ENRICHED METHODS ALLIED TO MODIFIED LOCAL GREEN'S FUNCTION METHOD FOR ELASTO STATIC PROBLEMS

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Abstract. *The Modified Local Green's Function Method (MLGFM) has been extensively employed during last decades to solve a large number of solid mechanics problems. The MLGFM is an integral method that uses finite element approaches in the domain, to automatically generate fundamental solutions to be applied in a boundary system. It uses conventional Lagrange shape functions to approximate the solution space. In this paper, the solution space is now enriched using, in this first attempt, a hierarchical approach with Lobatto shape functions. Once MLGFM uses domain and boundary meshes, the solution space can be enriched only in the domain, on the boundary or in both meshes at the same time. Some 2D elasto static problems are solved by this new approach showing excellent accuracy and convergence.*

Keywords: *MLGFM, Enriched methods, Hierarchical finite element, Lobatto shape functions.*

1. INTRODUCTION

The Modified Local Green's Function Method (MLGFM) was proposed by de Barcellos & Silva (1987) as an alternative to extend the Boundary Element Method (BEM) methodologies to problems which have no known fundamental solution in its explicit form or are very intricate. One of the most important features of the MLGFM is the advantage of taking the Green's function adjoint operator properties without the knowledge of its explicit form, but by evaluating its projections on appropriate subspaces spanned by domain and boundary interpolation functions. This allows the solution of problems where the fundamental solution is unknown. These projections can be calculated with appropriate numerical techniques. The Finite Element Method (FEM) is an interesting technique to obtain the discrete values for the Green's Function Projections (GFp) because they do not use fundamental solutions and/or Green's Functions. However, it is necessary to use an additional operator, $\mathcal{N}^{\#}$ prescribed on the boundary by the user in order to avoid the singularity of the final system of equations.

The MLGFM, in a global approach also known as unicell-multielement, has been used by several authors in the last decades and its implementation has been extensively studied. Some examples of this approach are: for membranes, Barcellos and Silva (1987); for Mindlin´s plate, Barbieri and Barcellos (1991, 1993); for singular potential, Barcellos and Barbieri (1991); for *h* and *p* convergence, Filippin et al. (1992); for semi-thick shell, Barbieri et al. (1993), for non-homogenous potentials problems, Barbieri and Barcellos (1993); for laminated plates with and without cracks, Machado et al. (1993, 2008, 2012); for 3D elasticity, Barcellos et al. (1995) and for mathematical formulation details, Barbieri et al. (1998,b).

In parallel, there have been efforts to improve the efficiency and convergence of the conventional finite element method – FEM (Oden and Reddy, 1976; Bathe, 1996) – during several decades. One of these efforts are the enriched finite element formulations and as an example, the development of the Hierarchical Finite Element Method – HFEM (Zienkiewicz, 1983).

The HFEM employs hierarchical polynomials as enrichment functions differently the conventional Lagrange polynomials in FEM. In other words, the set of functions of a lower degree approximation "*p*", constitutes a subset of the set of functions of the degree approximation "*p+1*". Further developments in this field originated the Generalized Finite Element Method and its improvements (Strouboulis et al., 2000; Babuska & Banerjee., 2012; Zhang et al., 2014). This method as well as the HFEM is based on the Partition of Unity Method, PUM (Melenk & Babuska, 1996) where the partition of unity is provided by the Lagrange FEM shape functions. Actually, besides those quoted methods, other ones based on PUM where developed and it's worth to mention the Extended Finite Element Method (Fries & Belytschko, 2000) and the hp Clouds Method (Duarte & Oden, 1995).

In this paper, the MLFGM approximation space is enriched using hierarchical Lobatto shape functions as described by Solín (Solín et al., 2004). In order to test the accuracy and convergence of the proposed method, some 2D elasto static problems are presented and compared with the MLGFM, the classic FEM and the analytical solution (MacNeal and Harder, 1985).

2. THE MODIFIED LOCAL GREEN'S FUNCTION METHOD

The modified local Green's function method was developed by Barcellos and Silva (1987) to solve continuum mechanics problems. Many researches have worked with this method, which has shown good results in elasticity, composite laminated plates, nonlinear fields, and thermal problems as earlier mentioned in this text.

The MLGFM is used to solve boundary value problems, supposing that Ω is any domain limited by boundaries $\Gamma_{\mathscr{D}}$ and $\Gamma_{\mathscr{N}}$ which are related to Dirichlet and Neumann boundary conditions respectively, such as $\Gamma_{\mathscr{D}} \cup \Gamma_{\mathscr{N}} = \partial\,\Omega$. The MLGFM solves any problem by two steps, the first one is the real problem, which is governed by the differential operators $\mathscr{L}$, $\mathscr{D}$, $\mathscr{N}$, and the second one is an auxiliary or adjoint problem, which is governed by the differential adjoint operators $\mathscr{L}^*$ and $\mathscr{N}^*$. The two steps are drafted in the "Table 1", where $\mathbf{u}(P)$ is the unknown variables, $\mathbf{a}(P)$ is the independent terms, $\mathbf{G}(P,Q)$ is the Green's function matrix, $\boldsymbol{\delta}(P,Q)$ is the Dirac delta generalized function, $\mathbf{I}$ is the identity operator, $\mathscr{N}^{\#}$ is an additional operator, P and Q are the domain source point and domain field point respectively, and p and q are the correspondent points on the boundary.

Table 1. The real and the auxiliary problems

Main (real) problem			Auxiliary (adjoint) problem		
Find u(P) such that			Find $\mathbf{G}(P,Q)$ such that		
$\mathscr{L}\mathbf{u}(P) = \mathbf{a}(P)$;	$P \in \Omega$	(a)	$\mathscr{L}^*\,\mathbf{G}(P,Q) = \boldsymbol{\delta}(P,Q)\mathbf{I}$;	$P, Q \in \Omega$	(b)
Subjected to			Subjected to		(21)
$\mathscr{D}\mathbf{u}(p) = \mathbf{b}(p)$;	$p \in \Gamma_{\mathscr{D}}$	(c)	$(\mathscr{N}^* + \mathscr{N}^{\#})\,\mathbf{G}(p,Q) = 0$		(d)
$\mathscr{N}\mathbf{u}(p) = \mathbf{c}(p)$;	$p \in \Gamma_{\mathscr{N}}$				

A good choice for the additional operator is such that it satisfies $\mathscr{N}^{\#}u(p) = 0$ on $\Gamma_{\mathscr{N}}$. For the auxiliary adjoint problem, the Green's tensor $\mathbf{G}(P,Q)$ must be understood as the generalized displacement of point P in the direction of an unitary vector $\mathbf{n_P}$ when a generalized force is applied on point Q, in the direction of an unitary vector $\mathbf{n_Q}$.

Some algebraic manipulations are necessary to determine the main expressions of the MLGFM. The reader is kindly referred to go through our reference (Barbieri et al, 1998a, b) for further details on the discretization procedure. It must be emphasized that the direct treatment of the Newmann operator may cause numerical troubles. This drawback can be avoided by introducing a vector $\mathbf{f}(p)$, associated to the boundary fluxes, and given by:

$$\mathbf{f}(p) = (\mathscr{N}^* + \mathscr{N}^{\#})\mathbf{u}(p) \tag{1}$$

In such a way, one is possible to write:

$$\mathbf{u}(Q) = \int_{\Omega} [\mathbf{G}^{\mathbf{T}}(P,Q)\mathbf{a}(P)]\mathrm{d}\Omega + \int_{\Gamma} [\mathbf{G}^{\mathbf{T}}(p,Q)\mathbf{f}(p)]\mathrm{d}\Gamma\,; \quad P,\, Q \in \Omega; \quad p \in \Gamma; \tag{2}$$

To extend the integral equation into the boundary, the trace operator (Oden and Reddy, 1976) is applied:

$$\mathbf{u}(q)=\int_{\Omega}[\mathbf{G}^{\mathbf{T}}(P,q)\mathbf{a}(P)]\mathrm{d}\Omega+\int_{\Gamma}[\mathbf{G}^{\mathbf{T}}(p,q)\mathbf{f}(p)]\mathrm{d}\Gamma\;;\qquad P\in\Omega;\qquad p,\,q\in\Gamma; \tag{3}$$

The “Equation (2)” and “Equation (3)” describe completely the problem. Since these equations involve domain and boundary integrals, two types of meshes are necessary, one in the domain and the other on the boundary, using FEM and BEM methods, respectively. The FEM domain approximation is also used to develop the Green’s functions which are associated to the matrices $\mathbf{G}(P,Q)$, $\mathbf{G}(p,Q)$, $\mathbf{G}(P,q)$ and $\mathbf{G}(p,q)$.

The basis shape functions are the same as in the conventional FEM and BEM methods. For the present work, the bilinear quadrilateral element is enriched by new shape functions in the domain and the linear element is enriched by a new shape function on the boundary.

By developing discrete equations from the nodal values, one obtains:

$$\mathbf{Au}_{\Omega}=\mathbf{Bf}+\mathbf{Ca}\qquad\text{(in the domain)} \tag{4}$$

$$\mathbf{Du}_{\Gamma}=\mathbf{Ef}+\mathbf{Fa}\qquad\text{(on the boundary)} \tag{5}$$

, where $\mathbf{u}_{\Omega}$ and $\mathbf{u}_{\Gamma}$ are the domain and the boundary displacements, respectively, **a** and **f** are the independent and the fluxes variables vectors. The matrices **A**, **B**, **C**, **D**, **E** and **F** can be written as:

$$\mathbf{A}=\int_{\Omega}\boldsymbol{\psi}(Q)^{\mathrm{T}}\boldsymbol{\psi}(Q)\mathrm{d}\Omega\;; \tag{6}$$

$$\mathbf{B}=\int_{\Omega}\boldsymbol{\psi}(Q)^{\mathrm{T}}\mathbf{G}_{\Gamma}(Q)\mathrm{d}\Omega\;; \tag{7}$$

$$\mathbf{C}=\int_{\Omega}\boldsymbol{\psi}(Q)^{\mathrm{T}}\mathbf{G}_{\Omega}(Q)\mathrm{d}\Omega\;; \tag{8}$$

$$\mathbf{D}=\int_{\Gamma}\boldsymbol{\phi}(q)^{\mathrm{T}}\boldsymbol{\phi}(q)\mathrm{d}\Gamma\;; \tag{9}$$

$$\mathbf{E}=\int_{\Gamma}\boldsymbol{\phi}(q)^{\mathrm{T}}\mathbf{G}_{\Gamma}(q)\mathrm{d}\Gamma\;; \tag{10}$$

$$\mathbf{F}=\int_{\Gamma}\boldsymbol{\phi}(q)^{\mathrm{T}}\mathbf{G}_{\Omega}(q)\mathrm{d}\Gamma\;; \tag{11}$$

, where $\boldsymbol{\psi}(Q)$ and $\boldsymbol{\phi}(q)$ are matrices with the shape functions in the domain and on the boundary, respectively, and $\mathbf{G}_{\Gamma}(Q)$, $\mathbf{G}_{\Omega}(Q)$, $\mathbf{G}_{\Gamma}(q)$, $\mathbf{G}_{\Omega}(q)$ are the Green’s function projections over the boundary Γ and the domain Ω, evaluated on the points Q and q. Note that $\boldsymbol{\phi}(q)$ must be the trace of $\boldsymbol{\psi}(Q)$. The Green’s projections can be written as:

$$\mathbf{G}_{\Gamma}(Q) = \int_{\Gamma} \mathbf{G}^{\mathrm{T}}(p,Q)\boldsymbol{\phi}(q)\mathrm{d}\Gamma \; ; \tag{12}$$

$$\mathbf{G}_{\Omega}(Q) = \int_{\Omega} \mathbf{G}^{\mathrm{T}}(P,Q)\boldsymbol{\psi}(P)\mathrm{d}\Omega \; ; \tag{13}$$

$$\mathbf{G}_{\Gamma}(q) = \int_{\Gamma} \mathbf{G}^{\mathrm{T}}(p,q)\boldsymbol{\phi}(q)\mathrm{d}\Gamma \; ; \tag{14}$$

$$\mathbf{G}_{\Omega}(q) = \int_{\Omega} \mathbf{G}^{\mathrm{T}}(P,q)\boldsymbol{\psi}(P)\mathrm{d}\Omega \; ; \tag{15}$$

To determine the Green's functions automatically, one should consider the following functional $\mathscr{F}$, which depends on $\mathbf{G}_{\Omega}$ or $\mathbf{G}_{\Gamma}$:

$$\mathscr{F}(\mathbf{G}_{\Omega},\mathbf{G}_{\Gamma}) = \mathscr{B}(G,G) - \alpha\mathscr{B}_1(\mathbf{G}_{\Omega},\boldsymbol{\psi}) - \beta\mathscr{B}_2(\mathbf{G}_{\Gamma},\boldsymbol{\phi}) + \mathscr{B}_3(\mathbf{G},\mathbf{G}) \; ; \tag{16}$$

, where $\mathbf{G}$ corresponds to $\mathbf{G}_{\Omega}$ or $\mathbf{G}_{\Gamma}$ depends on the case of interest; $\mathscr{B}$ is a bilinear form, developed to $\mathbf{G}_{\Omega}$ or $\mathbf{G}_{\Gamma}$; α and β are constants whose values are: $\alpha = 1$ and $\beta = 0$ to determine $\mathbf{G}_{\Omega}$, $\alpha = 0$ and $\beta = 1$ to determine $\mathbf{G}_{\Gamma}$; and $\mathscr{B}_1$, $\mathscr{B}_2$ and $\mathscr{B}_3$ are bilinear forms which can be written as:

$$\mathscr{B}_1(\mathbf{G}_{\Omega},[\boldsymbol{\psi}]) = \int_{\Omega} \mathbf{G}_{\Omega}(Q)\boldsymbol{\psi}(Q)\mathrm{d}\Omega \; ; \tag{17}$$

$$\mathscr{B}_2(\mathbf{G}_{\Gamma},[\boldsymbol{\phi}]) = \int_{\Gamma} \mathbf{G}_{\Gamma}(q)\boldsymbol{\phi}(q)\mathrm{d}\Gamma \; ; \tag{18}$$

$$\mathscr{B}_3(\mathbf{G},\mathbf{G}) = \frac{1}{2}\int_{\Gamma} \mathscr{N}^{\#}(q)\mathbf{G}(q).\mathbf{G}(q)\mathrm{d}\Gamma \; ; \tag{19}$$

The minimization of functional $\mathscr{F}(\mathbf{G}_{\Omega}, \mathbf{G}_{\Gamma})$ in "Eq. (16)" results a linear equation system which can be solved to determine the Green's projections:

$$[\mathbf{K}][\mathbf{G}_{\Omega}(Q) \;\vdots\; \mathbf{G}_{\Gamma}(Q)] = [\mathbf{A} \;\vdots\; \mathbf{D}]; \tag{20}$$

, where $[\mathbf{K}]$ is the global stiffness matrix, evaluated as the same way of the conventional finite element stiffness matrix; $\mathbf{A}$ and $\mathbf{D}$ are the matrices of "Eq. (6)" and "Eq. (9)", respectively. In such a way, the Green's projections are determined directly from "Eq. (20)", and they can be applied in "Eq. (7)", "Eq. (8)", "Eq. (10)" and "Eq. (11)" to complete the matrices of "Eq. (4)" and "Eq. (5)", which are the main system of the MLGFM.

3. THE POLYNOMIAL HIERARCHICAL FINITE ELEMENT

The conventional FEM uses Lagrange polynomials as local approximation functions. Actually, it's relatively easy to apply the boundary condition and to process the nodal quantities with these polynomials because they respect the FEM δ condition (delta property), e.g. these polynomials assume the unit value at the associated node and zero in all other nodes. However, the Lagrange polynomials of degree "*p*" are all different of degree "*p+1*" making a hard task to improve the approximation only increasing the polynomial degree.

The enriched method employed in this work known by Hierarchical Finite Element Method (HFEM) uses hierarchical functions to enrich the solution space. By hierarchical we mean the following: when increasing the approximation space degree from "*p*" to "*p+1*", the shape functions space of degree "*p*" remains the same. This is clearly not the case of the conventional FEM.

The construction of hierarchical mathematical space by polynomials is described in detail by Solín et al. (2004). In this case, a simple way to construct the hierarchical mathematical space is to use the Lobatto shape functions in the place of Lagrange polynomials.

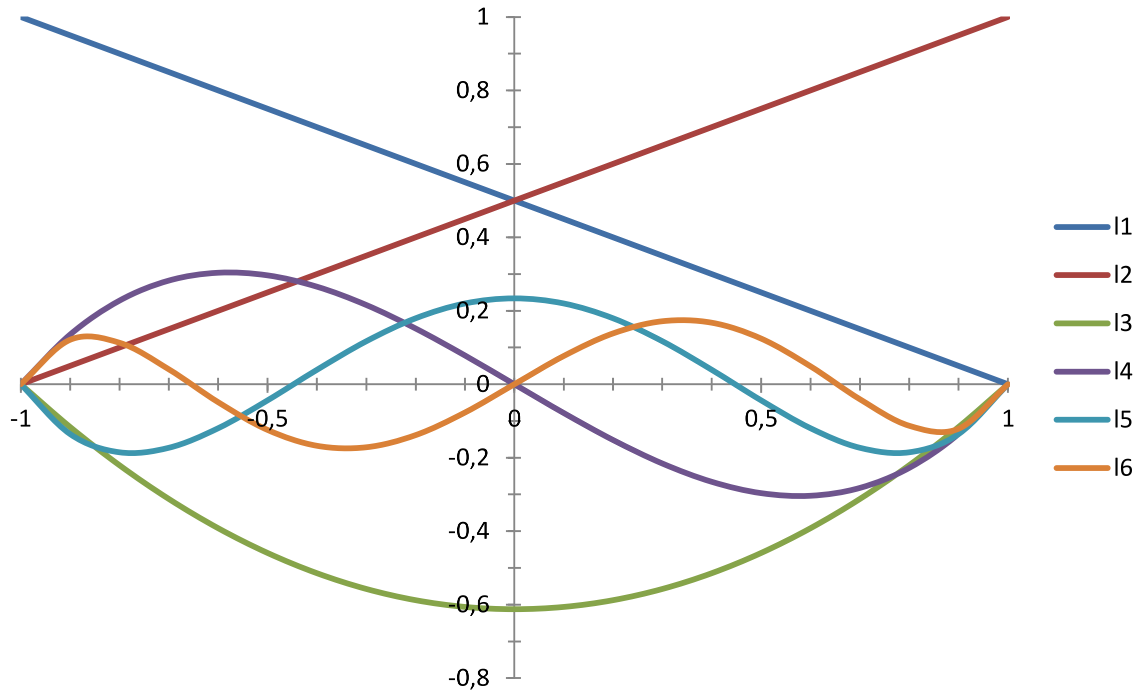

Figure 1. Six first Lobatto shape functions (Solín et al., 2004).

In this work, we have considered 1D 2-node linear element on the boundary and 2D 4-node linear quadrilateral element in the domain. The hierarchical concept is developed using

Lobatto shape functions as mentioned before. For instance, the first six Lobatto shape functions are given by:

$$l_1 = \frac{1-\xi}{2}, \tag{21}$$

$$l_2 = \frac{1+\xi}{2}, \tag{22}$$

$$l_3 = \frac{1}{2}\sqrt{\frac{3}{2}}(\xi^2 - 1), \tag{23}$$

$$l_4 = \frac{1}{2}\sqrt{\frac{5}{2}}(\xi^2 - 1)\xi, \tag{24}$$

$$l_5 = \frac{1}{8}\sqrt{\frac{7}{2}}(\xi^2 - 1)(5\xi^2 - 1), \tag{25}$$

And,

$$l_6 = \frac{1}{8}\sqrt{\frac{9}{2}}(\xi^2 - 1)(7\xi^2 - 3)\xi. \tag{26}$$

, where ξ = [−1, 1] is the finite element natural coordinate, also known as local or parameterized coordinates. These shape functions are showed in “Fig. 1”.

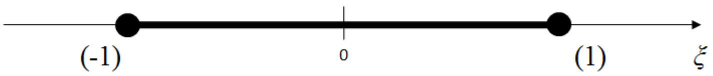

Figure 2. A 1D linear element with local coordinate ξ = [−1, 1].

In the case of the 1D linear element shape functions on the boundary (“Fig. 2”), using the Lobatto shape functions, the resulting boundary shape functions $\phi_b(\xi)$ are:

$$\phi_b(\xi) = l_i(\xi), \ \forall\ i = 1, 2, \ldots, n; \tag{27}$$

, where $\xi = [-1, 1]$ is the natural coordinate of a 1D finite element and n is the number of Lobatto shape functions to be used.

Taking $n = 2$, for example, results in the following shape functions:

$$\phi_1(\xi) = l_1(\xi) = \frac{1-\xi}{2}\,; \tag{28}$$

$$\phi_2(\xi) = l_2(\xi) = \frac{1+\xi}{2}\,; \tag{29}$$

In the case of the 2D 4-node linear quadrilateral element shape functions in the domain ("Fig. 3"), we can multiply the Lobatto shape functions defined above for the one dimensional case. The resulting domain shape functions $\psi_d(\xi,\eta)$ are:

$$\psi_d(\xi,\eta) = l_i(\xi) l_j(\eta)\,,\ \forall\ i, j = 1, 2, \ldots, n; \tag{30}$$

, where $\xi = [-1, 1]$ and $\eta = [-1, 1]$ are the natural coordinates of a square master finite element and n is the number of Lobatto shape functions to be used.

Taking $n = 2$, for example, results in the following shape functions:

$$\psi_1(\xi,\eta) = l_1(\xi) l_1(\eta) = \frac{1}{4}(1-\xi)(1-\eta)\,; \tag{31}$$

$$\psi_2(\xi,\eta) = l_1(\xi) l_2(\eta) = \frac{1}{4}(1+\xi)(1-\eta)\,; \tag{32}$$

$$\psi_3(\xi,\eta) = l_2(\xi) l_1(\eta) = \frac{1}{4}(1+\xi)(1+\eta)\,; \tag{33}$$

And,

$$\psi_4(\xi,\eta) = l_2(\xi) l_2(\eta) = \frac{1}{4}(1-\xi)(1+\eta)\,. \tag{34}$$

Note that, for the above examples, the resulting shape functions are the ones commonly used in the 1D linear element and 2D bilinear quadrilateral element. The same procedure can be used to obtain the shape functions for an arbitrary value of n. Note that the degree of the approximation will be $p = n - 1$. For cases where $n \geq 2$, the above functions from "Eq. (31) to (34)" are called as "*vertex*" shape functions. The other shape functions that can be used to enrich the approximation are called "*edge*" shape functions – related to the element boundary – and "*bubble*" shape functions – related to the element field (interior).

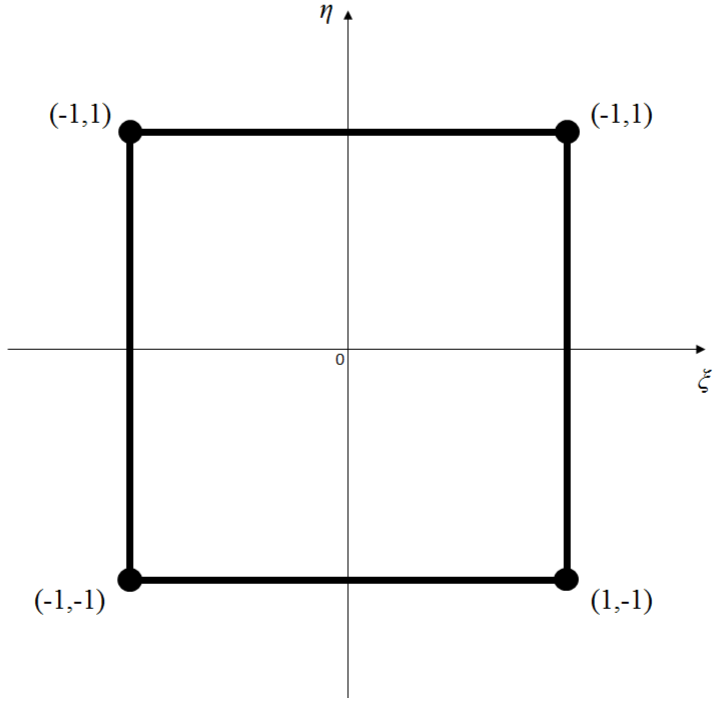

Figure 3. A 2D quadrilateral finite element with local coordinates ξ = [−1, 1] and η = [−1; 1].

4. RESULTS AND DISCUSSION

This section presents the first applications of enriched MLGFM using a 4-node bilinear quadrilateral element in the domain and a 2-node linear element on the boundary in plane stress problems. A hierarchical enrichment methodology was adopted using Lobatto shape functions (Solín et al., 2004). For this present work, we have used three different approaches: the domain and boundary enrichment, only the domain enrichment and only the boundary enrichment (3 ways to enrich the MLGFM). The results from this enriched approach are compared to the MLGFM using a 4-node bilinear quadrilateral element in the domain and a 2-node linear element on the boundary, a 9-node biquadratic quadrilateral element in the domain and a 3-node quadratic element on the boundary, the conventional FEM and the analytical solution (MacNeal and Harder, 1985).

The tensile and bending tests for beams proposed by MacNeal (MacNeal and Harder, 1985) are taken as reference in this work. The main goal of these problems is to verify the efficiency of the 2D elements relation to the distortion and aspect ratio. The geometry, material properties, boundary conditions, loading, and element meshing for each problem are described in the next sessions in order to construct a finite element model of a straight and a curved beam, respectively. Theoretical results for the problems are also given by MacNeal (MacNeal and Harder, 1985).

Discussions for each application studied in this work are better detailed in the next sessions.

4.1 Straight beam application

For this first application, we use the same straight cantilever beam problems suggested by McNeal (MacNeal and Harder, 1985). Their geometries and properties are showed in "Fig. 4".

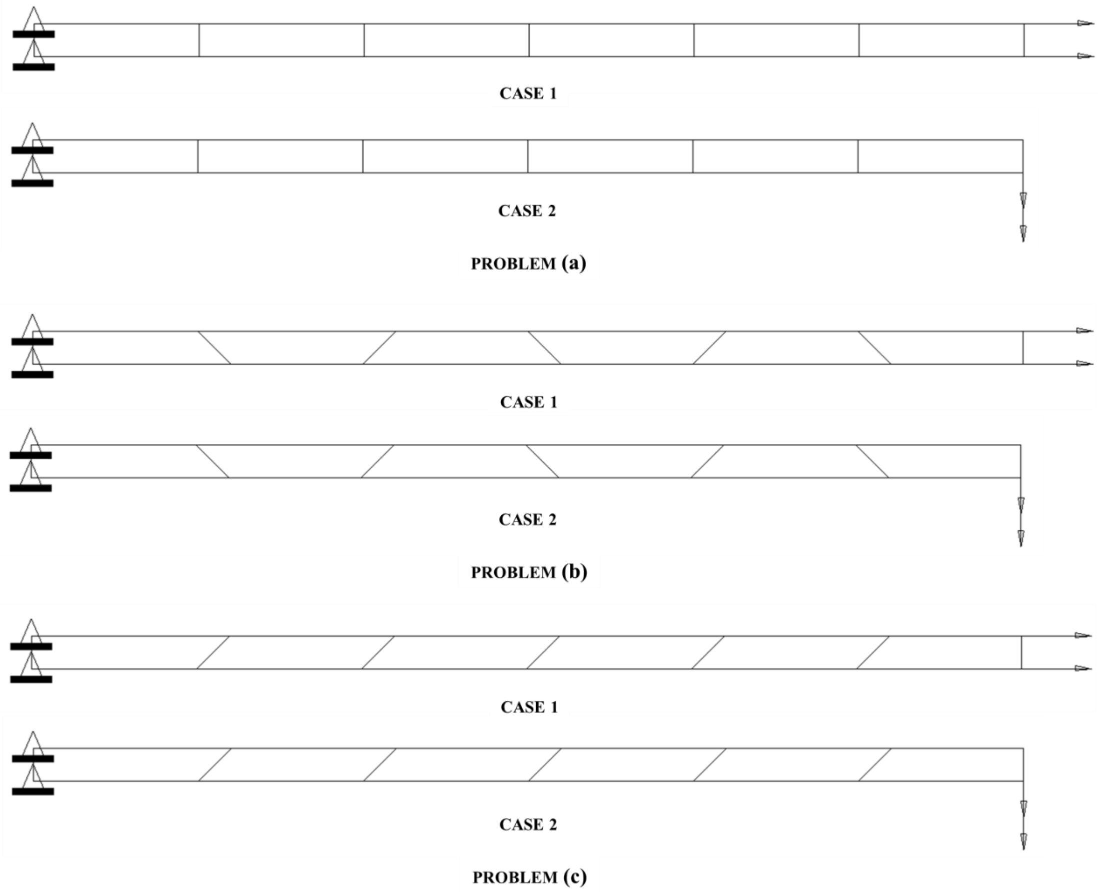

Figure 4. Straight cantilever beam: problem (a) - regular shape elements; problem (b) - trapezoidal shape elements; problem (c) - parallelogram shape elements. Length = 6.0; width (height) = 0.2; depth (thickness): 0.1; E = 1.0x107; ν = 0.30; mesh = 6 x 1. Loading: unit forces at free end (tip). Constraint: fixed support.

As we can see in the "Table 2" (displacements results) and "Table 3" (normalized displacement results), for straight beam at tensile load (extension – case 1), the results presented good approximation compared to the theoretical solution even with irregular elements shape. Anyway, it's important to highlight that there is no variation among the problems A, B and C for all approaches (domain/boundary, only domain and only boundary).

Table 2. Displacements results and theoretical solution for straight cantilever beam

STRAIGHT BEAM								
PROBLEM (a): REGULAR ELEMENT								
CASE TYPE	**TIP LOAD**	**THEORETICAL***	**CLASSIC FEM**	**MLGFM**		**ENRICHED MLGFM**		
				4-NODE	**9-NODE**	**DOMAIN/ BOUNDARY**	**DOMAIN**	**BOUNDARY**
CASE 1	**EXTENSION**	**3,0000E-05**	2,9863E-05	2,9863E-05	2,9943E-05	2,9925E-05	3,0614E-05	2,9863E-05
CASE 2	**IN-PLANE SHEAR**	**-1,0810E-01**	-1,0088E-02	-1,0088E-02	-1,0703E-01	-1,0755E-01	-1,0755E-01	-1,0088E-02
PROBLEM (b): TRAPEZOIDAL ELEMENT								
CASE TYPE	**TIP LOAD**	**THEORETICAL***	**CLASSIC FEM**	**MLGFM**		**ENRICHED MLGFM**		
				4-NODE	**9-NODE**	**DOMAIN/ BOUNDARY**	**DOMAIN**	**BOUNDARY**
CASE 1	**EXTENSION**	**3,0000E-05**	2,9927E-05	2,9927E-05	3,0338E-05	2,9917E-05	3,0638E-05	2,9839E-05
CASE 2	**IN-PLANE SHEAR**	**-1,0810E-01**	-2,6877E-03	-2,6860E-03	-7,1101E-02	-1,0747E-01	-1,0746E-01	-9,4011E-03
PROBLEM (c): PARALELLOGRAM ELEMENT								
CASE TYPE	**TIP LOAD**	**THEORETICAL***	**CLASSIC FEM**	**MLGFM**		**ENRICHED MLGFM**		
				4-NODE	**9-NODE**	**DOMAIN/ BOUNDARY**	**DOMAIN**	**BOUNDARY**
CASE 1	**EXTENSION**	**3,0000E-05**	2,9919E-05	2,9918E-05	3,0313E-05	2,9925E-05	3,0614E-05	2,9863E-05
CASE 2	**IN-PLANE SHEAR**	**-1,0810E-01**	-3,4277E-03	-3,4274E-03	-4,9135E-02	-1,0755E-01	-1,0755E-01	-1,0088E-02

* MacNeal and Harder, 1985.

Table 3. Normalized tip displacements results in direction of load

STRAIGHT BEAM							
PROBLEM (a): REGULAR ELEMENT							
CASE TYPE	**TIP LOAD**	**CLASSIC**	**MLGFM**		**ENRICHED MLGFM**		
			4-NODE	**9-NODE**	**DOMAIN/ BOUNDARY**	**DOMAIN**	**BOUNDARY**
CASE 1	**EXTENSION**	0,99543	0,99543	0,99810	0,99749	1,02048	0,99543
CASE 2	**IN-PLANE SHEAR**	0,09332	0,09332	0,99010	0,99494	0,99488	0,09332
PROBLEM (b): TRAPEZOIDAL ELEMENT							
CASE TYPE	**TIP LOAD**	**CLASSIC**	**MLGFM**		**ENRICHED MLGFM**		
			4-NODE	**9-NODE**	**DOMAIN/ BOUNDARY**	**DOMAIN**	**BOUNDARY**
CASE 1	**EXTENSION**	0,99757	0,99757	1,01127	0,99724	1,02127	0,99465
CASE 2	**IN-PLANE SHEAR**	0,02486	0,02485	0,65773	0,99413	0,99406	0,08697
PROBLEM (c): PARALELLOGRAM ELEMENT							
CASE TYPE	**TIP LOAD**	**CLASSIC**	**MLGFM**		**ENRICHED MLGFM**		
			4-NODE	**9-NODE**	**DOMAIN/ BOUNDARY**	**DOMAIN**	**BOUNDARY**
CASE 1	**EXTENSION**	0,99728	0,99727	1,01043	0,99749	1,02048	0,99543
CASE 2	**IN-PLANE SHEAR**	0,03171	0,03171	0,45453	0,99494	0,99488	0,09332

On the other hand, for bending load (in-plane shear - case 2) problems applied for straight beam a very low variation was observed with domain/boundary enrichment. It's also interesting to notice that only the boundary enrichment approach presents results very close to the 4-node MLGFM. Besides the great disturbance visualized for 4-node MLGFM element if compared to the plate elements tested by MacNeal (MacNeal and Harder, 1985), the results are much better. Anyway, this approach – only the boundary enrichment – reveals the low accuracy for shear problems. When it comes to only the domain enrichment approach, it becomes easily visible it's more effective to enrich the domain on the contrary of the boundary. The Enriched MLGFM reaches the best performance when the domain and boundary enrichment occur at the same time.

4.2 Curved beam application

For this application, the geometry and properties are showed in "Fig. 5".

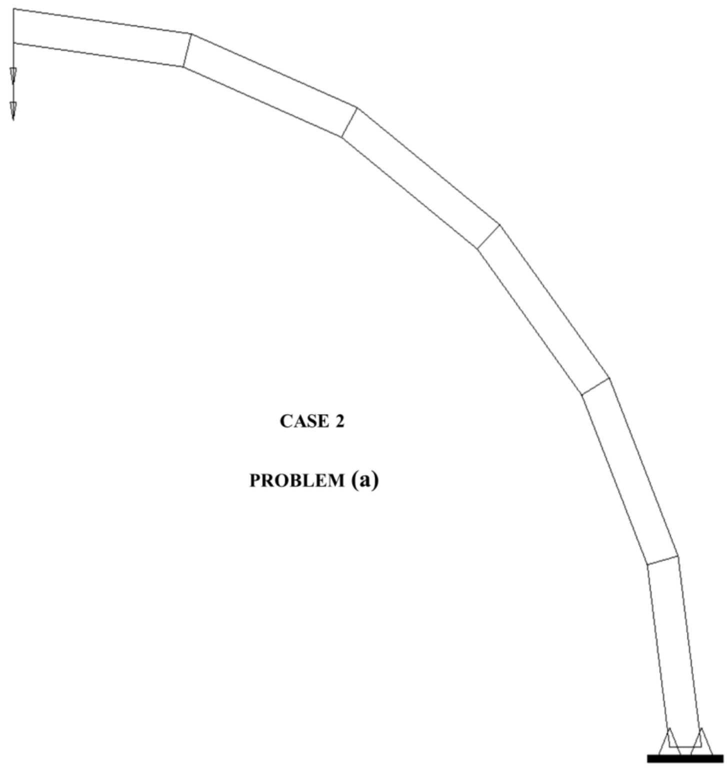

Figure 5. Curved beam: inner radius = 4.12; outer radius = 4.32; arc = 90°; depth (thickness): 0.1; E = 1.0x107; ν = 0.25; mesh = 6 x 1. Loading: unit forces at free end (tip). Constraint: fixed support.

As we can see in the "Table 4" (displacements results) and "Table 5" (normalized displacement results), for curved beam where only the bending load is applied (in-plane shear – case 2), it's important to point out that the element shape is not quite rectangular, which will also test the effect of slight irregularity. However, the enriched MLGFM in domain/on boundary demonstrates high accuracy as well as the enrichment only in the domain for this kind of shear problem.

Table 4. Displacements results and theoretical solution for curved beam

CURVED BEAM								
PROBLEM (a): REGULAR ELEMENT								
CASE TYPE	TIP LOAD	THEORETICAL*	CLASSIC FEM	MLGFM		ENRICHED MLGFM		
				4-NODE	9-NODE	DOMAIN/ BOUNDARY	DOMAIN	BOUNDARY
CASE 2	**IN-PLANE SHEAR**	**-8,7340E-02**	-6,4138E-03	-6,4137E-03	-7,7253E-02	-8,8285E-02	-8,8400E-02	-6,4137E-03

* MacNeal and Harder, 1985.

Table 5. Normalized tip displacements results in direction of load

CURVED BEAM							
PROBLEM (a): REGULAR ELEMENT							
CASE TYPE	TIP LOAD	CLASSIC	MLGFM		ENRICHED MLGFM		
			4-NODE	9-NODE	DOMAIN/ BOUNDARY	DOMAIN	BOUNDARY
CASE 2	**IN-PLANE SHEAR**	0,07343	0,07343	0,88451	1,01082	1,01214	0,07343

5. CONCLUSION

This paper presented the first results for MLGFM using an enriched method to enrich the domain, the boundary or both at the same time. The approach proposed here uses Lobatto shape functions as hierarchical shape functions since these ones naturally obey the delta property (δ condition) as in the context of FEM (only one shape function is non-null at each node).

It's given an overview of the Modified Local Green's Function Method (MLGFM) and the Hierarchical Finite Element Method (HFEM) with an emphasis on the mathematical approach of those two theories.

The results presented in this paper as for straight beam as for curved beam proved the Enriched MLGFM is superior compared to conventional FEM or MLGFM and its results are very close to the analytical solutions. Taking into account that, the examples indicate a strong potential of the Enriched MLGFM using both regular and distorted course mesh. Its high accuracy is first verified in extension straight beam as foreseen but more expressive for shear problems. Even in curved problems, where conventional FEM fails with coarse meshes, the Enriched MLGFM succeeds with a very low level of enrichment.

Once quoted that, it’s really important to point out the high possibilities of this method since only one level of enrichment was used on the examples explored in this paper. There is a huge field to be explored with MLGFM when it concerns to apply other enriched methods. For instance: other hierarchical level of enrichments, other shape functions, their numerical stability and convergence rate are consequently object for future researches as well as other problems in continuum mechanics.

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