Element Method (FEM). Shape functions with polynomial terms are generally used in FEM, however in problems like as dynamic analysis of structures the solutions have non-polynomial terms, and the FEM become inaccurate in relation to the real behavior. The addition of non-polynomial functions in the approximate method can be performed with enriched methods like as the Generalized Finite Element Method (GFEM). Therefore, this work seeks to present how the semi-rigid connections are coupled to GFEM from two different formulations proposed by Chen and Lui (1991) and Chan and Chui (2000). The analyses will be performed by a code developed in the Maple® program. An example encountered in the literature will be modeled and the GFEM results will be compared with the literature results.*

Keywords: *Semi-rigid connections, Finite element method, Generalized finite element method, Dynamic analysis.*

1 INTRODUCTION

The structural analysis in civil engineering has as objective the determination of the effects caused by the loads applied in the structures. The degree of analysis of the structural system basically depends on the simplifications adopted in the structural model, which will be an approximation of the real system. Among the important simplifications of the structural model, it is possible to highlight the behavior of the structural connections, which are usually based on rigid and flexible connections. The popularity of the use of these idealized models comes from the fact that they are simple to use in the structural analyzes of the projects of reticulated metal structures (Chen, Kishi and Komuro, 2011). However, experimental tests carried out on different types of connection configurations have indicated that in practice all connections have rotation and transmission of bending moments and are then classified as semi-rigid connections.

In addition to approaching the real physical behavior of the structure, the study of semi-rigid connections can generate better distributed efforts and the possibility of generate slender frames. The connections have a negligible weight, but the high cost of manufacturing and assembly makes it necessary to treat them with care and to avoid possible wastage, obtaining a financial saving of the project. In the framework of steel structures design, there is a growing need to dimension the structural elements with better use of material (Chen, Goto and Liew, 1996). In addition, the development of the civil industry led to the design of more resistant materials, leading to the interest of designing slender and lighter structures in order to make the products more competitive and achieve the desired economy. The design of structural systems with such characteristics can generate failures if advanced verifications are not performed (Silva, 2009). These changes in structural conceptions and verification requirements generate the need to develop computational systems with resources for nonlinear analysis of the structure when it is submitted to static and dynamic requests.

Soriano (2014) complements that these characteristics, applied to the civil structures, make the structural system more susceptible to vibrations and with reduced capacity to dissipate energy. These dynamic effects are generated by external requests such as vehicle and equipment traffic, or by natural phenomena such as winds, tides and earthquakes. In some cases, it is imperative to perform the dynamic analysis for the correct design of the structure. However, only some types of structural systems with simple geometries and classical boundary conditions have known analytical solutions for their dynamic analysis (Arndt, 2009), and it is ideal to use approximate computational methods to solve the problem.

The Finite Element Method (FEM) is the most known approximate method for continuum problems. Assan (2003) states that FEM emerged as an alternative to solve complex problems of elasticity and engineering theory, overcoming difficulties of existing models. In this method, it is considered that the continuous problem can be discretized in finite elements, so that they share the properties of the continuous medium and respond to the mathematical equation that governs the physical problem. Shape functions with polynomial terms are generally used in FEM, however in problems with the oscillatory phenomena, like as dynamic analysis of structures, the solutions have non-polynomial terms, and the approximate method with polynomials become inaccurate in relation to the real behavior. The addition of non-polynomial functions in the approximate method can be performed with enriched methods like as the Generalized Finite Element Method (GFEM). An adequate choice allows that some known behavior about the phenomena can be represented in the model.

Therefore, this work seeks to present how the semi-rigid connections are coupled to GFEM from two different formulations proposed by Chen and Lui (1991) and Chan and Chui (2000). These formulations differ in aspects as the influence of the stiffness in the normal forces of connection, and the method of deduction of stiffness matrix in the hybrid finite element. Thus, the problem analysis can be performed including some knowledge of the solution of the governing differential equation due to the use of GFEM, and it can also evaluate the physical nonlinearity of the semi-rigid connections due to the hybrid element. The analyses will be performed by a code developed in the Maple® program. An example encountered in the literature will be modeled and the GFEM results will be compared with the literature results.

2 ANALYSIS OF STRUCTURAL FREE VIBRATION PROBLEM WITH APPROXIMATE METHODS

For the structural free vibration problem in plane frames, according to the one-dimensional bar and Euler-Bernoulli beam theories, the finite element can be obtained by the governing equations of motion (Shang, Machado and Abdalla Filho, 2016) expressed by Eq. 1 and Eq. 2, and the generalized problem is showed in Fig. 1 and Fig. 2.

$$m(x)\frac{\partial^2 u_1(x,t)}{\partial t^2} - \frac{\partial}{\partial x}\left[E(t)A(x)\frac{\partial u_1(x,t)}{\partial x}\right] = 0 \tag{1}$$

$$m(x)\frac{\partial^2 u_2(x,t)}{\partial t^2} + \frac{\partial^2}{\partial x^2}\left[E(t)I(x)\frac{\partial u_2(x,t)}{\partial x^2}\right] = 0 \tag{2}$$

The variables involved are all written as a function of the axial coordinate (*x*) of the finite element and time (*t*): *m(x)* is the mass distributed along the element; u_1 and u_2 are the displacements in the axial and transverse directions, respectively; *E(t)* is the modulus of elasticity of the material, which can vary over time (physical non-linearity); and *A(x)* and *I(x)* are the functions that describe the cross-sectional area and the moment of inertia, respectively, along the element. The material used in the finite element is isotropic and homogeneous.

The element field displacements are evaluated according to Eq. 3, where $\boldsymbol{u}_e$ is the vector of nodal values in element coordinates, and $\boldsymbol{H}$ is the matrix of the shape functions. Thus, the displacement field is an approximation of the real displacement field, including errors in the numerical response. In general, the errors reduce as the refinement of the mesh increases.

$$\boldsymbol{u} = \boldsymbol{H}\boldsymbol{u}_e \tag{3}$$

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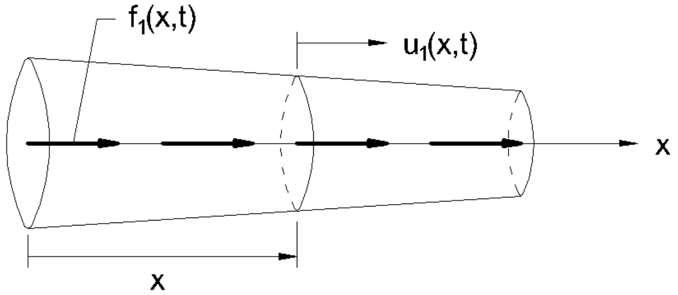

Figure 1. One-dimensional bar subjected to axial displacements (see Arndt, 2009).

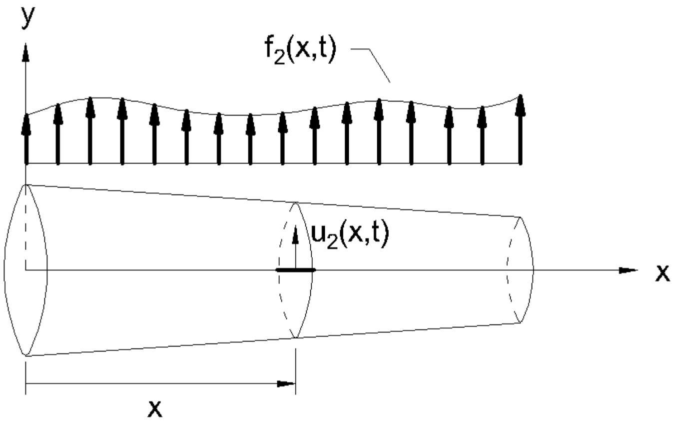

Figure 2. Euler-Bernoulli beam subjected to lateral displacements (see Arndt, 2009).

The Total Lagrangian Formulation is used to describe the behavior of the solid (Shang, Machado and Abdalla Filho, 2016). With some mathematical manipulation, it can be obtained the mass matrix $\boldsymbol{M^e}$ and the stiffness matrix $\boldsymbol{K^e}$ of the finite element:

$$\boldsymbol{M^e} = \int_{L_e} \boldsymbol{H}^T \rho A \boldsymbol{H} dx \tag{4}$$

$$\boldsymbol{K^e} = \int_{L_e} \boldsymbol{B^L}^T \boldsymbol{D^p} \boldsymbol{B^L} dx \tag{5}$$

where $\boldsymbol{D^p}$ is the constitutive matrix, $\boldsymbol{B^L}$ is the linear strain-displacement matrix, ρ is the density and A is the cross-section area. With proper coordinate transformation and element assembling, Eqs. 1 and 2 can be reduced to following equilibrium equation:

$$\boldsymbol{M\ddot{u}} + \boldsymbol{Ku} = 0 \tag{4}$$

where $\boldsymbol{M}$ is the global mass matrix and $\boldsymbol{K}$ is the global stiffness matrix.

A structural modal analysis has the objective to found the natural vibration frequencies and modes. This theme is treated in detail by authors such as Chopra (2011) and Soriano (2014). A modal analysis is a matrix eigenvalue and eigenvector problem, where the natural frequencies ω and modes ϕ must satisfy the following algebraic equation:

$$\boldsymbol{K}\phi = \omega^2 \boldsymbol{M}\phi \tag{5}$$

A steel structure with semi-rigid connections has lower natural frequencies than the same steel structure with rigid connections (Da Silva *et al.*, 2008; Silva, 2009; Silva, Silveira and Neves, 2010; Nguyen and Kim, 2013). This behavior leads to the need for further studies about the steel structures behavior, considering effects like the semi-rigid connections in the design step. Da Silva *et al.* (2008) presented that a connection analysis can be crucial for the security of a structure subjected to dynamical efforts. If the frequency of a periodic excitation approaches a natural frequency, the structure displacements increase considerably, and it can generate the resonance phenomena.

3 GENERALIZED FINITE ELEMENT METHOD

The Generalized Finite Element Method (GFEM) is a Galerkin method whose main goal is the construction of a finite dimensional subspace of approximating functions using local knowledge about the solution that ensures accurate local and global results. The GFEM local enrichment in the approximation subspace is incorporated by the partition of unity approach, introduced by Melenk and Babuska (1996). The Generalized Finite Element Method (GFEM) was developed in Duarte, Babuska and Oden (2000) and Strouboulis, Babuska and Copps (2000). The researchers observed the need to approximate the results obtained by FEM to the exact solutions of the governing equations of the problems. When analyzing engineering problems, special functions can be added to the code previously written in FEM, which reflect the nature of the exact solution, and allow the use of the same code without major changes.

In this section, it is presented topics about the partition of unity, and the generalized element used in the computational code for analysis of plane steel frames.

3.1 Partition of unity

Let $u \in \mathrm{H}^1(\Omega)$ be the function to be approximated and $\{\Omega_i\}$ be an open cover of domain Ω (Fig. 3) satisfying an overlap condition:

$$\exists M_S \in \mathbf{N} \ \text{ so that } \ \forall x \in \Omega \quad card\left\{ i \middle| x \in \Omega_i \right\} \leq M_S \, . \tag{6}$$

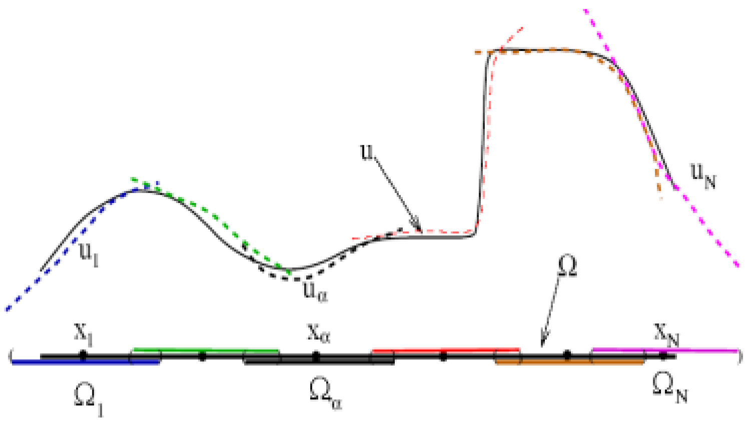

Figure 3. Open cover $\{\Omega_i\}$ of domain Ω (see Duarte, Babuska and Oden, 2000).

A Lipschitz partition of unity subordinate to the cover $\{\Omega_i\}$ is the set of functions $\{h_i\}$ satisfying the conditions:

$$\text{supp}\,(h_i) = \{\, x \in \Omega \ |h_i(x) \neq 0\} \subset [\Omega_i], \quad \forall i\,, \tag{7}$$

$$\sum_i h_i \equiv 1 \ \text{ on } \ \Omega\,, \tag{8}$$

where $\text{supp}\,(h_i)$ denotes the support of definition of the function h_i and $[\Omega_i]$ is the closure of the patch Ω_i.

The partition of unity set $\{h_i\}$ allows obtaining an enriched set of approximating functions. Let $L_i \subset \mathrm{H}^1(\Omega_i \cap \Omega)$ be a set of functions that locally well represents *u*:

$$L_i = \{l_i^j\}_{j=1}^m . \tag{9}$$

Then the enriched set is formed by multiplying each partition of unity function h_i by the corresponding l_i^j, i.e.,

$$L = \sum_i h_i \ L_i = \left\{ \sum_i h_i \ l_i^j \ \ |l_i^j \in L_i \right\} \subset H^1(\Omega). \tag{10}$$

Accordingly, the function *u* can be approximated by the enriched set as:

$$u_h(x) = \sum_i \sum_{l_i^j \in L_i} h_i l_i^j(x) \ a_{ij}\,, \tag{11}$$

where a_{ij} are the degrees of freedom.

In the proposed GFEM, the cover $\{\Omega_i\}$ corresponds to the finite element mesh and each patch Ω_i corresponds to the sub domain of Ω formed by the union of elements that contain the node x_i (Fig. 4).

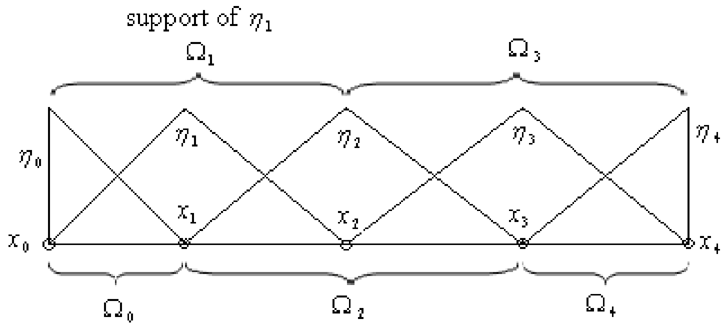

Figure 4. Patchs and partition of unity set for one-dimensional GFEM finite element mesh (see Arndt, 2009).

3.2 Generalized element

The equation of the unknown nodal approximation is given by the conventional finite element shape function and enrichment monomial. This expression can be shown by (Shang, Machado and Abdalla Filho, 2016):

$$u(x) = \sum_{i=1}^{N} h_i(x) \left\{ u_i + \sum_{j=1}^{qj} \sum_{k=1}^{\delta} L_{ij}^{k}(x) b_{ij}^{k} \right\} = \phi^T U \tag{12}$$

The generalized elements use the classical linear FEM shape functions as the partition of unity, i.e. the Lagrangian and the Hermitian polynomials. They are presented in Eq. 13 to Eq. 18, and in Fig. 5.

$$h_1 = \frac{(1-\xi)}{2} \tag{13}$$

$$h_2 = \frac{1}{2} - \frac{3}{4}\xi + \frac{1}{4}\xi^3 \tag{14}$$

$$h_3 = \frac{L_e}{8}\left(1 - \xi - \xi^2 + \xi^3\right) \tag{15}$$

$$h_4 = \frac{(1+\xi)}{2} \tag{16}$$

$$h_5 = \frac{1}{2} + \frac{3}{4}\xi - \frac{1}{4}\xi^3 \tag{17}$$

$$h_6 = \frac{L_e}{8}\left(-1 - \xi + \xi^2 + \xi^3\right) \tag{18}$$

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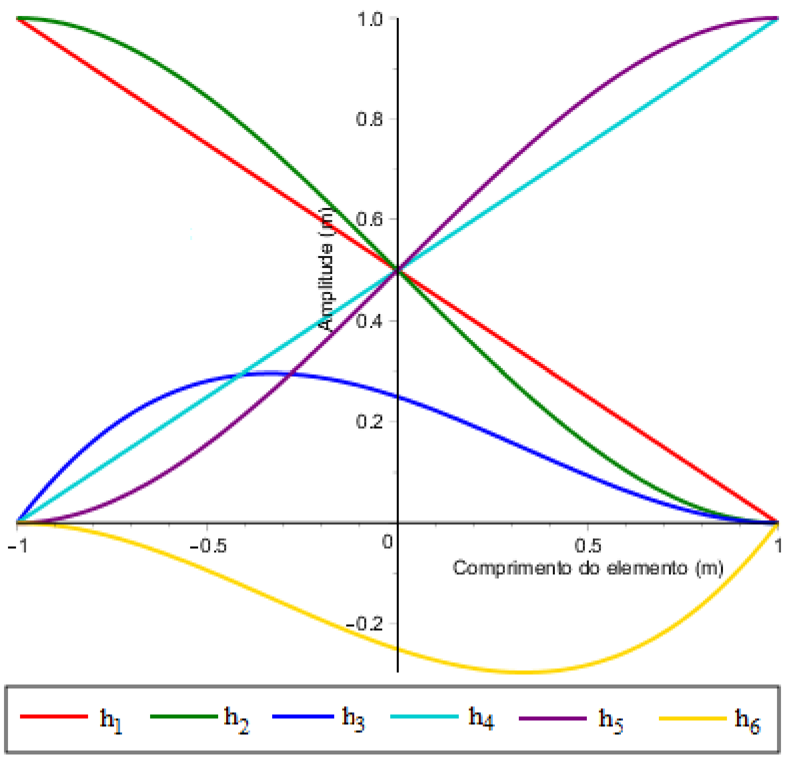


Figure 5. Plot of FEM shape functions.

The group of enrichment functions used in this work is initially proposed by Arndt (2009), who studied the solution of the governing equations of motion presented above and performed analysis including some knowledge of the solution due to the use of GFEM. These functions were also used by Torii (2012), Weinhardt (2016) and Shang, Machado and Abdalla Filho (2016) in the free vibration analysis of bars and Euler-Bernoulli beams, and they are presented in Eq. 19 and Eq. 20.

$$L_{1j}^{1} = \cos\left(\beta_j \frac{(1+\xi)}{2}\right) - 1 \tag{19}$$

$$L_{2j}^{1} = \cos\left(\beta_j \frac{(1-\xi)}{2}\right) - 1 \tag{20}$$

where β_j is a parameter related to the enrichment level *j*. The enriched set *L*, so proposed, vanishes at element nodes 1 and 2, which allows the imposition of boundary conditions in the same fashion of the finite element procedure.

Different parameters β_j produce different enriched elements. The increase in the number of elements in the mesh with only one level of enrichment (*j* = 1) and a fixed parameter β_1, for example $\beta_1 = \pi$, produces an *h* refinement. Otherwise the increase in the number of levels of enrichment, with a different parameter β_j each, for example, $\beta_j = j\pi$, produces a hierarchical *p* refinement (Arndt, 2009; Arndt, Machado and Scremin, 2010). Fig. 6 shows the enrichment shape functions for $\beta_1 = \pi$.

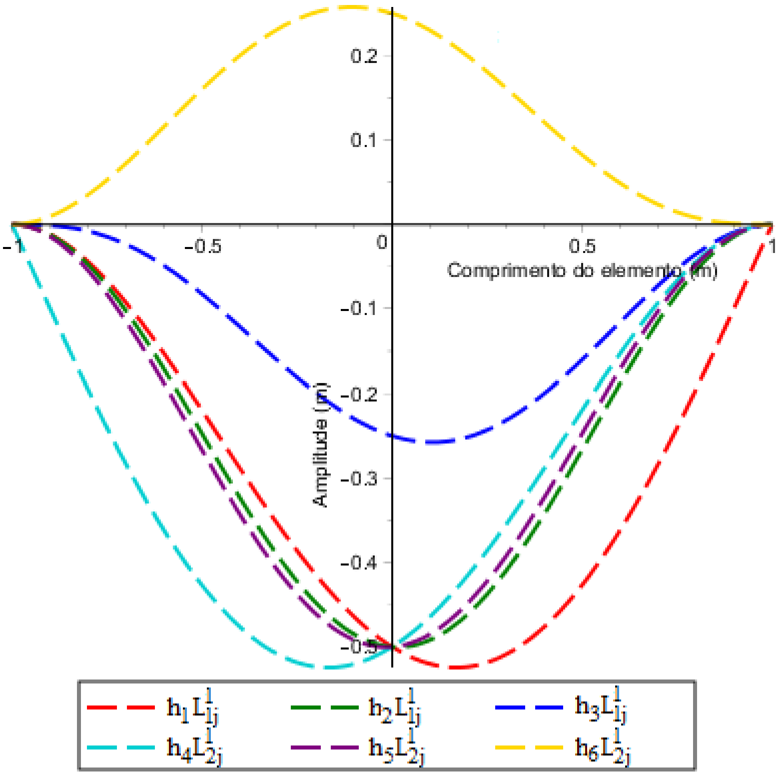


Figure 6. Plot of GFEM enrichment shape functions.

The enriched FE shape functions can be added into ***H*** to obtain the modified ***H***, formulated by GFEM and showed in Eq. 21 (Shang, Machado and Abdalla Filho, 2016).

$$\boldsymbol{H} = \begin{bmatrix} h_1 & 0 & 0 & h_4 & 0 & 0 & h_1 L^1_{1j} & 0 & 0 & \ldots & h_4 L^\delta_{2j} & 0 & 0 \\ 0 & h_2 & h_3 & 0 & h_5 & h_6 & 0 & h_2 L^1_{1j} & h_3 L^1_{1j} & \ldots & 0 & h_5 L^\delta_{2j} & h_6 L^\delta_{2j} \end{bmatrix} \quad (21)$$

where h_i are the conventional finite element shape functions, and L^δ_{ij} represents the enrichment monomials. Sub-index *i* is associated to the enriched shape function to indicate node 1 or 2. Sub-index *j* is the level of enrichment adopted, and the super-index δ indicates the group of enrichment function used. The conventional ***H*** matrix in FEM has only the first six columns.

The linear strain-displacement matrix $\boldsymbol{B^L}$ with enrichment can be represented by Eq. 22. The conventional $\boldsymbol{B^L}$ matrix in FEM has only the first six columns.

$$\boldsymbol{B^L} = \begin{bmatrix} h_1^{'} & 0 & 0 & h_4^{'} & 0 & 0 & \left(h_1 L_{1j}^1\right)^{'} & 0 & 0 & \ldots & \left(h_4 L_{2j}^{\delta}\right)^{''} & 0 & 0 \\ 0 & h_2^{''} & h_3^{''} & 0 & h_5^{''} & h_6^{''} & 0 & \left(h_2 L_{1j}^1\right)^{''} & \left(h_3 L_{1j}^1\right)^{''} & \ldots & 0 & \left(h_5 L_{2j}^{\delta}\right)^{''} & \left(h_6 L_{2j}^{\delta}\right)^{''} \end{bmatrix} \quad (22)$$

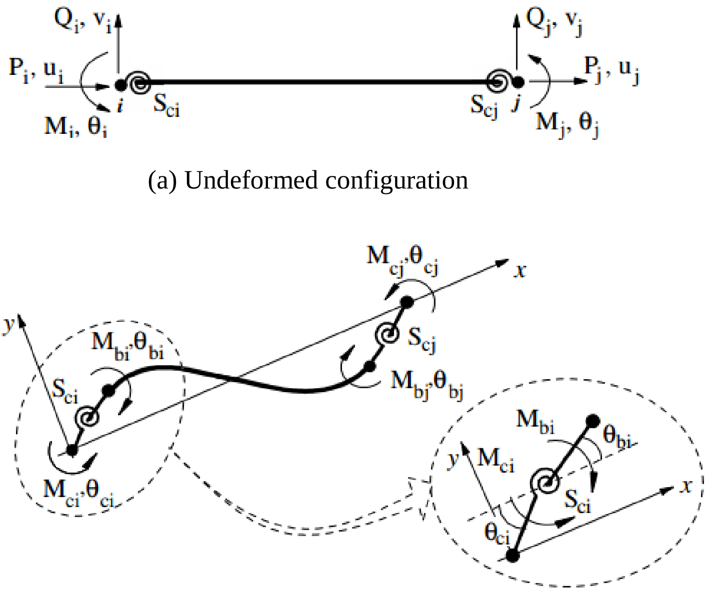
4 SEMI-RIGID CONNECTIONS

The behavior of steel structures depends on the way the connections are designed in the structural design step. Connections are traditionally considered as rigid or flexible. In rigid connections, it is considered that the bending moment acting on the connection is totally transferred between the connected elements, and it is assumed that there is a perfect continuity, leading to the situation that the relative angle formed between the connected structural elements is essentially the same after loading. In flexible connections, it is assumed that the relative rotation between the elements, in which the transfer of bending moment by the connection of the elements is not considered. However, a connection can present a diversity of configurations and connection devices, which directly influence the local connection behavior and, consequently, the global behavior of the structure.

Experimental studies have evaluated different connection configurations and demonstrated that all of them have a non-linear behavior in the moment-rotation relationship (Chan and Chui, 2000; Chen, Kishi and Komuro, 2011). Chen, Goto and Liew (1996) justify that the non-linear behavior is due to numerous factors such as the material discontinuity between the connection components and buckling effects on the connection parts. In addition, the flexibility effect of connections is known to be one of the main causes of non-linearity in the behavior of steel frames under static and dynamic loads (Chan and Chui, 2000). In this way, it is not possible to say that there is a perfect idealization between connected elements. The connections introduce local effects and imperfections that can induce a behavior of partial rigidity of the connections, called semi-rigid connections.

In this work, semi-rigid connections are considered in the analysis by the coupling in the finite element stiffness matrix, obtained by the Finite Element Method (FEM) or by the Generalized Finite Element Method (GFEM). In this way, the original design of the plane steel frame stiffness matrix is altered, being this one obtained by the approximate methods studied in this work. It is considered that the formulations of hybrid elements modifies only the stiffness matrix elements originated by the FEM.

The semi-rigid connection can be modeled as a spring element between the beam and column elements. The hybrid finite element used is shown in Fig. 7, in its undeformed and deformed configuration.



(a) Undeformed configuration

(b) Deformed configuration

Figure 7. Hybrid finite element (see Silva, 2009).

4.1 Formulation of Chen and Lui (1991)

The formulation of Chen and Lui (1991) idealizes the hybrid element by coupling the connecting element to the plane steel frame element, considering in a first moment that the degrees of freedom of rotation of the spring element are independent of degrees of rotation of the plane steel frame element, as shown in Eq. 23, where S_{ci} and S_{cj} are the connection stiffness. In addition to this consideration, the spring elements have null lengths. The hybrid element is shown in Fig. 8.

$$\begin{Bmatrix} \Delta M_{ci} \\ \Delta M_{bi} \\ \boldsymbol{\Delta f}^{\boldsymbol{e}} \\ \Delta M_{bj} \\ \Delta M_{cj} \end{Bmatrix} = \begin{bmatrix} S_{ci} & -S_{ci} & 0 & 0 & 0 \\ -S_{ci} & S_{ci} & 0 & 0 & 0 \\ 0 & 0 & \boldsymbol{K}^{\boldsymbol{e}} & 0 & 0 \\ 0 & 0 & 0 & S_{cj} & -S_{cj} \\ 0 & 0 & 0 & -S_{cj} & S_{cj} \end{bmatrix} \begin{Bmatrix} \Delta\theta_{ci} \\ \Delta\theta_{bi} \\ \boldsymbol{\Delta u} \\ \Delta\theta_{bj} \\ \Delta\theta_{cj} \end{Bmatrix} = \boldsymbol{K}_{\boldsymbol{ac}}^{\boldsymbol{e}} \boldsymbol{\Delta u}_{\boldsymbol{ac}}^{\boldsymbol{e}} \tag{23}$$

where $\boldsymbol{K}_{\boldsymbol{ac}}^{\boldsymbol{e}}$ is the stiffness matrix of the coupled element, and $\boldsymbol{\Delta u}_{\boldsymbol{ac}}^{\boldsymbol{e}}$ is the vector of degrees of freedom of the coupled element.

The transformation matrix between the coupled element and the auxiliary element of Chen and Lui (1991) is defined as the matrix $\boldsymbol{C}$:

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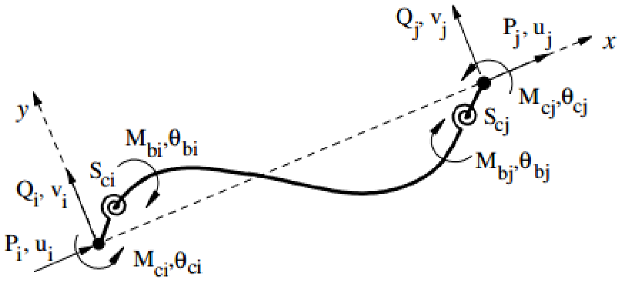

Figure 8. Hybrid finite element for Chen and Lui (1991) formulation (see Silva, 2009).

$$
\boldsymbol{C} = \begin{bmatrix}
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0
\end{bmatrix} \tag{24}
$$

It can use the matrix $\boldsymbol{C}$ to rearrange the stiffness matrix of the coupled element, generating the auxiliary stiffness matrix $\boldsymbol{K}_a^e$ indicated in the Eq. 25.

$$
\boldsymbol{K}_a^e = \boldsymbol{C}^T \boldsymbol{K}_{ac}^e \boldsymbol{C} = \begin{bmatrix}
k_{11} & k_{12} & 0 & k_{14} & k_{15} & 0 & k_{13} & k_{16} \\
k_{21} & k_{22} & 0 & k_{24} & k_{25} & 0 & k_{23} & k_{26} \\
0 & 0 & S_{ci} & 0 & 0 & 0 & -S_{ci} & 0 \\
k_{41} & k_{42} & 0 & k_{44} & k_{45} & 0 & k_{43} & k_{46} \\
k_{51} & k_{52} & 0 & k_{54} & k_{55} & 0 & k_{53} & k_{56} \\
0 & 0 & 0 & 0 & 0 & S_{cj} & 0 & -S_{cj} \\
k_{31} & k_{32} & -S_{ci} & k_{34} & k_{35} & 0 & S_{ci}+k_{33} & k_{36} \\
k_{61} & k_{62} & 0 & k_{64} & k_{65} & -S_{cj} & k_{63} & S_{cj}+k_{66}
\end{bmatrix} \tag{25}
$$

where the matrix elements k_{mn} are the stiffness matrix elements from the FEM.

A static condensation is performed in the stiffness matrix obtained in Eq. 25, aiming to return to six degrees of freedom. The auxiliary stiffness matrix $\boldsymbol{K}_a^e$ can be partitioned into four sub-matrices, as follows:

$$\begin{Bmatrix} \boldsymbol{\Delta f_i} \\ \boldsymbol{\Delta f_m} \end{Bmatrix} = \begin{bmatrix} \boldsymbol{K_{b1}} & \boldsymbol{K_{b2}} \\ \boldsymbol{K_{b3}} & \boldsymbol{K_{b4}} \end{bmatrix} \begin{Bmatrix} \boldsymbol{\Delta u} \\ \boldsymbol{\Delta u_b} \end{Bmatrix} \tag{26}$$

Considering that the elements of the vector $\boldsymbol{\Delta f_m}$ are zero, the static condensation can be performed with the matrix operation given by:

$$\boldsymbol{K^{e*}} = \boldsymbol{K_{b1}} - \boldsymbol{K_{b2}} \boldsymbol{K_{b4}^{-1}} \boldsymbol{K_{b3}} \tag{27}$$

Finally, the force-displacement relationship of the hybrid element with semi-rigid connections can be written using the stiffness matrix $\boldsymbol{K^{e*}}$. In this formulation of Chen and Lui (1991), all terms of the stiffness matrix $\boldsymbol{K^{e*}}$ depend on the stiffness parameters S_{ci} and S_{cj} due to the use of static condensation. Therefore, the semi-rigid connection effect affects all terms of stiffness and consequently all components of displacements and forces of the finite element.

4.2 Formulation of Chan and Chui (2000)

Chan and Chui (2000) develop the hybrid element by coupling the connection element to the plane frame element, writing the equations as a function of only the degrees of freedom of translation and rotation. The formulation of Chan and Chui (2000) simplify the plane steel frame element to a beam element. Only after the couplings and operations, the terms of the stiffness matrix of the beam element are replaced in the stiffness matrix of the plane steel frame. In addition to this consideration, the spring elements have zero lengths. Most of the variables used in this formulation were defined in the formulation of Chen and Lui (1991), and only the new variables are defined here.

The formulation considers the moment-rotation equilibrium relationship in the inner sections of the plane frame element, and uses the coupling of two spring elements, resulting in:

$$\begin{Bmatrix} \Delta M_{ci} \\ \Delta M_{bi} \\ \Delta M_{bj} \\ \Delta M_{cj} \end{Bmatrix} = \begin{bmatrix} S_{ci} & -S_{ci} & 0 & 0 \\ -S_{ci} & S_{ci} + k_{33} & k_{36} & 0 \\ 0 & k_{63} & S_{cj} + k_{66} & -S_{cj} \\ 0 & 0 & -S_{cj} & S_{cj} \end{bmatrix} \begin{Bmatrix} \Delta\theta_{ci} \\ \Delta\theta_{bi} \\ \Delta\theta_{bj} \\ \Delta\theta_{cj} \end{Bmatrix} \tag{28}$$

Considering that there are no moments internally applied to the hybrid element, that is, ΔM_{bi} and ΔM_{bj} are null, we can rewrite the Eq. 28:

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$$\begin{Bmatrix} \Delta M_{ci} \\ \Delta M_{cj} \end{Bmatrix} = \left(\begin{bmatrix} S_{ci} & 0 \\ 0 & S_{cj} \end{bmatrix} - \frac{1}{\beta} \begin{bmatrix} S_{ci} & 0 \\ 0 & S_{cj} \end{bmatrix} \begin{bmatrix} S_{cj} + k_{66} & -k_{36} \\ -k_{63} & S_{ci} + k_{33} \end{bmatrix} \begin{bmatrix} S_{ci} & 0 \\ 0 & S_{cj} \end{bmatrix} \right) \begin{Bmatrix} \Delta \theta_{ci} \\ \Delta \theta_{cj} \end{Bmatrix} \tag{29}$$

where $\beta = (S_{ci} + k_{33})(S_{cj} + k_{66}) - k_{63} k_{36}$.

Chan and Chui (2000) write the equilibrium relations between the nodal shear forces and the nodal bending moments of the hybrid element with the bending moments of the spring element in the matrix form:

$$\begin{Bmatrix} \Delta Q_i \\ \Delta M_i \\ \Delta Q_j \\ \Delta M_j \end{Bmatrix} = \begin{bmatrix} 1/L & 1/L \\ 1 & 0 \\ -1/L & -1/L \\ 0 & 1 \end{bmatrix} \begin{Bmatrix} \Delta M_{ci} \\ \Delta M_{cj} \end{Bmatrix} \tag{30}$$

where ΔQ_i and ΔQ_j are the shear forces applied at the nodes of the hybrid element; ΔM_i and ΔM_j are the bending moments applied at the nodes of the hybrid element; and L is the length of the hybrid element.

The hybrid element of Chan and Chui (2000) is presented in Fig. 9. It can be observed some relationships between the rotations of the spring element with the rotations and displacements (degrees of freedom) of the hybrid element:

$$\begin{Bmatrix} \Delta \theta_{ci} \\ \Delta \theta_{cj} \end{Bmatrix} = \begin{bmatrix} 1/L & 1 & -1/L & 0 \\ 1/L & 0 & -1/L & 1 \end{bmatrix} \begin{Bmatrix} \Delta v_i \\ \Delta \theta_i \\ \Delta v_j \\ \Delta \theta_j \end{Bmatrix} \tag{31}$$

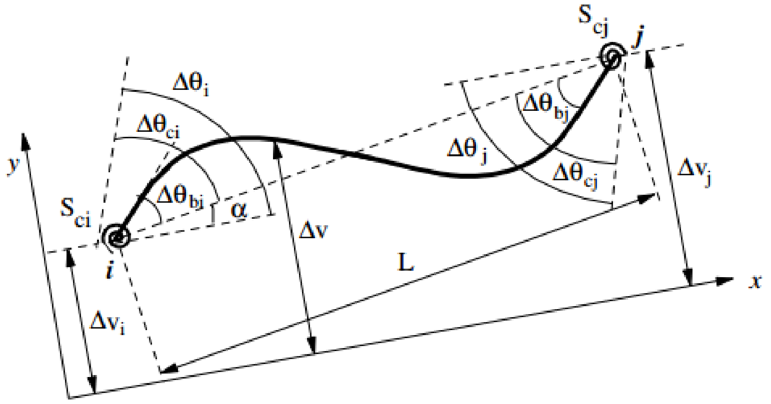

Figure 9. Hybrid finite element for Chan and Chui (2000) formulation (see Silva, 2009).

Substituting Eq. 31 into Eq. 29, and using the result obtained to replace the variables in Eq. 30, and realizing the matrix operations involved, we obtain the equation of the hybrid beam element with the formulation of Chan and Chui (2000), given by:

$$\begin{Bmatrix} \Delta Q_i \\ \Delta M_i \\ \Delta Q_j \\ \Delta M_j \end{Bmatrix} = \begin{bmatrix} k_{22}^* & k_{23}^* & k_{25}^* & k_{26}^* \\ k_{32}^* & k_{33}^* & k_{35}^* & k_{36}^* \\ k_{52}^* & k_{53}^* & k_{55}^* & k_{56}^* \\ k_{62}^* & k_{63}^* & k_{65}^* & k_{66}^* \end{bmatrix} \begin{Bmatrix} \Delta v_i \\ \Delta \theta_i \\ \Delta v_j \\ \Delta \theta_j \end{Bmatrix} \tag{32}$$

where the elements k_{mn}^* consider the effect of the semi-rigid connections. All of them are function of the parameters S_{ci} and S_{cj}.

Finally, the hybrid element for the plane steel frame is obtained by substituting the modified terms k_{mn}^* in the respective lines and columns of the FEM or GFEM stiffness matrix. Due to this type of coupling, the formulation of Chan and Chui (2000) does not change values of forces or axial displacements.

5 NUMERICAL APPLICATION

The methodology presented with the Generalized Finite Element Method (GFEM) is used to obtain the natural frequencies and modes of vibration of an L-shape plane frame, in comparison to the Finite Element Method (FEM). Any effect caused by structural viscous damping has been disregarded, and the material is considered elastic. The analyzes are carried out considering a linear behavior of the beam-column semi-rigid connections, and a comparison of the results between the FEM and the GFEM is performed, with consideration of the rigid and semi-rigid connections.

The structure is an L-shape plane steel frame, studied by Kawashima and Fujimoto (1984), Chan and Chui (2000) and Silva (2009), and is shown in Fig. 10. The plane steel frame has the ends clamped, being constituted by a beam and a column with 150cm and 100cm in length, respectively. The connection between these two structural elements is considered rigid and semi-rigid. For the semi-rigid connection, an initial stiffness S_c was assumed equal to 137.3 N.m/rad, equivalent to a header plate connection.

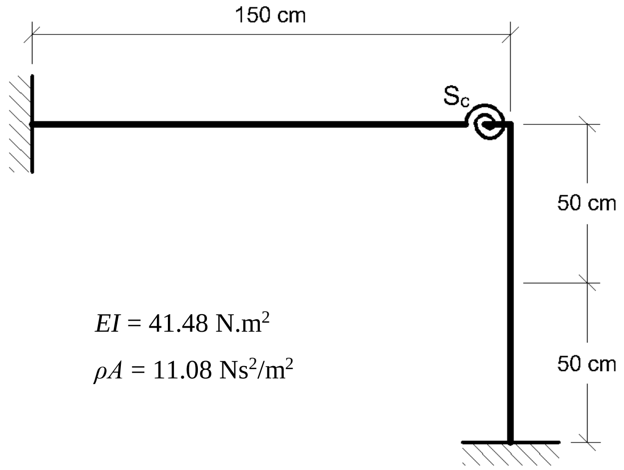

Figure 10. L-shape plane steel frame: geometry and properties.

The analyzes performed with FEM and GFEM took into account the variation of number of finite elements, the level of enrichment of the enrichment functions, and the type of hybrid element used. For the case of rigid connections, conventional finite elements were used. This strategy will be designated as FEM-0 (without enrichment) or GFEM-0 (with trigonometric enrichment). The hybrid element of Chen and Lui (1991) was referred to as FEM-1 (without enrichment) or GFEM-1 (with trigonometric enrichment), and the hybrid element of Chan and Chui (2000) was referred to as FEM-2 (without enrichment) or GFEM-2 (with trigonometric enrichment). When the GFEM is used, the level of enrichment was modified between 1 and 4 to verify how the GFEM becomes more accurate than the FEM.

A study of the natural frequencies of the structure is accomplished. Kawashima and Fujimoto (1984) measured experimentally the natural frequency considering the behavior of connection as rigid, and after they evaluated the natural frequency with semi-rigid connections by numerical method. Chan and Chui (2000) and Silva (2009) used this structure to test their numerical formulations, using 10 finite elements by structural element. These results are presented in Table 1 for the first two modes of vibration, and the values will be used to compare the results of the code developed in the Maple program (Maplesoft, 2015).

Table 1. The two lower natural frequencies, ω_1 and ω_2, in rad/s.

Connection type	**Kawashima and Fujimoto (1984)**		**Chan and Chui (2000)**	**Silva (2009)**
	Numerical	**Experimental**		
Rigid	ω_1 = 16,30	ω_1 = 15,50		ω_1 = 15,81
	ω_2 = 35,90	ω_2 = 30,80		ω_2 = 34,74
Semi-rigid	ω_1 = 14,90		ω_1 = 15,22	ω_1 = 14,90
	ω_2 = 33,00		ω_2 = 33,52	ω_2 = 32,77

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In Table 2, the results of analysis performed with the Finite Element Method and the Generalized Finite Element Method are shown. The number of degrees of freedom (D.O.F.'s) is counted before the imposition of the essential boundary conditions.

Table 2. Analysis performed in code developed in Maple 2015.

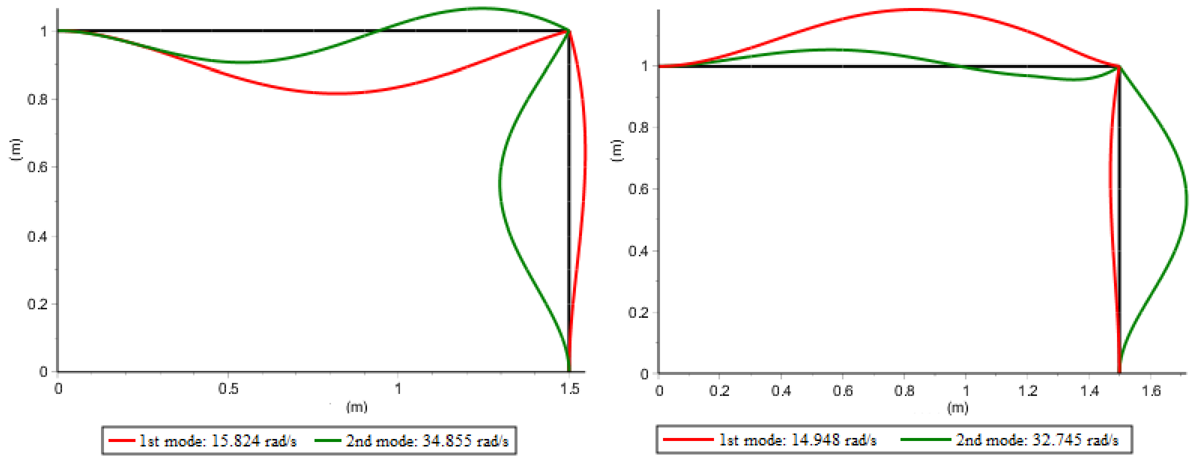
Type of element (conventional or hybrid)	**Number of elements / DOF's**	**Level of enrichment in GFEM**	**Connection type**	ω_1 **[rad/s]**	ω_2 **[rad/s]**	**Total time for analysis [seg]**
1) FEM-0	20 elements / 63 d.o.f.	0	Rigid	15,82416	34,85547	48,92
2) GFEM-0	2 elements / 21 d.o.f.	1	Rigid	15,82384	34,85496	29,359
3) GFEM-0	2 elements / 57 d.o.f.	4	Rigid	15,82380	34,85401	97,356
4) GFEM-0	10 elements / 93 d.o.f.	1	Rigid	15,82380	34,85401	106,88
5) FEM-1	20 elements / 63 d.o.f.	0	Semi-rigid	14,91516	32,85410	55,223
6) GFEM-1	2 elements / 21 d.o.f.	1	Semi-rigid	15,22918	34,04016	33,217
7) GFEM-1	2 elements / 57 d.o.f.	4	Semi-rigid	15,22914	34,03916	105,087
8) GFEM-1	10 elements / 93 d.o.f.	1	Semi-rigid	14,94806	32,74549	108,486
9) FEM-2	20 elements / 63 d.o.f.	0	Semi-rigid	14,91516	32,85410	61,231
10) GFEM-2	2 elements / 21 d.o.f.	1	Semi-rigid	15,22918	34,04016	35,49
11) GFEM-2	2 elements / 57 d.o.f.	4	Semi-rigid	15,22914	34,03916	120,791
12) GFEM-2	10 elements / 93 d.o.f.	1	Semi-rigid	14,94806	32,74549	121,925

When comparing the results from the conventional FEM (Table 2) with the results found in the literature (Table 1), it is noted that the natural frequency values are consistent.

Regarding the rigid connection, the first four analysis reached to the same natural frequencies. However, it can be highlighted the second analysis (GFEM-0, 2 elements, 21 d.o.f.) which took less computational time for the modal analysis than the FEM conventional, reaching results with more precision with only 2 elements and 21 degrees of freedom. At the same time in this example, the GFEM generally took more computational time to perform all calculations, however the high frequencies can be better evaluated by the GFEM analysis than the FEM analysis.

It can be also analyzed that the different formulations for semi-rigid connections (Chen and Lui, 1991; Chan and Chui, 2000) do not influence the natural frequencies for this example (FEM-1, FEM-2, GFEM-1 and GFEM-2). The major influence is caused by the *h* refinement, reaching values close to those obtained by the FEM conventional analysis. About the influence of the level of enrichment in GFEM for the enrichment functions, the 3 different analysis (2, 3, 6, 7, 10 and 11) shows that increasing the level of enrichment of the enrichment functions did not significantly modify the result of the natural frequencies.

The first two modes of vibration of this structure for the rigid connection and for the semi-rigid connection are shown in Fig. 11. Fig. 11a is from the analysis nº. 1 (FEM-0), and Fig. 11b is from the analysis nº. 12 (GFEM-2). There is not a notable variation of results in the modes of vibration between the analysis for rigid connections and for the semi-rigid connections.



(a) Rigid connection (analysis nº. 1 – FEM-0) (b) Semi-rigid connection (analysis nº. 12 – GFEM-2)

Figure 11. The first two modes of vibration for the L-shape plane steel frame.

6 CONCLUSIONS

The present work presented a comparative numerical study of the influence of different semi-rigid hybrid elements in the Generalized Finite Element Method. It was presented a brief literature about the structural free vibration problem, the Generalized Finite Element Method, and the semi-rigid connections. In this paper, it is used the formulations of hybrid elements from Chen and Lui (1991) and Chan and Chui (2000).

It was analyzed the changes in the values of the natural frequencies and the natural modes of vibration for an L-shape plane steel frame. The results obtained leads to the conclusion that hybrid element is very useful in GFEM when the *h* refinement is also used. For this example, the modification of level of enrichment in the GFEM enrichment functions did not influence significantly the result of the first natural frequencies.

It is intended to perform further analysis with the computational code developed in Maple 2015, analyzing other structures found in the literature regarding the effect of the semi-rigid connections, in order to verify the consistency of the code. Other point to be analyzed is about the efficiency of hybrid elements with GFEM. In this work, the use of hybrid elements proved better with the *h* refinement, which is a resource that can normally be disregarded with the use of GFEM.

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REFERENCES

Arndt, M., 2009. *O método dos elementos finitos generalizados aplicado à análise de vibrações livres de estruturas reticuladas*. Doctoral thesis (Doutorado em Engenharia Civil) – Programa de Pós-Graduação em Métodos Numéricos em Engenharia, Universidade Federal do Paraná, Curitiba.

Arndt, M., Machado, R. D., Scremin, A., 2010. An adaptive generalized finite element method applied to free vibration analysis of straight bars and trusses. *Journal of Sound and Vibration 329*, pp. 659-672.

Assan, A. E., 2003. *Método dos Elementos Finitos*: primeiros passos. 2. ed. Campinas: Editora da Unicamp.

Chan, S. L., Chui, P. P. T., 2000. Non-linear Static and Cyclic Analysis of Steel Frames with Semi-Rigid Connections. Oxford: Elsevier.

Chen, W. F., Goto, Y., Liew, J. Y. R., 1996. *Stability design of semi-rigid frames*. 1. ed. New Jersey: John Wiley & Sons, Inc.

Chen, W. F., Lui, E. M., 1991. *Stability design of steel frames*. 1. ed. Boca Raton, Florida: CRC Press.

Chen, W. F., Kishi, N., Komuro, M., 2011. *Semi-rigid connections handbook*. 1. ed. Fort Lauderdale: J. Ross Publishing Inc.

Chopra, A. K., 2011. *Dynamics of structures*: theory and applications to earthquake engineering. 4. ed. New Jersey: Prentice Hall.

Da Silva, J. G. S., De Lima, L. R. O., Vellasco, P. C. G. S., Andrade, S. A. L., Castro, R. A., 2008. Nonlinear dynamic analysis of steel portal frames with semi-rigid connections. *Engineering Structures 30*, pp. 2566-2579.

Duarte, C. A., Babuska, I., Oden, J. T., 2000. Generalized finite element methods for three-dimensional structural mechanics problems. *Computers and Structures 77*, pp. 215-232.

Kawashima, S., Fujimoto, T., 1984. Vibration analysis of frames with semi-rigid connections. *Engineering Structures 19*, n. 1-2, pp. 85-92.

Maplesoft. *Maple 2015.0*. Software para programação e cálculos matemáticos.

Melenk, J. M., Babuska, I., 1996. The partition of unity finite element method: basic theory and applications. *Computer Methods in Applied Mechanics and Engineering 139*, pp. 289-314.

Nguyen, P., Kim, S., 2013. Nonlinear elastic dynamic analysis of space steel frames with semi-rigid connections. *Journal of Constructional Steel Research 84*, pp. 72-81.

Shang, H. Y.; Machado, R. D.; Abdalla Filho, J. E., 2016 Dynamic analysis of Euler-Bernoulli beam problems using the Generalized Finite Element Method. *Computers and Structures 173*, pp. 109-122.

Silva, A. R. D., 2009. *Sistema Computacional para Análise Avançada Estática e Dinâmica de Estruturas Metálicas*. Doctoral thesis (Doutorado em Engenharia Civil) – Programa de Pós-Graduação do Departamento de Engenharia Civil da Escola de Minas, Universidade Federal de Ouro Preto, Ouro Preto.

Silva, A. R. D., Silveira, R. A. M., Neves, F. A., 2010. Efeito das ligações semirrígidas na resposta dinâmica de estruturas de aço. *Mecánica Computacional Vol XXIX*, pp. 7583-7607.

Soriano, H. L., 2014. *Introdução à dinâmica das estruturas*. 1. ed. Rio de Janeiro: Elsevier.

Strouboulis, T.; Babuska, I.; Copps, K., 2000. The design and analysis of the Generalized Finite Element Method. *Computer Methods in Applied Mechanics and Engineering 181*, pp. 43-69.

Torii, A. J., 2012. *Análise dinâmica de estruturas com o método dos elementos finitos generalizado*. Doctoral thesis (Doutorado em Métodos Numéricos em Engenharia) – Programa de Pós-Graduação em Métodos Numéricos em Engenharia, Universidade Federal do Paraná, Curitiba.

Weinhardt, P. O., 2016. *Estudo da estabilidade do método dos elementos finitos generalizados aplicado à análise dinâmica*. Master dissertation (Mestrado em Métodos Numéricos em Engenharia) – Programa de Pós-Graduação em Métodos Numéricos em Engenharia, Universidade Federal do Paraná, Curitiba.