ASPECTS OF THE STABILITY OF NONCONSERVATIVE STRUCTURAL SYSTEMS

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Abstract. *Several aspects concerning the dynamic stability of beams were analysed over the years, including paradoxical results such as the indifference of the flutter load of the Beck's column to the rigidity of a Winkler foundation. Such results have been criticized because of the difficulty in obtaining idealized follower loads in experiments. The aim of this paper is to clarify some of these results and explore different examples evaluating the influence of certain parameters in classical problems and the actual application of nonconservative loads. A computer program has been developed based on the Rayleigh-Ritz method, using nodal shape functions enriched with internal additional functions. Therefore, validation examples are presented and novel results are introduced to clarify some of the issues raised in the recent literature.*

Keywords: *Dynamic stability, Nonconservative, Follower force, Rayleigh-Ritz method*

1. INTRODUCTION

A considerable number of researches on elastic stability in systems subjected to non-conservative loads has been conducted for several years. Such loads have as a common characteristic the absence of a potential associated, inducing an energy variation of the system, such as aerodynamic, hydrodynamic and frictional forces. Recent research in bioengineering has also shown that follower forces are the best representation of compressive loads carried by the human lumbar spine (Rohlmann, Zander, Rao, & Bergmann, 2009).

It has been well known that non-conservative systems subjected to follower forces (rotation-dependent) forces or moments, may lose stability either by divergence, a static response, or by flutter, a dynamic response in the form of oscillations with exponentially growth of amplitude. Initial studies of elastic stability in non-conservative systems, mentioned by (Bolotin, 1963), conducted by Nicolai and Beck are worth referring. The former, in 1930, studied torsion in bars under follower moment and after a paradoxical result elucidated the need of a dynamic method and the latter, in 1952, was the first to determine the critical load for a cantilever column subjected to a follower compressive force at the end.

Thereafter, numerous works have been conducted standing out (Bolotin, 1963) and (Ziegler, 1977) for the general understanding. Intriguing characteristics for Beck's column were also obtained, such as: the indifference to the rigidity of a Winkler foundation (Smith & Hermann, 1972), the influence of elastic support (Sundararajan, 1974), the influence of shear deformation and mass distribution (Suanno & Rosas e Silva, 1989) and (Suanno, 1988).

The application of realistic follower forces has always been a concern and has received close scrutiny in recent papers (Elishakoff, 2005; Langthjem & Sugiyama, 2000). Motivated by these survey papers, the aim of this paper is to clarify some aspects and evaluate the influence of certain parameters in the classical Beck's column for a better understanding of the limitations and applications of this model, part of a more fundamental study on the origin and effects of flutter loads.

The present study investigates the influence of elastic support and rigidity of the Winkler foundation on Beck's column through the Rayleigh-Ritz method. The use of shape function in an association to nodal displacement enriched with internal functions it is an appealing approach due to the reduced number of degrees of freedom and ability to handle a variety of different boundary conditions easily imposed. The methodology was implemented in a computer program developed in MATLAB R2016b for the Euler-Bernoulli theory. Composed with several subroutines, the program accepts different types of loads, boundary conditions and Winkler foundation stiffness. Through a dynamic approach, the critical load is evaluated and the behaviour of the frequencies analysed to verify the occurrence of flutter or divergence.

2. MATHEMATICAL MODEL

Figure 1 depicts a cantilever column of length $\boldsymbol{L}$, mass distribution through the column length $\boldsymbol{m}$, flexural rigidity $\boldsymbol{EI}$, Winkler foundation $\boldsymbol{k}$ and compressive force $\boldsymbol{P}$. Neglecting axial and shear deformation, the displacement in the y-direction of the column is represented as a function dependent on the axial coordinate, $\boldsymbol{x}$ and time, $\boldsymbol{t}$:

$$\bar{w}(x,t) = \sum_{i=1}^{n} w_i(x)\, e^{i\omega_i t}, \tag{1}$$

where $\boldsymbol{\omega_i}$ is the natural frequency and $\boldsymbol{w(x)}$ represents the transverse displacement of the beam axis. This deflection is approximated by the Rayleigh-Ritz method, combining the beam element nodal shape functions and additional functions to enrich the interpolation within the element. Thus, the function is represented as

$$w(x) = \begin{bmatrix} N_B & N_A \end{bmatrix} \begin{bmatrix} q_B \\ q_A \end{bmatrix}, \tag{2}$$

where $\boldsymbol{N_B}$ and $\boldsymbol{N_A}$ are the nodal shape functions and additional functions related with the nodal displacements $\boldsymbol{q_B}$ and generalized displacements $\boldsymbol{q_A}$, respectively.

The nodal displacements (two at each end) are associated with the usual beam shape functions and the additional functions are required to provide zero displacement and zero rotation at the end nodes of the element. This characteristic is guaranteed, for instance, by the association of a third-degree polynomial and a sine term. This set additional functions can be viewed as a generalized form of Fourier series. Consequently, the amount of additional functions used on the model is defined by a convergence process, the higher the amount, the higher accuracy of the element deformation.

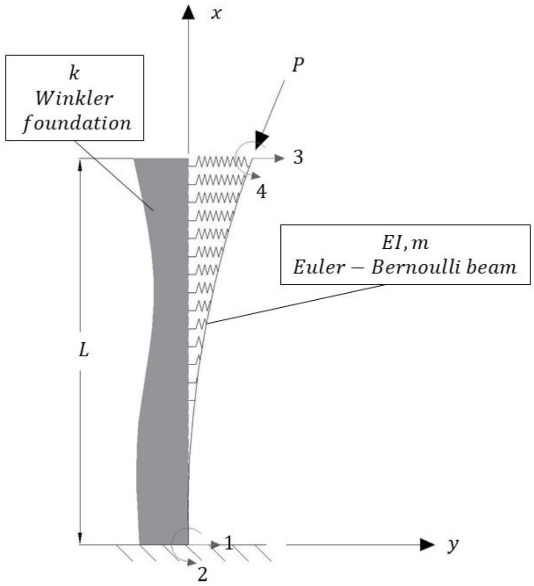

Figure 1 – Schematic representation of the beam under follower force resting on a Winkler foundation

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The boundary conditions for a cantilever beam may be imposed just by adding adequate spring stiffness value for each nodal degree of freedom, i.e. zero represents a free displacement and the greater the value the closer will be to a restraint showed in Fig. 2. This procedure is analogous to the penalty method and corresponds to a simpler way to impose the conditions and evaluate the influence of elastic supports at the free end, $x = L$.

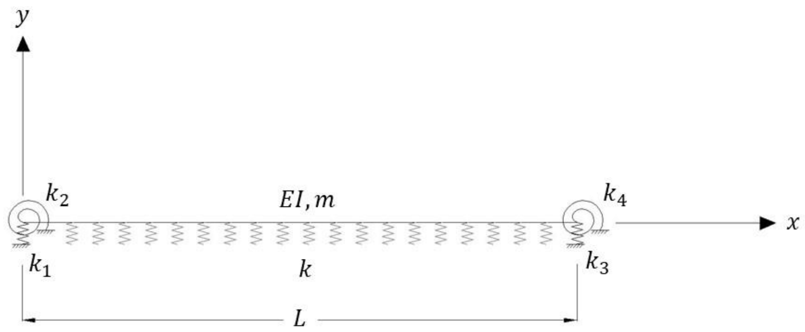

Figure 2 – Beam element

The equation of motion for a non-conservative system is determined by a generalized form of Hamilton's principle, formulated as

$$\delta \int_{t_1}^{t_2} (T - \pi)\, dt + \int_{t_1}^{t_2} \delta W_{NC}\, dt = 0, \tag{3}$$

where $\boldsymbol{T}$ is the kinetic energy, $\boldsymbol{\pi}$ is the elastic potential energy and $\boldsymbol{\delta W_{NC}}$ is the non-conservative virtual work. The kinematic and potential energy are under the assumption of small displacements and linear elastic material behaviour.

The first part of Eq. (3) results in the usual system equation of motion for buckling analysis through dynamic approach and the second one represents the load correction considering the behaviour of the non-conservative force. Thereby, the equation of motion results in the following form:

$$[(K_e - \lambda(K_g + K_{nc})) - \omega^2 M]q = 0, \tag{4}$$

where $\boldsymbol{M}$ is the mass matrix, $\boldsymbol{K}$ is the stiffness matrix subdivided in elastic stiffness ($\boldsymbol{K_e}$), geometric stiffness ($\boldsymbol{K_g}$) and a matrix corresponding to the non-conservative load behaviour ($\boldsymbol{K_{nc}}$), $\boldsymbol{q}$ is the generalized displacement vector and **λ** a load parameter.

All the matrices are obtained by the energy functional and symmetrical, except $\boldsymbol{K_{nc}}$ matrix, which is determined considering the equilibrium of a deformed configuration, Fig. 3. Appling a variation of displacements, d_1 and d_2, in the undisturbed configuration Figure 3 – (a), a variation of the applied force is obtained and investigated Fig. 3 – (b) and (c). Consequently, the $\boldsymbol{K_{nc}}$ matrix for a follower force applied at the tip of the column is

$$[\delta P] = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{Bmatrix} \delta d_1 \\ \delta d_2 \end{Bmatrix}. \tag{5}$$

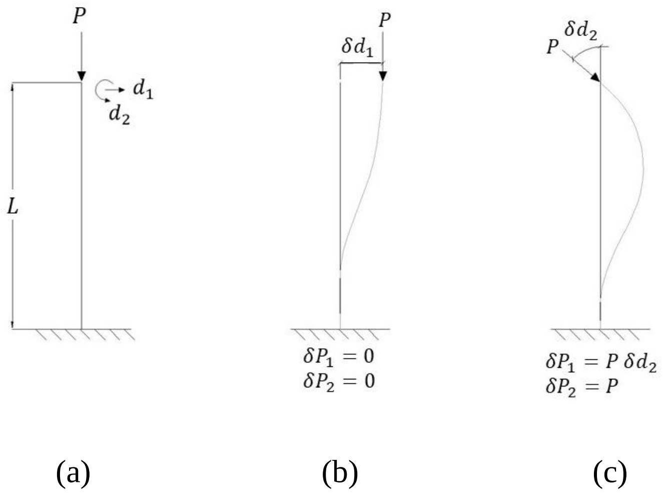

Figure 3 - Load behaviour matrix for nodal follower force *Cf.* (Suanno, 1988)

The generalized eigenvalue problem in Eq. (4) is fundamental to the evaluation of the critical load. This process must be repeated for increasing load values and for each step the frequencies must be analysed until system loose stability. The frequency behaviour determines the type of instability: a coalescence of frequencies indicates a flutter instability while the vanishing of ae frequencies, indicates a divergence.

3. RESULTS AND DISCUSSION

The follow dimensionless parameters are used in the representation of the results:

$$\bar{x} = \frac{x}{L_{[1]}};\ p = P\ \frac{L_{[1]}^2}{EI_{[1]}};\ \kappa = k\frac{{L_{[1]}}^4}{EI_{[1]}};\ \tau = t\sqrt{\frac{EI_{[1]}}{m_{[1]}{L_{[1]}}^4}};\ k_3 \propto \frac{EI_{[2]}}{L_{[2]}^3};\ k_4 \propto \frac{EI_{[2]}}{L_{[2]}} \qquad (6)$$

The numbers 1 and 2 in square brackets were used to distinguish the column, object of study, and the beam supported at the end of the column, $x = L$, to provide a more realistic understanding for the effect of elastic supports.

The most determinant aspects of the program to achieve adequate results are the number of degrees of freedom and the initial load step. Therefore, their influence was investigated to ensure the best combination of processing time and accuracy of the results. The number of d.o.f. to incorporate the model was defined by a convergence test for each of the examples studied, and the initial load step were limited by the distance of the first and second critical load to make sure that the result corresponds to the lowest critical load.

The validity of the program and the methodology associated with it were checked by comparing the results with the well-known critical load of Beck's column (Bolotin, 1963; Ziegler, 1977),

$$P_{cr} = 20.05\,\frac{EI}{L^2}, \tag{7}$$

and the paradoxical result obtained by (Smith & Hermann, 1972) that the buckling load $\boldsymbol{P}$ is indifferent to the stiffness of the foundation.

Both examples designated for the validation test lead to satisfactory results, Fig. 4 and Fig. 5. Moreover, the variation of the first and second frequencies through the load increment, an important characteristic of the non-conservative system, was as expected and the critical load was estimated by their coalescence.

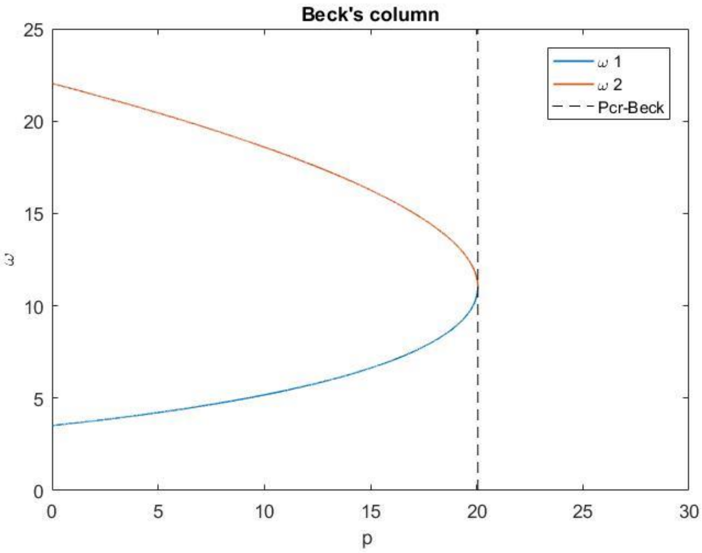

Figure 4 - First and second frequencies variation with load magnitude

The flutter load is indifferent to the Winkler foundation, Fig. 5, a paradoxical result since the tendency expected would be a raise in the flutter load with increasing foundation stiffness. However, it should be mentioned that the system frequencies do increase with increasing stiffness foundation, as well as the coalescence frequency.

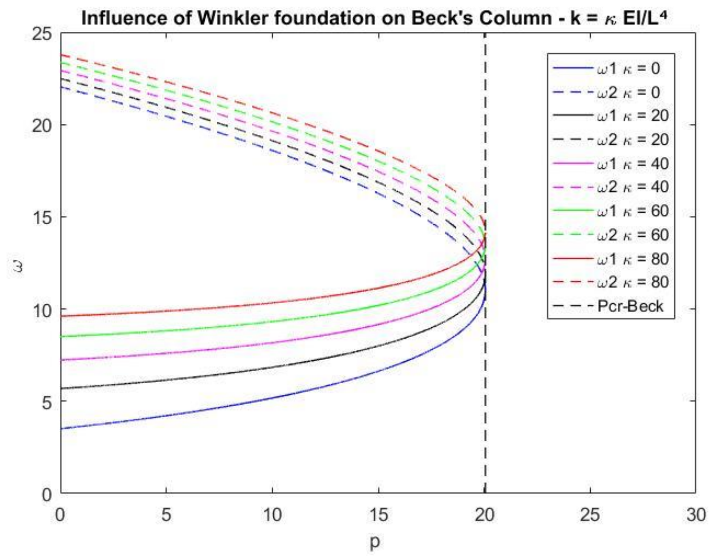

Figure 5 - First and second frequencies vs. load magnitude for different foundation constant

The influence of an elastic support at the free end of Beck's column was first analysed separately for the transverse displacement restraint and rotation restraint. The columns were then clamped-elastically pinned and a clamped-elastically clamped. Fig. 6 and Fig. 7 The variation of critical load with increasing spring rigidity is presented in Figure 6 and Figure 7. Their behaviours are similar: for lower rigidity values the critical condition corresponds to flutter and at a certain point the condition changes from flutter to divergence.

$$k_{3-lim} = 34.85\,\frac{EI_{[2]}}{L_{[2]}^{3}} \quad and \quad k_{4-lim} = 4.60\,\frac{EI_{[2]}}{L_{[2]}}. \tag{8}$$

The transition point, k_{3-lim}, and the stability behaviour of the clamped-elastically pinned column is in agreement with the reported results (Sundararajan, 1976) and (Suanno, 1988).

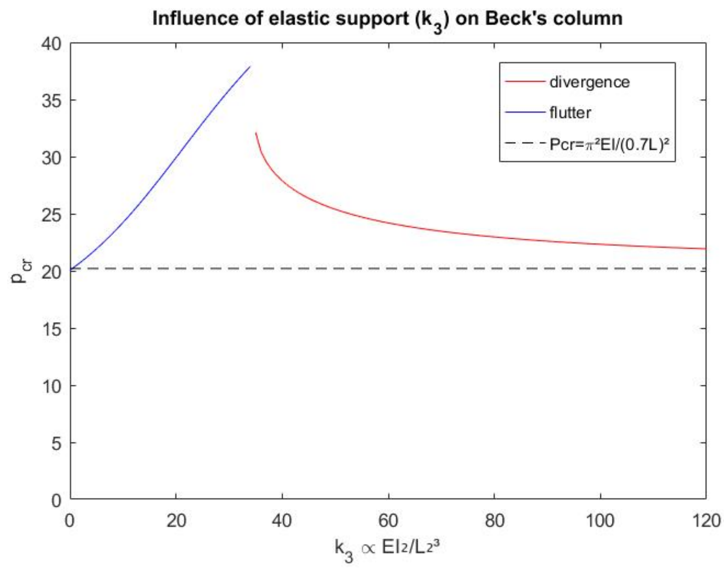

Figure 6 – Critical load vs. transversal displacement spring stiffness

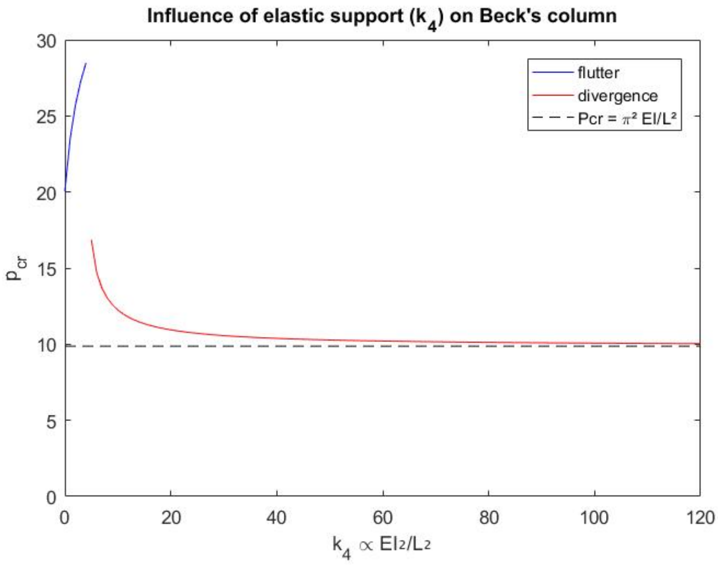

Figure 7 - Critical load vs. rotational spring stiffness

These points represent also a change on the stabilizing effect for an increasing of rigidity. At first, a stabilizing effect is observed, with the critical load increasing with the rigidity. Beyond the transition point a sudden decrease appears and then the critical load converges to:

$$P_{cr-1} = \frac{\pi^2 EI_{[1]}}{(0.7\ L_{[1]})^2} \quad and \ \ P_{cr-2} = \frac{\pi^2 EI_{[1]}}{{L_{[1]}}^2}, \tag{9}$$

which are the static critical loads for clamped-pinned and a clamped-guided column, respectively. The horizontal component of the force loses its intensity and its stabilizing effect vanishes with the increasing spring rigidity, until the system becomes conservative for $k_3 \to \infty$ and $k_4 \to \infty$.

Despite their similar responses, the influence of elastic support on clamped-elastically pinned column and clamped-elastically clamped systems still presents some differences, mainly on the magnitude of the transition point Eq. (8) and the rate of convergence on the divergence path to Eq. (9). Therefore, the sensitivity of Beck's column to rotation restraint appears to be more pronounced than to transverse displacement restraint.

For a better understanding of the behaviour of the system with an increase of the spring stiffness of the elastic support, the variation of the frequencies was analysed. Figure 8 shows the variation of the first and second frequencies of the clamped-elastically clamped column and due to their similarity, this analysis is representative for both. The flutter instability occurs until the transition point, Eq. (8), and for grater values of rigidity, the first frequency vanishes, corresponding to divergence instability. It is evident that there is a sudden destabilization effect when the system changes the instability type, observed in Figure 6 and Figure 7 and clarified in Figure 8. The first frequency vanishes at a considerable smaller load value when the transition point is reached, and the divergence critical loads continues to present this effect until converging. These characteristics are related to the slope of the first

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frequency curve. A change of slope sign appeared as the stiffness increases in the first frequency, whereas the second frequency remains with the same. For a free end, $k_4 = 0$, the slope of the first frequency curve is positive and for an increasing rigidity the slope becomes negative and the rate of increase of critical load decreases until the first frequency vanishes. Therefore, after the instability type changes, the critical load decreases for an increasing stiffness and the slope of the first frequency curve becomes more negative.

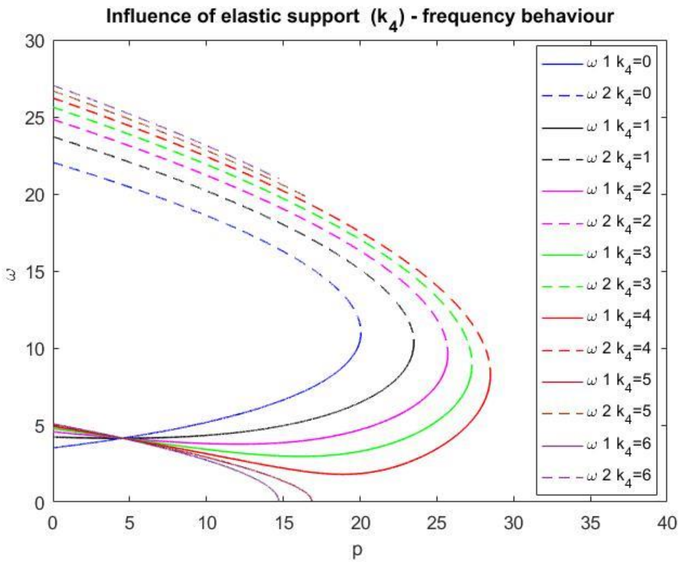

Figure 8 - First and second frequencies vs. load magnitudefor different rotational spring stiffnesses

The influence analysis of elastic supports and Winkler foundation combined summarizes the former results. The overall system's behaviour, shown in Fig. 9 and Fig. 10, is similar as described in Fig. 6 and Fig. 7 two distinct paths related to different types of instabilities and for higher values of spring rigidity the critical load converges to a conservative response.

Although the influence of Winkler foundation is minimal on the Beck's column, the relevance in the elastically supported system is elucidated by:

- the increasing of the flutter instability path due to the foundation stiffening;
- the pronounced stability effect on the divergence path.

The indifference of Winkler foundation and growth of the coalescence frequency described for the Beck's column, Fig. 5, are the reasons for the increase in the transition points and the common flutter path for different foundation stiffness. Moreover, the stability effect on the divergence path is related to the standard behaviour known for conservative systems in presence of Winkler foundation. This stabilizing effect is even more evident if the type of instability changes increasing the foundation stiffness, observed in Fig. 9 e Fig. 10. For example, with $\boldsymbol{k_4} = 6$ and $\boldsymbol{\kappa} = 0$ the system loses stability by divergence with $\boldsymbol{p_{cr}} = 14.72$, whereas with the same elastic support and $\boldsymbol{\kappa} = 50$ the system loses stability by flutter and its critical load increases significantly, $\boldsymbol{p_{cr}} = 30.26$.

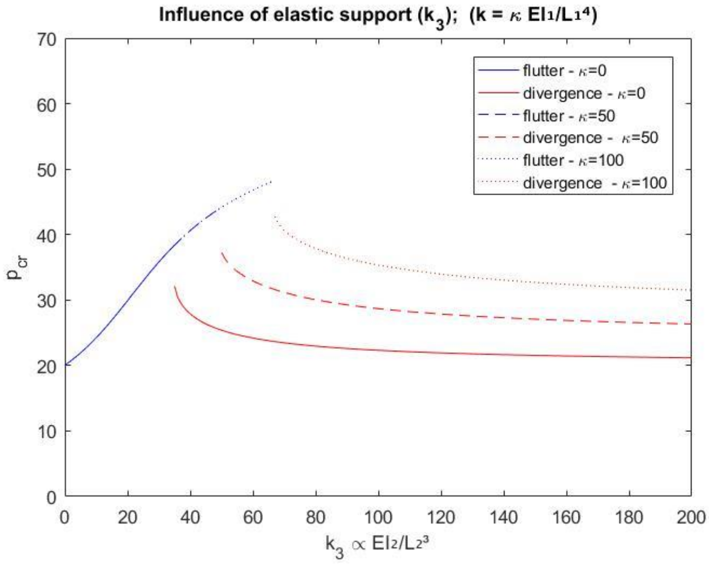

Figure 9 – Critical load vs. translational spring stiffness, for different foundation constants

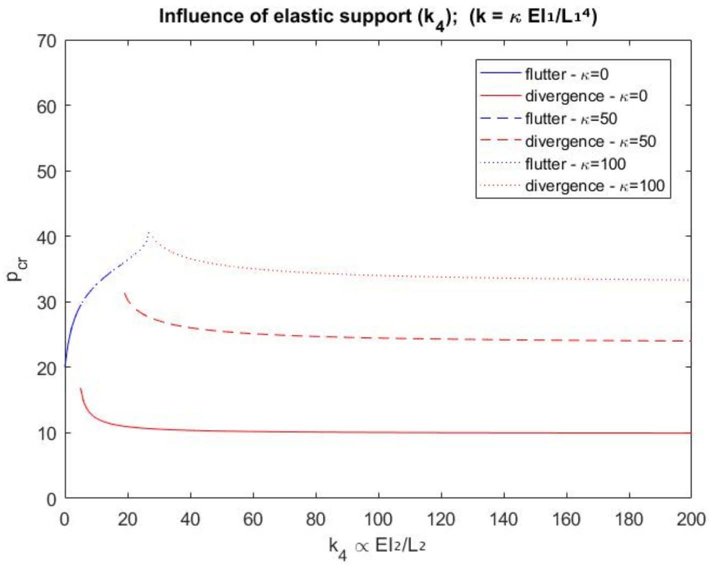

Figure 10 – Critical load vs. rotational spring stiffness, for different foundation constants

4. CONCLUSION

The paper presents an analysis of the influence of different aspects, Winkler foundation and elastic support, on the classical problem of Beck's column. Rayleigh-Ritz method presented good correlation to the analytical solutions used to validate the method implemented herein. The advantages of this method were mostly observed on the imposition of the boundary conditions, allowing to easily change the rigidity of the elastic supports, and the smaller number of d.o.f. comparing to FEM.

The Winkler foundation and elastic support influence on Beck's column were analysed separately and combined. The results of the separate analysis confirmed the paradoxical result of the critical load indifference to Winkler foundation constant: for different foundation stiffness only the frequency of coalescence increases. In the case of the influence of elastic support, the increase in the transversal and rotational constraints leads to a stabilizing effect when the system loses stability by flutter and to a destabilizing effect if it loses stability by divergence.

The combined analysis reinforces the conclusions of the separate analysis. The increase on the frequency of coalescence occurs for larger range of stiffness of elastic support, when the system loses stability by flutter. The influence of Winkler foundation is distinct for each type of instability. If the system loses stability by flutter the system is indifferent to the presence of the foundation. However, if the critical condition is divergence, the foundation has a stabilizing effect, more evident for a spring stiffness close to the transition point.

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